ON NEW FAMILY OF KIES BURR III DISTRIBUTION:

Development, Properties, Characterizations and Applications

Fiaz Ahmad Bhatti and Munir Ahmad
National College of Business Administration and Economics, Lahore Pakistan.
Email: fiazahmad72@gmail.com, munirahmaddr@yahoo.co.uk

ABSTRACT

In this paper, a new lifetime family of distribution called new family of Kies Burr III (NFKBIII) distribution is developed from T-X family technique. The NFKBIII distribution is very flexible and its hazard rate function accommodates various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. The density function of the NFKBIII is arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shaped. Some structural and mathematical properties including quantiles, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions and reliability measures are derived. Two characterizations for the NFKBIII distribution are studied. The maximum likelihood estimates (MLE) for unknown parameters of NFKBIII distribution are obtained. A simulation study is performed to evaluate the behavior of the maximum likelihood estimators. The NFKBIII distribution is applied to two real data sets to illustrate its potentiality and utility. The adequacy of the NFKBIII distribution is tested via different goodness of fit statistics.

Key Words: Moments; Reliability; Characterizations; Maximum Likelihood Estimation

1. INTRODUCTION

Many univariate continuous distributions have been established in recent decades but many data sets from reliability, life testing, risk analysis, finance, ecology, climatology, geology, hydrology and other fields do not fit to these distributions. Therefore, applications of the modified distributions to problems in these fields are a vibrant necessity of day.

The modified, generalized and extended distributions are attained by adding one or more parameters or the introduction of some transformation to the parent distribution. Therefore, the new proposed distributions provide best fit than the sub and competing models.

Burr [1] proposed a family of 12 distributions by fitting cumulative frequency functions to frequency data called Burr family. Burr distributions III, VI, X and XII have wide applications. Burr-III (BIII) distribution is commonly applied to model risk data in business and finance, crop rice in market, failure time data in life testing and reliability and ozone data in environmental sciences.

Many modified, generalized and extended types of BIII distribution are presented in statistical literature such as two parameter family of distributions (Mielke; [2]), inverse Burr (Kleiber and Kotz; [3]), BIII type (Gove et al.;[4]), extended Burr III (Shao et al.; [5]), Dagum (Benjamin et al.; [6]) modified BIII(Ali et al.;[7]), McDonald BIII (Gomes et al.; [8]), interpolating family (Sinner et al.;[9]), mixture of two BIII (Moisheer; [10]), generalized gamma BIII (Olobatuyi et al.; [11]), four parameter gamma BIII (Cordeiro et al.; [12]), odd BIII family (Jamil et al.[13]), Kumaraswamy odd Burr G family (Nasir et al. [14]) and generalized BIII (Kehinde et al.; [15]).

Marshall and Olkin [16] presented a new technique to add a parameter to a family of distribution. Cordeiro and Castro [17] established Kumaraswamy generalized family with its distributional properties. Alizadeh et al. [18] studied Burr generalized family with various properties. Cordeiro et al. [19] developed generalization of odd log-logistic family with properties. Haghbin et al. [20] presented a new generalization odd log-logistic family of distributions. Korkmaz and Genç [21 and 22] studied generalized two-sided class of distributions along with applications. Cordeiro et al. [23] studied a new family based on the Burr XII density with detailed properties. Alizadeh et al. [24] studied odd log-logistic logarithmic class of continuous distributions. Yousof et al. [25] developed Burr Hatke- G family of distributions. Korkmaz et al. [26] presented the Weibull Marshall–Olkin family along with properties.

The main motivation of this article is to develop and study a flexible lifetime family of BIII type distribution with two extra shape parameters and two location parameters called the NFKBIII distribution. The shapes of NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shapes. The hazard rate function for the NFKBIII distribution has various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. The NFKBIII distribution is the best model for modeling data such as times to failures of items in life testing, maximum annual flood discharges in hydrology and other various fields. The NFKBIII distribution offers better fits than sub and competing models.

The article is organized follows. In Section 2, the NFKBIII distribution is derived from T-X family technique, transformation and compounding mixture of distributions. Structural properties, quantile function, sub-models and various plots for density and hazard rate functions are discussed. In Section 3, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions and reliability measures are derived. The characterizations for the NFKBIII distribution are studied in Section 4. In Section 5, the maximum likelihood estimates (MLEs) for unknown parameters of the NFKBIII distribution are obtained. In Section 6, a simulation study is performed to assess the behavior of the maximum likelihood estimators. In Section 7, the potentiality and utility of the NFKBIII distribution is illustrated via its applications to two real data sets: times to failures of devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution is tested via different goodness of fit statistics. The ultimate comments are given in Section 8.

