Coordinating a closed-loop green supply chain for remanufactured product under competition

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Appendices

According to our assumptions, the profit functions of the remanufacturer, retailers and collectors are given as follows:

$$\Pi_{RM}(w_1, w_2, m_1, m_2, \theta) = (w_1 - c)D_1 + (w_2 - c)D_2 - (m_1 + c_I)R_1 - (m_2 + c_I)R_2 - \lambda\theta^2; \quad (1)$$

$$\Pi_{Ri}(p_i) = (p_i - w_i)D_i; \quad (2)$$

$$\Pi_{Ci}(a_i) = (m_i - a_i)R_i. \quad (3)$$

Appendix A

Proof of Proposition 1

The centralized model is

$$\max_{(p_1,p_2,a_1,a_2,\theta)} \Pi^C(p_1, p_2, a_1, a_2, \theta) = (p_1 - c)D_1 + (p_2 - c)D_2 - (a_1 + c_I)R_1 - (a_2 + c_I)R_2 - \lambda\theta^2$$

subject to $D^C = \tau R^C$

So, we consider the Lagrangian function as

$$\Pi_L^C = \Pi^C + L(\tau R^C - D^C), \quad (4)$$

where $L(\geq 0)$ is the Lagrangian multiplier.

The corresponding Hessian matrix is given by

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$$H^C = \begin{pmatrix} \frac{\partial^2\Pi_L^C}{\partial p_1^2} & \frac{\partial^2\Pi_L^C}{\partial p_1\partial p_2} & \frac{\partial^2\Pi_L^C}{\partial p_1\partial a_1} & \frac{\partial^2\Pi_L^C}{\partial p_1\partial a_2} & \frac{\partial^2\Pi_L^C}{\partial p_1\partial \theta} \\ \frac{\partial^2\Pi_L^C}{\partial p_2\partial p_1} & \frac{\partial^2\Pi_L^C}{\partial p_2^2} & \frac{\partial^2\Pi_L^C}{\partial p_2\partial a_1} & \frac{\partial^2\Pi_L^C}{\partial p_2\partial a_2} & \frac{\partial^2\Pi_L^C}{\partial p_2\partial \theta} \\ \frac{\partial^2\Pi_L^C}{\partial a_1\partial p_1} & \frac{\partial^2\Pi_L^C}{\partial a_1\partial p_2} & \frac{\partial^2\Pi_L^C}{\partial a_1^2} & \frac{\partial^2\Pi_L^C}{\partial a_1\partial a_2} & \frac{\partial^2\Pi_L^C}{\partial a_1\partial \theta} \\ \frac{\partial^2\Pi_L^C}{\partial a_2\partial p_1} & \frac{\partial^2\Pi_L^C}{\partial a_2\partial p_2} & \frac{\partial^2\Pi_L^C}{\partial a_2\partial a_1} & \frac{\partial^2\Pi_L^C}{\partial a_2^2} & \frac{\partial^2\Pi_L^C}{\partial a_2\partial \theta} \\ \frac{\partial^2\Pi_L^C}{\partial \theta\partial p_1} & \frac{\partial^2\Pi_L^C}{\partial \theta\partial p_2} & \frac{\partial^2\Pi_L^C}{\partial \theta\partial a_1} & \frac{\partial^2\Pi_L^C}{\partial \theta\partial a_2} & \frac{\partial^2\Pi_L^C}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} -2\alpha & 2\beta & 0 & 0 & \gamma \\ 2\beta & -2\alpha & 0 & 0 & \gamma \\ 0 & 0 & -2\delta & 2\eta & 0 \\ 0 & 0 & 2\eta & -2\delta & \mu \\ \gamma & \gamma & 0 & \mu & -2\lambda \end{pmatrix}$$

The leading principle minors are $|M_1| = -2\alpha < 0$, $|M_2| = 4(\alpha^2-\beta^2) > 0$, $|M_3| = -8\delta(\alpha^2-\beta^2) < 0$, $|M_4| = 16(\alpha^2-\beta^2)(\delta^2-\eta^2) > 0$, and $|H^C| = -16(\alpha+\beta)(\delta+\eta)[(\delta-\eta)\big(2\lambda(\alpha-\beta)-\gamma^2\big) - (\alpha-\beta)\mu^2] < 0$ if $\lambda > \frac{(\alpha-\beta)\mu^2+(\delta-\eta)\gamma^2}{2(\alpha-\beta)(\delta-\eta)}$. Therefore, the Hessian matrix is negative definite if $\lambda > \frac{(\alpha-\beta)\mu^2+(\delta-\eta)\gamma^2}{2(\alpha-\beta)(\delta-\eta)}$.