2. DEVELOPMENT OF NFKBIII DISTRIBUTION

The cumulative distribution function (cdf) of the generalized uniform distribution is given by

$$G(x;a,b,\kappa)=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-a^{\kappa}}, \qquad x\in[a,b], a>0, b>0, \kappa>0. \tag{1}$$

The odds ratio for the generalized uniform random variable X is

$$W\left(G(x)\right)=\frac{G(x;a,b,\kappa)}{\bar{G}(x;a,b,\kappa)}=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}. \tag{2}$$

Gurvich et al. [27] replaced ‘x’ with odds ratio in the Weibull distribution for the development of a class of extended Weibull distributions. Alzaatreh et al. [28] developed the cdf of the T-X family of distributions as

$$F\left(x\right)=\int_{a}^{W(G(x))} r\left(t\right)dt\ , \tag{3}$$

where $W\left(G(x)\right)$ is a function of $G\left(x\right)$ and $r\left(t\right)$ is the pdf of a non-negative random variable.

Bourguignon et al. [29] inserted the odds ratio of a baseline distribution in place of ‘x’ in the cdf of the Weibull distribution for the development of a new family of distributions.

The NFKBIII is developed by inserting the odds ratio for the generalized uniform in place of ‘x’ in the cdf of MBIII distribution. The cdf for the NFKBIII distribution is obtained as

$$F(x)=\int_{0}^{W(G(x))} \alpha\beta t^{-\beta-1}\left(1+\gamma t^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

or

$$F(x;\alpha,\beta,\gamma,\lambda)=\int_0^{\frac{x^\kappa-a^\kappa}{b^\kappa-x^\kappa}}\alpha\beta t^{-\beta-1}\left(1+\gamma t^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

or

$$F(x)=\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}},a\le x<b, \tag{4}$$

where $a,b,\ \alpha,\ \beta,\kappa$ and γ are positive parameters of which a,b are location parameters and $\alpha,\ \beta,\kappa\ and\ \gamma$ are shape parameters. Clearly, $F(x)$ is a strictly increasing and differential cdf on (*a*, b).

The pdf of the NFKBIII distribution is

$$f(x)=\alpha\beta\left(b^\kappa-a^\kappa\right)\kappa x^{\kappa-1}\frac{\left(b^\kappa-x^\kappa\right)^{\beta-1}}{\left(x^\kappa-a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1},x>a. \tag{5}$$

2.1 Transformation and Compounding

The NFKBIII model is also developed via (i) transformation between the ratio of exponential and gamma random variables and (ii) compounding generalized inverse Kies (GNIK) and gamma distributions.

2.1 (i) Let Z_1 be a random variable having exponential distribution with parameter value 1 and Z_2 be a random variable with gamma i.e., $Z_2\sim gamma\left(\frac{\alpha}{\gamma},1\right)$, then using relationship $Z_1=\gamma\left(\frac{b^\kappa-X^\kappa}{X^\kappa-a^\kappa}\right)^\beta Z_2$, we have

$$X=\left\{\left[a^\kappa+b^\kappa\left(\frac{Z_1}{\gamma Z_2}\right)^{-\frac{1}{\beta}}\right]\left[1+\left(\frac{Z_1}{\gamma Z_2}\right)^{-\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{\kappa}}\sim NFKBIII\left(a,b,\alpha,\beta,\gamma,\kappa\right).$$

(ii) Let X be a random variable with GNIK distribution i.e. $X\sim GNIK\left(x;a,b,\beta,\kappa,\gamma,\theta\right)$ and θ be a random variable with gamma distribution, i.e. $\theta\sim gamma\left(\theta;\alpha,\gamma\right)$, then after simplifying the integral,

$$f\left(x;a,b,\alpha,\beta,\gamma,\kappa\right)=\int_0^\infty GNIK\left(x/a,b,\beta,\kappa,\gamma,\theta\right)g\left(\theta/\alpha,\gamma\right)d\theta,$$

we have $X\sim NFKBIII\left(a,b,\alpha,\beta,\gamma,\kappa\right)$.

2.2 Structural Properties

The survival, hazard, cumulative hazard, reverse hazard functions and the Mills ratio of a random variable X with the NFKBIII distribution are given, respectively, by

$$S(x)=1-\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}},x\ge a, \tag{6}$$

$$h(x)=\alpha\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\frac{\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}}{\left\{1-\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right\}},x>a, \quad (7)$$

$$H(x)=-\ln\left\{1-\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right\},\ x\geq a, \quad (8)$$

$$\text{r(x)}=\frac{f(x)}{F(x)}=\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-1} \quad x>a, \quad (9)$$

and

$$m(x)=\frac{1-F(x)}{f(x)}=\frac{1-\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}}{\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}}. \quad (10)$$

The elasticity $e\left(x\right)=xr\left(x\right)=\frac{d\,\ln\text{F}(x)}{d\ln x}$, for the NFKBIII distribution is

$$e\left(x\right)=\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-1}. \quad (11)$$

The quantile function of NFKBIII distribution is $x_q=\left[\frac{a^{\kappa}\gamma^{-\frac{1}{\beta}}\left(q^{-\frac{\gamma}{\alpha}}-1\right)^{\frac{1}{\beta}}+b^{\kappa}}{\gamma^{-\frac{1}{\beta}}\left(q^{-\frac{\gamma}{\alpha}}-1\right)^{\frac{1}{\beta}}-1}\right]^{\frac{1}{\kappa}}$ and its random number generator is $X=\left[\frac{a^{\kappa}\gamma^{-\frac{1}{\beta}}\left(Z^{-\frac{\gamma}{\alpha}}-1\right)^{\frac{1}{\beta}}+b^{\kappa}}{\gamma^{-\frac{1}{\beta}}\left(Z^{-\frac{\gamma}{\alpha}}-1\right)^{\frac{1}{\beta}}-1}\right]^{\frac{1}{\kappa}}$, where the random variable Z has the uniform distribution on $\left(0,1\right)$.

2.3 Sub-Models

The NFKBIII distribution has applications in life testing, reliability concept, survival analysis and hydrology. The NFKBIII distribution has the subsequent nested models (**Table 1**).

2.4 Plots for the NFKBIII Density and Hazard Rate Functions

Fig.1 shows that the shapes of the NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical (**Fig.1**). The shapes of failure rate function for the NFKBIII distribution are increasing, decreasing, increasing-decreasing-increasing and bathtub (**Fig.2**).

3. MATHEMATICAL PROPERTIES

Some descriptive measures for the NFKBIII distribution such as ordinary and incomplete moments, inequality curves, means deviations, residual life functions and reliability measures are established in this section.

3.1 Moments of the NFKBIII Distribution

The r^{th} moment about origin of X with the NFKBIII distribution is

$$\mu_r' = E(X^r) = \int_a^b x^r f(x)dx,$$

$$E(X^r) = \alpha\beta\left(b^\kappa - a^\kappa\right)\kappa\int_a^b x^r x^{\kappa-1}\frac{\left(b^\kappa - x^\kappa\right)^{\beta-1}}{\left(x^\kappa - a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

Letting $\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta = w$ and $x = \left\{\frac{\left(a^\kappa - b^\kappa\right)}{\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]} + b^\kappa\right\}^{\frac{1}{\kappa}}$,

we arrive at $$E(X^r) = \frac{\alpha}{\gamma}\int_0^\infty\left\{\frac{\left(a^\kappa - b^\kappa\right)}{\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]} + b^\kappa\right\}^{\frac{r}{\kappa}}\left[1+w\right]^{-\frac{\alpha}{\gamma}-1}dw,$$

$$E(X^r) = \frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa - b^\kappa\right)^\ell b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\left(\ell\right)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}\left[\int_0^\infty w^{-\frac{\nu}{\beta}}\left[1+w\right]^{-\frac{\alpha}{\gamma}-1}dw\right].$$

Observe that

$$\mu_r' = \frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa - b^\kappa\right)^\ell b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\left(\ell\right)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right),\ \mathrm{r}=1,2,3,\ldots, \quad (12)$$

where $\left(\ell\right)_k = \frac{\Gamma\left(\ell+k\right)}{\Gamma\left(\ell\right)}$ is Pochhammar symbol.

The factorial moments $E[X]_n = \sum_{r=1}^{n} \vartheta_r E(X^r) =$ for the NFKBIII distribution are

$$E[X]_n = \sum_{r=1}^{n} \left[\vartheta_r \frac{\alpha}{\gamma} \sum_{\ell=0}^{r} \binom{r}{\ell} (a-b)^{\ell} b^{r-\ell} \sum_{k=0}^{\infty} \frac{(-1)^k (\ell)_k \gamma^{\frac{k}{\beta}}}{k!} B\left(1-\frac{k}{\beta}, \frac{\alpha}{\gamma}+\frac{k}{\beta}\right) \right], \tag{13}$$

where $[Z]_i = Z(Z+1)(Z+2)......(Z+i-1)$ and ϑ_r is Stirling number of the first kind.