Thus, we find that Π_L^C is jointly concave in $p_i,\ a_i,\ \theta$. So, the optimal solution can be determined by using KKT condition, *i.e.* $\frac{\partial\Pi_L^C}{\partial p_i} = 0$, $\frac{\partial\Pi_L^C}{\partial a_i} = 0$, $\frac{\partial\Pi_L^C}{\partial \theta} = 0$, $L(\tau R^C - D^C) = 0$, $(\tau R^C - D^C) = 0$, and $L \geq 0$. Now, from $\frac{\partial\Pi_L^C}{\partial p_i} = 0$, $\frac{\partial\Pi_L^C}{\partial a_i} = 0$, $\frac{\partial\Pi_L^C}{\partial \theta} = 0$, we can get the values of $p_i,\ a_i,\ \theta$ in terms of L which are given by

$$\begin{aligned} p_i^C &= \frac{D_{0i}\Omega_1 + D_{0j}\Omega_2 + (\alpha+\beta)\big(2(c+L)[2(\delta-\eta)\big(\lambda(\alpha-\beta)-\gamma^2\big) - 2\mu^2(\alpha-\beta)] - \gamma\mu[\Phi_2 + 2L\tau(\delta-\eta)]\big)}{4(\alpha+\beta)[(\delta-\eta)\big(2(\alpha-\beta)\lambda-\gamma^2\big) - (\alpha-\beta)\mu^2]}, \\ a_i^C &= \frac{-R_{0i}\Omega_3 - R_{0j}\Omega_4 + (\delta+\eta)\gamma\mu[\Phi_3 - 2L(\alpha-\beta)] - 2(c_I - L\tau)[(\delta-\eta)\big(2\lambda(\alpha-\beta)-\gamma^2\big) - 2(\alpha-\beta)\mu^2]}{4(\delta+\eta)[(\alpha-\beta)\big(2(\delta-\eta)\lambda-\mu^2\big) - (\delta-\eta)\gamma^2]}, \\ \theta^C &= \frac{\gamma(\delta-\eta)\Phi_3 - \mu(\alpha-\beta)\Phi_2 - 2L(\alpha-\beta)(\delta-\eta)(\gamma+\mu\tau)}{2[2(\alpha-\beta)(\delta-\eta)\lambda - \big((\alpha-\beta)\mu^2 + (\delta-\eta)\gamma^2\big)]}. \end{aligned}$$

where $\Omega_1 = [(\delta-\eta)(4\alpha\lambda-\gamma^2) - 2\alpha\mu^2]$, $\Omega_2 = [(\delta-\eta)(4\beta\lambda+\gamma^2) - 2\beta\mu^2]$, $\Omega_3 = [(\alpha-\beta)(4\delta\lambda-\mu^2) - 2\delta\gamma^2]$, $\Omega_4 = [(\alpha-\beta)(4\eta\lambda-\mu^2) - 2\eta\gamma^2]$.

Now, from $\tau R^C - D^C = 0$, we get $L = \frac{\Phi_3[(\alpha-\beta)\big(2\lambda(\delta-\eta)-\mu^2\big)+\tau\gamma\mu(\delta-\eta)] - \Phi_2[\tau(\delta-\eta)\big(2\lambda(\alpha-\beta)-\gamma^2\big)+\gamma\mu(\alpha-\beta)]}{\Psi_1}$

Putting this value of L in the above equations, the optimal values of the decision variables can be obtained as given in Proposition 1.

Appendix B

Proof of Proposition 2

We use backward induction method during the calculation of decentralized models. So, the followers *i.e.* the collectors and the retailers first determine their decisions. As $\frac{\partial^2\Pi_{Ci}^N}{\partial a_i^2} = -2\delta < 0$ and $\frac{\partial^2\Pi_{Ri}^N}{\partial p_i^2} = -2\alpha < 0$, there exists unique response for the collectors and the retailers which are given below:

$$\tilde{a_i}^N(m_i, m_j, \theta) = \frac{2\delta(-R_{0i} + \delta m_i + \mu\theta) + \eta(-R_{0j} + \delta m_j + \mu\theta)}{4\delta^2 - \eta^2}, \tag{5}$$

$$\tilde{p_i}^N(w_i, w_j, \theta) = \frac{2\alpha(D_{0i} + \alpha w_i + \gamma\theta) + \beta(D_{0j} + \alpha w_j + \gamma\theta)}{4\alpha^2 - \beta^2}. \tag{6}$$

Putting these decisions in the remanufacturer's profit function, we get

$$\begin{aligned}\Pi_{RM}^{N} &= \sum_{i=1}^{2}\frac{(w_i-c)\alpha[2\alpha D_{0i}+\beta D_{0j}-(2\alpha^2-\beta^2)w_i+\alpha\beta w_j+(2\alpha+\beta)\gamma\theta]}{4\alpha^2-\beta^2}\\ &\quad -\sum_{i=1}^{2}\frac{(m_i+c_I)\delta[2\delta R_{0i}+\eta R_{0j}+(2\delta^2-\eta^2)m_i-\delta\eta m_j-(2\delta+\eta)\mu\theta]}{4\delta^2-\eta^2}-\lambda\theta^2 \end{aligned} \tag{7}$$

So, the remanufacturer's problem becomes

$$\begin{aligned}&\max_{(w_1,w_2,m_1,m_2,\theta)} \Pi_{RM}^{N}(w_1,w_2,m_1,m_2,\theta)\\ &\text{subject to } D^N=\tau R^N\end{aligned}$$

Now, we consider the Lagrangian function as

$$\Pi_L^N = \Pi_{RM}^N + L(\tau R^N - D^N), \tag{8}$$

where $L(\geq 0)$ is the Lagrangian multiplier.

The corresponding Hessian matrix is given by

$$H^N = \begin{pmatrix} \frac{\partial^2\Pi_L^N}{\partial w_1^2} & \frac{\partial^2\Pi_L^N}{\partial w_1\partial w_2} & \frac{\partial^2\Pi_L^N}{\partial w_1\partial m_1} & \frac{\partial^2\Pi_L^N}{\partial w_1\partial m_2} & \frac{\partial^2\Pi_L^N}{\partial w_1\partial\theta}\\ \frac{\partial^2\Pi_L^N}{\partial w_2\partial w_1} & \frac{\partial^2\Pi_L^N}{\partial w_2^2} & \frac{\partial^2\Pi_L^N}{\partial w_2\partial m_1} & \frac{\partial^2\Pi_L^N}{\partial w_2\partial m_2} & \frac{\partial^2\Pi_L^N}{\partial w_2\partial\theta}\\ \frac{\partial^2\Pi_L^N}{\partial m_1\partial w_1} & \frac{\partial^2\Pi_L^N}{\partial m_1\partial w_2} & \frac{\partial^2\Pi_L^N}{\partial m_1^2} & \frac{\partial^2\Pi_L^N}{\partial m_1\partial m_2} & \frac{\partial^2\Pi_L^N}{\partial m_1\partial\theta}\\ \frac{\partial^2\Pi_L^N}{\partial m_2\partial w_1} & \frac{\partial^2\Pi_L^N}{\partial m_2\partial w_2} & \frac{\partial^2\Pi_L^N}{\partial m_2\partial m_1} & \frac{\partial^2\Pi_L^N}{\partial m_2^2} & \frac{\partial^2\Pi_L^N}{\partial m_2\partial\theta}\\ \frac{\partial^2\Pi_L^N}{\partial\theta\partial w_1} & \frac{\partial^2\Pi_L^N}{\partial\theta\partial w_2} & \frac{\partial^2\Pi_L^N}{\partial\theta\partial m_1} & \frac{\partial^2\Pi_L^N}{\partial\theta\partial m_2} & \frac{\partial^2\Pi_L^N}{\partial\theta^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2\alpha(2\alpha^2-\beta^2)}{4\alpha^2-\beta^2} & \frac{2\alpha^2\beta}{4\alpha^2-\beta^2} & 0 & 0 & \frac{\alpha\gamma}{2\alpha-\beta}\\ \frac{2\alpha^2\beta}{4\alpha^2-\beta^2} & -\frac{2\alpha(2\alpha^2-\beta^2)}{4\alpha^2-\beta^2} & 0 & 0 & \frac{\alpha\gamma}{2\alpha-\beta}\\ 0 & 0 & -\frac{2\delta(2\delta^2-\eta^2)}{4\delta^2-\eta^2} & \frac{2\delta^2\eta}{4\delta^2-\eta^2} & \frac{\delta\mu}{2\delta-\eta}\\ 0 & 0 & \frac{2\delta^2\eta}{4\delta^2-\eta^2} & -\frac{2\delta(2\delta^2-\eta^2)}{4\delta^2-\eta^2} & \frac{\delta\mu}{2\delta-\eta}\\ \frac{\alpha\gamma}{2\alpha-\beta} & \frac{\alpha\gamma}{2\alpha-\beta} & \frac{\delta\mu}{2\delta-\eta} & \frac{\delta\mu}{2\delta-\eta} & -2\lambda \end{pmatrix}$$