The Mellin transform is used to obtain moments of a probability distribution. By definition, the Mellin transform is

$$\mathrm{M}\{f(x);\mathrm{s}\} = f^*(s) = \int_0^{\infty} f(x)\, x^{s-1} dx.$$

The Mellin transform of X with the NFKBIII distribution is

$$\mathrm{M}\{f(x);\mathrm{s}\} = \alpha \sum_{\ell=0}^{s-1} \sum_{k=0}^{\infty} \binom{s-1}{\ell} \frac{(-1)^k (\ell)_k}{k!} \gamma^{\frac{k}{\beta}-1} b^{s-1} \left(\frac{a}{b}-1\right)^{\ell} B\left(1-\frac{k}{\beta}, \frac{\alpha}{\gamma}+\frac{k}{\beta}\right). \tag{14}$$

The r^{th} moment about means, Pearson's measures for skewness and kurtosis, moment generating function and cumulants of X for the NFKBIII distribution are attained from the relations

$$\mu_r = \sum_{i=1}^{r} \binom{r}{i} (-1)^i \mu'_i \mu'_{i-r},\ \gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\ \beta_2 = \frac{\mu_4}{(\mu_2)^2},\ M_X(t) = E\left[e^{tX}\right] = \sum_{r=1}^{\infty} \frac{t^r}{\mathrm{r}!} E(X)^r,$$

and $k_r = \mu'_r - \sum_{c=1}^{r-1} \binom{r-1}{c-1} k_c\ \mu'_{r-c}.$

Table 2 displays the numerical descriptive measures such as median, mean, standard deviation, skewness and kurtosis of the NFKBIII distribution for carefully chosen parameter values to describe their effect on these descriptive measures.

3.2 Moments of Order Statistics

Moments of order statistics have uses in life testing and reliability. Moments of order statistics are also aimed to anticipate the failure of future items obtained after few initial failures.

The pdf for m^{th} order statistic $X_{\mathrm{m}:n}$ is

$$f(x_{\mathrm{m}:n}) = \frac{1}{B(\mathrm{m}, n-m+1)} \left[F(x)\right]^{m-1} \left[1-F(x)\right]^{n-m} f(x). \tag{15}$$

The pdf of $X_{\mathrm{m}:n}$ for the NFKBIII distribution is

$$f_{X_{\mathrm{m}:n}}(x)=\left\{\begin{array}{l}\frac{1}{B(\mathrm{m},n-m+1)}\sum_{i=0}^{n-m}(-1)^i\binom{n-m}{i}\times \\ \qquad \alpha\beta\left(b^\kappa-a^\kappa\right)\kappa x^{\kappa-1}\frac{\left(b^\kappa-x^\kappa\right)^{\beta-1}}{\left(x^\kappa-a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}(m+i)-1}\end{array}\right\}. \qquad (16)$$

Moments about the origin of $X_{\mathrm{m}:n}$ for the NFKBIII distribution are

$$E\left(X^r{}_{\mathrm{m}:n}\right)=\int_a^b x^r f\left(x_{\mathrm{m}:n}\right)dx, \qquad (17)$$

$$E\left(X^r{}_{\mathrm{m}:n}\right)=\frac{\alpha}{\gamma}\frac{1}{B(\mathrm{m},n-m+1)}\sum_{i=0}^{n-m}\sum_{\ell=0}^{\frac{r}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa-b^\kappa\right)^\ell b^{(r-\kappa\ell)}\binom{n-m}{i}\frac{(-1)^{i+\nu}(\ell)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}(m+i)+\frac{\nu}{\beta}\right)$$

$$E\left(X^r{}_{\mathrm{m}:n}\right)=\frac{\alpha}{\gamma}\frac{\sum_{i=0}^{n-m}\sum_{\ell=0}^{\frac{r}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa-b^\kappa\right)^\ell b^{(r-\kappa\ell)}\binom{n-m}{i}\frac{(-1)^{i+\nu}(\ell)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}(m+i)+\frac{\nu}{\beta}\right)}{B(\mathrm{m},n-m+1)},\mathrm{r}=1,2,3.... \qquad 18)$$