The leading principle minors are $|M_1| = -\frac{2\alpha(2\alpha^2-\beta^2)}{4\alpha^2-\beta^2} < 0$, $|M_2| = \frac{4\alpha^2(\alpha^2-\beta^2)}{4\alpha^2-\beta^2} > 0$, $|M_3| = -\frac{8\alpha^2\delta(\alpha^2-\beta^2)(2\delta^2-\eta^2)}{(4\alpha^2-\beta^2)(4\delta^2-\eta^2)} < 0$, $|M_4| = \frac{16\alpha^2\delta^2(\alpha^2-\beta^2)(\delta^2-\eta^2)}{(4\alpha^2-\beta^2)(4\delta^2-\eta^2)} > 0$, and
$|H^N| = -\frac{16\alpha^2\delta^2(\alpha+\beta)(\delta+\eta)\left((\delta-\eta)(2\delta-\eta)[2\lambda(2\alpha-\beta)(\alpha-\beta)-\alpha\gamma^2]-\delta\mu^2(\alpha-\beta)(2\alpha-\beta)\right)}{(2\alpha-\beta)(2\delta-\eta)(4\alpha^2-\beta^2)(4\delta^2-\eta^2)} < 0$, if
$\lambda > \frac{(\alpha-\beta)(2\alpha-\beta)\delta\mu^2+(\delta-\eta)(2\delta-\eta)\alpha\gamma^2}{2(\alpha-\beta)(\delta-\eta)(2\alpha-\beta)(2\delta-\eta)}$. Thus the Hessian matrix is negative definite if
$\lambda > \frac{(\alpha-\beta)(2\alpha-\beta)\delta\mu^2+(\delta-\eta)(2\delta-\eta)\alpha\gamma^2}{2(\alpha-\beta)(\delta-\eta)(2\alpha-\beta)(2\delta-\eta)}$.

Thus, we find that Π_L^N is jointly concave in $w_i,\ m_i,\ \theta$. So, the optimal solution can be determined by using KKT condition, *i.e.* $\frac{\partial\Pi_L^N}{\partial w_i}=0$, $\frac{\partial\Pi_L^N}{\partial m_i}=0$, $\frac{\partial\Pi_L^N}{\partial\theta}=0$, $L(\tau R^N-D^N)=0$, $(\tau R^N-D^N)=0$, and $L\geq 0$. Now, from $\frac{\partial\Pi_L^N}{\partial w_i}=0$, $\frac{\partial\Pi_L^N}{\partial m_i}=0$, $\frac{\partial\Pi_L^N}{\partial\theta}=0$, we can get the values of $w_i,\ m_i,\ \theta$ in terms of L which are given by

$$w_i^N = \frac{\begin{pmatrix} D_{0i}\Theta_1 + D_{0j}\Theta_2 - \gamma\delta\mu(\alpha+\beta)(2\alpha-\beta)[\Phi_2 + 2L\tau(\delta-\eta)] \\ +2(c+L)(\alpha+\beta)[2(\delta-\eta)(2\delta-\eta)\big(\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big) - 2\delta\mu^2(\alpha-\beta)(2\alpha-\beta)] \end{pmatrix}}{4(\alpha+\beta)[2(\alpha-\beta)(\delta-\eta)(2\alpha-\beta)(2\delta-\eta)\lambda - \big((\alpha-\beta)(2\alpha-\beta)\delta\mu^2 + (\delta-\eta)(2\delta-\eta)\alpha\gamma^2\big)]}, \quad (9)$$

$$m_i^N = \frac{\begin{pmatrix} -R_{0i}\Theta_3 - R_{0j}\Theta_4 + (\delta+\eta)(2\delta-\eta)\alpha\gamma\mu[\Phi_3 - 2L(\alpha-\beta)] \\ -2(c_I - L\tau)(\delta+\eta)[(\delta-\eta)(2\delta-\eta)\big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big) - 2(\alpha-\beta)(2\alpha-\beta)\delta\mu^2] \end{pmatrix}}{4(\delta+\eta)[2(\alpha-\beta)(\delta-\eta)(2\alpha-\beta)(2\delta-\eta)\lambda - \big((\alpha-\beta)(2\alpha-\beta)\delta\mu^2 + (\delta-\eta)(2\delta-\eta)\alpha\gamma^2\big)]}, \quad (10)$$

$$\theta^N = \frac{\alpha\gamma(\delta-\eta)(2\delta-\eta)\Phi_3 + \delta\mu(\alpha-\beta)(2\alpha-\beta)\Phi_2 - 2L(\alpha-\beta)(\delta-\eta)\big(\alpha\gamma(2\delta-\eta) + \delta\mu\tau(2\alpha-\beta)\big)}{2[2(\alpha-\beta)(\delta-\eta)(2\alpha-\beta)(2\delta-\eta)\lambda - \big((\alpha-\beta)(2\alpha-\beta)\delta\mu^2 + (\delta-\eta)(2\delta-\eta)\alpha\gamma^2\big)]}. \quad (11)$$

$\Theta_1 = (\delta-\eta)(2\delta-\eta)[4\beta\lambda(2\alpha-\beta)+\alpha\gamma^2]-2\beta\delta\mu^2(2\alpha-\beta)$, $\Theta_2 = \alpha[(\delta-\eta)(2\delta-\eta)[4\lambda(2\alpha-\beta)-\gamma^2]-2\delta\mu^2(2\alpha-\beta)]$, $\Theta_3 = \delta[2(2\delta-\eta)[2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2]-\mu^2(\alpha-\beta)(2\alpha-\beta)]$, $\Theta_4 = 2\eta(2\delta-\eta)[2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2]+\delta\mu^2(\alpha-\beta)(2\alpha-\beta)$.

Now, from $\tau R^N - D^N = 0$, we get

$$L = \frac{\alpha\Phi_3[(\alpha-\beta)\Big(2\lambda(\delta-\eta)(2\delta-\eta)-\delta\mu^2\Big)+\tau\gamma\delta\mu(\delta-\eta)]-\delta\Phi_2[\tau(\delta-\eta)\Big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\Big)+\alpha\gamma\mu(\alpha-\beta)]}{\Psi_2}$$

Putting this value of L in the above equations, the optimal values of the decision variables can be obtained as given in Proposition 2.