3.3 Incomplete Moments

Bonferroni and Lorenz curves can be easily computed using first incomplete moment. The life testing features such as residual life and mean inactivity life functions can be obtained from incomplete moments. The lower incomplete moments for the random variable X with the NFKBIII distribution are

$$M_r'(z)=E_{X\le z}\left(X^r\right)=\alpha\beta\left(b^\kappa-a^\kappa\right)\kappa\int_a^z x^r x^{\kappa-1}\frac{\left(b^\kappa-x^\kappa\right)^{\beta-1}}{\left(x^\kappa-a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1}dx.$$

Letting $\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta=w$ and $x=\left\{\left(a^\kappa-b^\kappa\right)\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]^{-1}+b^\kappa\right\}^{\frac{1}{\kappa}}$,

we arrive at $$E(X^r)=\frac{\alpha}{\gamma}\int_{\gamma\left(\frac{b^\kappa-z^\kappa}{z^\kappa-a^\kappa}\right)^\beta}^{\infty}\left\{\left(a^\kappa-b^\kappa\right)\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]^{-1}+b^\kappa\right\}^{\frac{1}{\kappa}}[1+w]^{-\frac{\alpha}{\gamma}-1}dw,$$

$$E_{X\le z}\left(X^r\right)=\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^\kappa-b^\kappa\right)^\ell b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(\ell)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}\left[\int_{\gamma\left(\frac{b^\kappa-z^\kappa}{z^\kappa-a^\kappa}\right)^\beta}^{\infty}w^{-\frac{\nu}{\beta}}[1+w]^{-\frac{\alpha}{\gamma}-1}dw\right].$$

Observe that

$$M'_r(z)=\left(\begin{array}{l}\dfrac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell} b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\dfrac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}\times \\ \left\{B\left(1-\dfrac{\nu}{\beta},\dfrac{\alpha}{\gamma}+\dfrac{\nu}{\beta}\right)-B\left[\gamma\left(\dfrac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1-\dfrac{\nu}{\beta},\dfrac{\alpha}{\gamma}+\dfrac{\nu}{\beta}\right]\right\}\end{array}\right),\ \mathrm{r}=1,2,3,\ldots, \qquad (19)$$

where $B(z\ ;.,.)$ is incomplete beta function.

The upper incomplete moments for the random variable X with the NFKBIII distribution are

$$E_{X\geq z}\left(X^r\right)=\int_z^b x^r\,\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$E_{X\geq z}\left(X^r\right)=\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell} b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left[\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]. \qquad (20)$$

The mean deviation about the mean is $MD_{\bar{X}}=E\left|X-\mu_1^1\right|=2\mu_1^1F\left(\mu_1^1\right)-2\mu_1^1M'_1\left(\mu_1^1\right)$ and mean deviation about the median is $MD_M=E\left|X-M\right|=2MF(M)-2MM'_1(M)$, where $\mu'_1=E(X)\,and\,M=Q(0.5)$. Bonferroni and Lorenz curves for a specified probability p are computed by $\mathrm{B}(p)=\dfrac{M'_1(q)}{p\mu'}$ and $L(p)=\dfrac{M'_1(q)}{\mu'}$, where $q=Q(p)$.

3.4 Residual Life functions

The n[th] moment $m_n(z)$ of residual life for X with the NFKBIII distribution is

$$m_n(z)=E\left[\left(X-z\right)^n\middle|X>z\right]=\frac{1}{S(z)}\int_z^{\infty}(x-z)^s f(x)dx,$$

$$m_n(z)=\frac{1}{S(z)}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s}E_{X>z}\left(X^s\right),$$

$$m_n(z)=\frac{1}{S(z)}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s}\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell} b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left[\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]. \qquad (21)$$

The residual life (MRL) function $m_1(z)$ of a component at time *z*, or average remaining lifetime is also called the life expectancy given by

$$m_1(z)=\frac{1}{S(z)}\sum_{s=0}^{1}\binom{1}{s}(-z)^{1-s}\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell} b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left[\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]. \qquad (22)$$

The n[th] moment of reverse residual life $M_n(z)$ for X with the NFKBIII distribution is

$$M_n(z)=E\left[\left(z-X\right)^n/X\leq z\right]=\frac{1}{F(z)}\int_a^z(z-x)^n f(x)dx$$

$$M_n(z)=\frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}E_{X\le z}\left(X^s\right),$$

$$M_n(z)=\frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}\left(\begin{array}{c}\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^\kappa-b^\kappa\right)^\ell b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(\ell)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}\times\\ \left\{B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right)-B\left[\gamma\left(\frac{b^\kappa-z^\kappa}{z^\kappa-a^\kappa}\right)^\beta;1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]\right\}\end{array}\right). \quad (23)$$