Appendix C

Proof of Proposition 4

On simplification, we have

$$\begin{aligned}
\theta^N - \theta^J &= \frac{\alpha\delta\Phi_1\big((\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3\big)}{\Psi_2} - \frac{\alpha\Phi_1\big((\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3\big)}{\Psi_3} \\
&= \frac{4\alpha^2\eta\lambda(\alpha-\beta)^2(\delta-\eta)\Phi_1\big((\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3\big)}{\Psi_2\Psi_3} > 0 \\
p_i^J - p_i^N &= \frac{\alpha^2\eta\lambda(\alpha-\beta)(\delta-\eta)[(\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3]}{2(\alpha+\beta)(2\alpha+\beta)\Psi_2\Psi_3} \times \big[\lambda\tau(\alpha-\beta)(\delta-\eta) - \gamma\Phi_1\big] \\
&> 0, \quad \text{if } \lambda > \frac{\gamma\Phi_1}{\tau(\alpha-\beta)(\delta-\eta)}. \\
w_i^J - w_i^N &= \frac{\alpha\eta\lambda(\alpha-\beta)(\delta-\eta)[(\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3]}{2(\alpha+\beta)\Psi_2\Psi_3} \times \big[\tau\lambda(\delta-\eta)(2\alpha-\beta)(\alpha-\beta) - \alpha\gamma\Phi_1\big] \\
&> 0, \quad \text{as } \frac{\alpha}{2\alpha-\beta} < 1. \\
m_i^J - m_i^N &= \frac{\alpha\eta\lambda(\alpha-\beta)[(\alpha-\beta)\Phi_2 + (\delta-\eta)\tau\Phi_3]}{2(\delta+\eta)\Psi_2\Psi_3} \times \big[\alpha(\alpha-\beta)^2\mu^2 + \tau^2(\delta-\eta)^2\big(2\lambda(2\alpha-\beta)(\alpha-\beta) - \alpha\gamma^2\big)\big] \\
&> 0, \quad \text{if } \lambda > \frac{\alpha[\gamma^2\tau^2(\delta-\eta)^2 - \mu^2(\alpha-\beta)^2]}{2\tau^2(2\alpha-\beta)(\alpha-\beta)(\delta-\eta)^2}.
\end{aligned}$$

Appendix D

Proof of Proposition 5

$\Pi_{RM}^T \geq \Pi_{RM}^N$

$\Rightarrow (w_1^T - c)D_1^C + (w_2^T - c)D_2^C - (m_1^T + c_I)R_1^C - (m_2^T + c_I)R_2^C - \lambda(\theta^C)^2 + F_{RM} \geq (w_1^N - c)D_1^N + (w_2^N - c)D_2^N - (m_1^N + c_I)R_1^N - (m_2^N + c_I)R_2^N - \lambda(\theta^N)^2$

$\Rightarrow F_{RM} \geq (w_1^N - c)D_1^N + (w_2^N - c)D_2^N - (m_1^N + c_I)R_1^N - (m_2^N + c_I)R_2^N - \lambda(\theta^N)^2 - (w_1^T - c)D_1^C - (w_2^T - c)D_2^C + (m_1^T + c_I)R_1^C + (m_2^T + c_I)R_2^C + \lambda(\theta^C)^2 (= F_{RM}^{min})$.

$\Pi_{Ri}^T \geq \Pi_{Ri}^N$

$\Rightarrow (p_i^C - w_i^T)D_i^C - F_{Ri} \geq (p_i^N - w_i^N)D_i^N$

$\Rightarrow F_{Ri} \leq (p_i^C - w_i^T)D_i^C - (p_i^N - w_i^N)D_i^N (= F_{Ri}^{max})$.

$\Pi_{Ci}^T \geq \Pi_{Ci}^N$

$\Rightarrow (m_i^T - a_i^C)R_i^C - F_{Ci} \geq (m_i^N - a_i^N)R_i^N$

$\Rightarrow F_{Ci} \leq (m_i^T - a_i^C)R_i^C - (m_i^N - a_i^N)R_i^N (= F_{Ci}^{max})$.

Additional symbols

Symbols related to Proposition 1

$\Phi_1 = \gamma\tau(\delta - \eta) - \mu(\alpha - \beta), \Phi_2 = R_{01} + R_{02} - 2c_I(\delta - \eta), \Phi_3 = D_{01} + D_{02} - 2c(\alpha - \beta)$,

$\Phi_4 = (\alpha - \beta)(3\alpha + \beta)\big(2\lambda(\delta - \eta) - \mu^2\big) + 4\alpha\gamma\mu\tau(\delta - \eta) + (\delta - \eta)^2\tau^2(4\alpha\lambda - \gamma^2)$,

$\Phi_5 = (\alpha - \beta)(\alpha + 3\beta)\big(2\lambda(\delta - \eta) - \mu^2\big) + 4\beta\gamma\mu\tau(\delta - \eta) + (\delta - \eta)^2\tau^2(4\beta\lambda + \gamma^2)$,

$\Phi_6 = (\delta - \eta)(3\delta + \eta)\tau^2\big(2\lambda(\alpha - \beta) - \gamma^2\big) + 4\delta\gamma\mu\tau(\alpha - \beta) + (\alpha - \beta)^2(4\delta\lambda - \mu^2)$,

$\Phi_7 = (\delta - \eta)(\delta + 3\eta)\tau^2\big(2\lambda(\alpha - \beta) - \gamma^2\big) + 4\eta\gamma\mu\tau(\alpha - \beta) + (\alpha - \beta)^2(4\eta\lambda + \mu^2)$,

$\Psi_1 = 4\lambda(\alpha - \beta)(\delta - \eta)\big((\alpha - \beta) + (\delta - \eta)\tau^2\big) - 2\Phi_1^2$.

Symbols related to Proposition 2

$\Xi_1 = [(\alpha - \beta)(3\alpha + \beta)\big(2\lambda(\delta - \eta)(2\delta - \eta) - \delta\mu^2\big) + 4\alpha\gamma\delta\mu\tau(\delta - \eta) + \delta(\delta - \eta)^2\tau^2(4\lambda(2\alpha - \beta) - \gamma^2)]$,

$\Xi_2 = [\alpha(\alpha - \beta)(\alpha + 3\beta)\big(2\lambda(\delta - \eta)(2\delta - \eta) - \delta\mu^2\big) + 4\alpha\beta\gamma\mu\tau(\delta - \eta) + \delta(\delta - \eta)^2\tau^2(4\lambda(2\alpha - \beta) + \alpha\gamma^2)]$,

$\Xi_3 = [(\delta - \eta)(3\delta + \eta)\tau^2\big(2\lambda(\alpha - \beta)(2\alpha - \beta) - \alpha\gamma^2\big) + 4\alpha\delta\gamma\mu\tau(\alpha - \beta) + \alpha(\alpha - \beta)^2(4\lambda(2\delta - \eta) - \mu^2)]$,

$\Xi_4 = [\delta(\delta - \eta)(\delta + 3\eta)\tau^2\big(2\lambda(\alpha - \beta)(2\alpha - \beta) - \alpha\gamma^2\big) + 4\alpha\gamma\delta\eta\mu\tau(\alpha - \beta) + \alpha(\alpha - \beta)^2(4\eta\lambda(2\delta - \eta) + \delta\mu^2)]$,

$\Xi_5 = [(\alpha - \beta)(7\alpha + 5\beta)\big(2\lambda(\delta - \eta)(2\delta - \eta) - \delta\mu^2\big) + 2\gamma\delta\mu\tau(\delta - \eta)(5\alpha^2 + 2\alpha\beta - \beta^2) + \delta(\delta - \eta)^2\tau^2(12\lambda(2\alpha^2 - \beta^2) - \gamma^2(3\alpha + 2\beta))]$,

$\Xi_6 = [\alpha(\alpha - \beta)(\alpha^2 + 7\alpha\beta + 4\beta^2)\big(2\lambda(\delta - \eta)(2\delta - \eta) - \delta\mu^2\big) - 2\alpha\gamma\delta\mu\tau(\delta - \eta)(\alpha^2 - 4\alpha\beta - 3\beta^2) + \delta(\delta - \eta)^2\tau^2(4\beta\lambda(5\alpha^2 - 2\beta^2) + \alpha(3\alpha + 2\beta)\gamma^2)]$,

$\Xi_7 = [(\delta - \eta)(7\delta + 5\eta)\tau^2\big(2\lambda(\alpha - \beta)(2\alpha - \beta) - \alpha\gamma^2\big) + 2\alpha\gamma\mu\tau(\alpha - \beta)(5\delta^2 + 2\delta\eta - \eta^2) + \alpha(\alpha - \beta)^2(12\lambda(2\delta^2 - \eta^2) - \mu^2(3\delta + 2\eta))]$,