The waiting time z for the failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity time. The waiting time z for the failure of a component of X with the NFKBIII distribution is defined by

$$M_1(z)=\frac{1}{F(z)}\sum_{s=0}^{1}(-1)^s\binom{1}{s}z^{1-s}\left(\begin{array}{c}\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^\kappa-b^\kappa\right)^\ell b^{(r-\kappa\ell)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(\ell)_\nu\gamma^{\frac{\nu}{\beta}}}{\nu!}\times\\ \left\{B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right)-B\left[\gamma\left(\frac{b^\kappa-z^\kappa}{z^\kappa-a^\kappa}\right)^\beta;1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]\right\}\end{array}\right). \quad (24)$$

3.5 Stress-strength Reliability for the NFKBIII Distribution

Let X_1 be strength and X_2 be stress and X_1 follows NFKBIII distribution $(\alpha_1,\beta,\gamma,\kappa,a,b)$ and X_2 follows NFKBIII distribution $(\alpha_2,\beta,\gamma,\kappa,a,b)$, then $\mathrm{R}=\Pr\left(\mathrm{X}_2<X_1\right)=\int_a^b f_{x_1}(x)F_{x_2}(x)\,dx$ is reliability parameter (Kotz et al.; [32]). The reliability of the component is computed as

$$R=\int_a^b\alpha_1\beta\left(b^\kappa-a^\kappa\right)\kappa x^\kappa\frac{\left(b^\kappa-x^\kappa\right)^{\beta-1}}{\left(x^\kappa-a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}-1}\left[1+\gamma\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}}dx,$$

$$R=\frac{\alpha_1}{\alpha_1+\alpha_2}\quad. \quad (25)$$

Therefore R is independent of a,b,β,κ *and* γ.

3.6 Estimation of Multicomponent Stress-Strength System Reliability with NFKBIII Distribution

Consider a system that has m identical components out of which s components are functioning. The strengths of m components are $\mathrm{X}_i, i=1,2...m$ with common cdf F while, the stress Y imposed on the components has cdf G. The strengths $\mathrm{X}_i, i=1,2...m$ and stress Y are i.i.d. distributed. The probability that system operates properly is reliability of the system i.e.

$$R_{s,m}=P[strengths(\mathrm{X}_i,i=1,2...m)>stress(Y)],$$

$$R_{s,m}=P[\text{at the minimum}\ "s"\ of\ (\mathrm{X}_i,i=1,2...m)\ exceed\ Y]. \quad (26)$$

$$R_{s,\mathrm{m}}=\sum_{l=s}^{m}\binom{m}{l}\int_{-\infty}^{\infty}[1-F(y)]^l[F(y)]^{m-l}dG(y)\quad \text{(Bhattacharyya and Johnson; [33])} \quad (27)$$

Let $X \sim NFKBIII(\alpha_1,\beta,\gamma,\kappa,a,b)$ and $\Upsilon \sim NFKBIII(\alpha_2,\beta,\gamma,\kappa,a,b)$ such that α_1,α_2 are unknown shape parameters and *a*, *b*, are common location parameters. X and Y are independently distributed. The reliability that system operates properly in multicomponent stress- strength for the NFKBIII distribution is

$$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_a^b\left(1-\left[1+\gamma\left(\frac{b^\kappa-y^\kappa}{y^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right)^\ell\left(\left[1+\gamma\left(\frac{b^\kappa-y^\kappa}{y^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right)^{(m-\ell)}\alpha_2\beta\left(b^\kappa-a^\kappa\right)\kappa y^\kappa\frac{\left(b^\kappa-y^\kappa\right)^{\beta-1}}{\left(y^\kappa-a^\kappa\right)^{\beta+1}}\left[1+\gamma\left(\frac{b^\kappa-y^\kappa}{y^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}-1}dy.$$

Letting $\left[1+\gamma\left(\frac{b^\kappa-y^\kappa}{y^\kappa-a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}}=u$,

we obtain $$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_0^1\left(1-u^\Upsilon\right)^\ell u^{\Upsilon(m-\ell)}du,\quad where\ \ \Upsilon=\frac{\alpha_2}{\alpha_1}.$$

Again letting $u^\Upsilon=w,$

we reach at $$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_0^1(1-u)^\ell w^{(m-\ell)}\frac{1}{\Upsilon}w^{\frac{1}{\Upsilon}-1}dw,$$

$$R_{s,m}=\frac{1}{\Upsilon}\sum_{\ell=s}^{m}\binom{m}{\ell}B\left[1+\ell,\ \mathrm{m}+\frac{1}{\Upsilon}-\ell\right], \tag{28}$$

is reliability in multicomponent stress-strength model (Bhattacharyya and Johnson; [33]).

4. CHARACTERIZATIONS

In this section, two essential characterizations for the NFKBIII distribution are planned via: (i) conditional expectation and (ii) ratio of truncated moments.

4.1 Characterization Based on Conditional Expectation

Here the NFKBIII distribution is characterized via conditional expectation.

Proposition 4.1.1: Let $X:\Omega\to(a,\mathrm{b})$ be a continuous random variable with cdf $F(x)$, ($0<F(x)<1$ for $x\geq a$), then for $\alpha>\gamma$, X has cdf (4) if and only if

$$E\left[\left(\frac{X^\kappa-a^\kappa}{b^\kappa-X^\kappa}\right)^{-\beta}\middle| X<z\right]=\frac{1}{(\alpha-\gamma)}\left[1+\alpha\left(\frac{z^\kappa-a^\kappa}{b^\kappa-z^\kappa}\right)^{-\beta}\right],\quad for\ z>a. \tag{29}$$

Proof If (5) is pdf of X, then $E\left[\left(\frac{b^\kappa-X^\kappa}{X^\kappa-a^\kappa}\right)^\beta\middle| X<z\right]=\left(F(z)\right)^{-1}\int_a^z\left(\frac{b^\kappa-x^\kappa}{x^\kappa-a^\kappa}\right)^\beta f(x)dx$

$$= \left(F(z)\right)^{-1} \int_a^z \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta \times$$

$$\frac{\alpha}{\gamma}\gamma\beta\left(b^\kappa - a^\kappa\right)\kappa x^{\kappa-1} \frac{\left(b^\kappa - x^\kappa\right)^{\beta-1}}{\left(x^\kappa - a^\kappa\right)^{\beta+1}} \left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} dx$$

After integrating and simplifying, we arrive at

$$E\left[\left(\frac{b^\kappa - X^\kappa}{X^\kappa - a^\kappa}\right)^\beta \middle| X < z\right] = \frac{1}{(\alpha-\gamma)}\left[1+\alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right], \quad for\ z > a.$$

Conversely if proposition 4.1.1 holds then

$$\int_a^z \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta f(x)\,dx = \frac{F(z)}{\gamma\left(\frac{\alpha}{\gamma}-1\right)}\left[1+\alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right]. \tag{30}$$

Differentiate (30) with respect to t, we achieve

$$\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta f(z) = \frac{f(z)}{(\alpha-\gamma)}\left[1+\alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right] - \frac{F(z)}{\gamma\left(\frac{\alpha}{\gamma}-1\right)}\left[\alpha\beta\left(b^\kappa - a^\kappa\right)\kappa x^{\kappa-1}\frac{\left(b^\kappa - z^\kappa\right)^{\beta-1}}{\left(z^\kappa - a^\kappa\right)^{\beta+1}}\right]$$

After simplification and integration, we work out $F(z) = \left[1+\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}}$, $z \geq a$.

4.2 Characterization of the NFKBIII Distribution through Ratio of Truncated Moments

The NFKBIII distribution is characterized using Theorem G (Glänzel; [34]) from a simple relationship between two truncated moments of functions of X.

Proposition 4.2.1: Let $X : \Omega \to (a, \mathrm{b})$ be a continuous random variable. Let $h_1(x) = \frac{1}{\alpha}\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{\frac{\alpha}{\gamma}+1}, x > a$ and $h_2(x) = 2\alpha^{-1}\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{\frac{\alpha}{\gamma}+1}, x > a$. According to theorem G, the random variable X has pdf (5) if and only if that the function $p(x)$ has the form $p(x) = \left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta, \quad x > a.$