$\Xi_8 = [\delta(\delta - \eta)(\delta^2 + 7\delta\eta + 4\eta^2)\tau^2\big(2\lambda(\alpha - \beta)(2\alpha - \beta) - \alpha\gamma^2\big) + 2\alpha\gamma\delta\mu\tau(\alpha - \beta)(-\delta^2 + 4\delta\eta + 3\eta^2) + \alpha(\alpha - \beta)^2(4\eta\lambda(5\delta^2 - 2\eta^2) + \delta(3\delta + 2\eta)\mu^2)]$,

$\Psi_2 = 4\lambda(\alpha - \beta)(\delta - \eta)\big(\alpha(\alpha - \beta)(2\delta - \eta) + (\delta - \eta)(2\alpha - \beta)\delta\tau^2\big) - 2\alpha\delta\Phi_1^2$.

Symbols related to Proposition 3

$\Delta_1 = [(\alpha-\beta)(3\alpha+\beta)\big(4\lambda(\delta-\eta)-\mu^2\big)+4\alpha\gamma\mu\tau(\delta-\eta)+(\delta-\eta)^2\tau^2(4\lambda(2\alpha-\beta)-\gamma^2)]$,

$\Delta_2 = [\alpha(\alpha-\beta)(\alpha+3\beta)\big(4\lambda(\delta-\eta)-\mu^2\big)+4\beta\gamma\mu\tau(\delta-\eta)+(\delta-\eta)^2\tau^2(4\beta\lambda(2\alpha-\beta)+\alpha\gamma^2)]$,

$\Delta_3 = [(\delta-\eta)(3\delta+\eta)\tau^2\big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big)+4\alpha\delta\gamma\mu\tau(\alpha-\beta)+\alpha(\alpha-\beta)^2(8\delta\lambda-\mu^2)]$,

$\Delta_4 = [(\delta-\eta)(\delta+3\eta)\tau^2\big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big)+4\alpha\gamma\eta\mu\tau(\alpha-\beta)+\alpha(\alpha-\beta)^2(8\eta\lambda+\mu^2)]$,

$\Delta_5 = [(\alpha-\beta)(7\alpha+5\beta)\big(4\lambda(\delta-\eta)-\mu^2\big)+2\gamma\mu\tau(\delta-\eta)(5\alpha^2+2\alpha\beta-\beta^2)+(\delta-\eta)^2\tau^2(12\lambda(2\alpha^2-\beta^2)-\gamma^2(3\alpha+2\beta))]$,

$\Delta_6 = [\alpha(\alpha-\beta)(\alpha^2+7\alpha\beta+4\beta^2)\big(4\lambda(\delta-\eta)-\mu^2\big)-2\alpha\gamma\mu\tau(\delta-\eta)(\alpha^2-4\alpha\beta-3\beta^2)+(\delta-\eta)^2\tau^2(4\beta\lambda(5\alpha^2-2\beta^2)+$

$\alpha(3\alpha+2\beta)\gamma^2)]$,

$\Delta_7 = [(\delta-\eta)(7\delta+\eta)\tau^2\big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big)+2\alpha\gamma\mu\tau(\alpha-\beta)(5\delta-\eta)+3\alpha(\alpha-\beta)^2(8\delta\lambda-\mu^2)]$,

$\Delta_8 = [(\delta-\eta)(\delta+7\delta)\tau^2\big(2\lambda(\alpha-\beta)(2\alpha-\beta)-\alpha\gamma^2\big)-2\alpha\gamma\mu\tau(\alpha-\beta)(\delta-5\eta)+3\alpha(\alpha-\beta)^2(8\eta\lambda+\mu^2)]$,

$\Psi_3 = 4\lambda(\alpha-\beta)(\delta-\eta)\big(2\alpha(\alpha-\beta)+(\delta-\eta)(2\alpha-\beta)\tau^2\big)-2\alpha\Phi_1^2$.