Proof. For random variable X with pdf (5), then

$$\left(1-F(x)\right)E\left(h_1(x) \middle| X \geq x\right) = \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta, \quad x > a,$$

$$\left(1-F(x)\right)E\left(h_2(x)\middle| X\geq x\right)=\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{2\beta},\ x>a,$$

$$\frac{E[h_1(x)| X\geq x]}{E[h_2(x)| X\geq x]}=p(x)=\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta},\ \ x>a$$

and

$$p'(x)=\beta\kappa\left(b^{\kappa}-a^{\kappa}\right)x^{\kappa-1}\left(x^{\kappa}-a^{\kappa}\right)^{\beta-1}\left(b^{\kappa}-x^{\kappa}\right)^{-\beta-1},\ \ x>a.$$

The differential equation $s'(x)=\dfrac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)}=2\beta\left(b^{\kappa}-a^{\kappa}\right)\dfrac{\kappa x^{\kappa-1}}{\left(b^{\kappa}-x^{\kappa}\right)^2}\left[\dfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right]^{-1}$

has solution $s(x)=\ln\left(\dfrac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{2\beta},\ \ x>a$. So, in the light of theorem G, X has pdf (5)

Corollary 4.2.1: Let $X:\Omega\rightarrow(a,\mathrm{b})$ be a continuous random variable and let

$h_2(x)=2\alpha^{-1}\left(\dfrac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\left[1+\gamma\left(\dfrac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{\frac{\alpha}{\gamma}+1},x>a$. The pdf of X is (5) if and only if there exists functions $p(x)$ and $h_1(x)$ (defined in Theorem G), satisfying the differential equation

$$\frac{p'(x)}{p(x)h_2(x)-h_1(x)}=\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\frac{\kappa x^{\kappa-1}}{\left(b^{\kappa}-x^{\kappa}\right)^2}\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{\beta-1}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}. \qquad (31)$$

Remarks 4.2.1: The solution of (31) is

$$p(x)=\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{2\beta}\left[\int\left(-\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\frac{\kappa x^{\kappa-1}}{\left(b^{\kappa}-x^{\kappa}\right)^2}\left(\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}\right)^{-\beta-1}\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}h_1(x)\right)dx\right]+D,$$

where D is constant.

5. MAXIMUM LIKELIHOOD ESTIMATION

In this section, parameters estimates are derived using maximum likelihood method. The log likelihood function for the NFKBIII distribution with the vector of parameters $\Phi=(a,b,\alpha,\beta,\gamma,\kappa)$ is

$$\ln L\left(x_i,\Phi\right)=n\ln\alpha+n\ln\beta+n\ln\left(b^{\kappa}-a^{\kappa}\right)-\left(\beta+1\right)\sum_{i=1}^{n}\ln\left(x_i^{\kappa}-a^{\kappa}\right)$$
$$+\left(\beta-1\right)\sum_{i=1}^{n}\ln\left(b^{\kappa}-x_i^{\kappa}\right)-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\right]. \quad (32)$$

where a and b are assumed to be known, since its minimum and maximum likelihood are equal to minimum and maximum order statistics. The MLEs of the parameters for the NFKBIII distribution can be computed from the simultaneously solution of the following nonlinear equations:

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}-\frac{1}{\gamma}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\right]=0, \quad (33)$$

$$\frac{\partial\ln L}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\ln\left(x_i^{\kappa}-a^{\kappa}\right)+\sum_{i=1}^{n}\ln\left(b^{\kappa}-x_i^{\kappa}\right)+\left(\alpha+\gamma\right)\left[1+\gamma\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-1}\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\ln\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)=0, \quad (34)$$

$$\frac{\partial\ln L}{\partial\gamma}=\alpha\gamma^{-2}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\right]-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\left[1+\gamma\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-1}=0, \quad (35)$$

$$\frac{\partial\ln L}{\partial\kappa}=n\left[\frac{b^{\kappa}\ln b-a^{\kappa}\ln a}{b^{\kappa}-a^{\kappa}}\right]-\left(\beta+1\right)\sum_{i=1}^{n}\left[\frac{x_i^{\kappa}\ln x_i-a^{\kappa}\ln a}{x_i^{\kappa}-a^{\kappa}}\right]+\left(\beta-1\right)\sum_{i=1}^{n}\left[\frac{b^{\kappa}\ln b-x_i^{\kappa}\ln x_i}{b^{\kappa}-x_i^{\kappa}}\right]$$
$$-\left(\alpha+\gamma\right)\beta\sum_{i=1}^{n}\left(a^{\kappa}\ln a+b^{\kappa}\ln b-2x_i^{\kappa}\ln x_i\right)\left[\left(\frac{b^{\kappa}-x_i^{\kappa}}{x_i^{\kappa}-a^{\kappa}}\right)^{-\beta}+\gamma\right]^{-1}. \quad (36)$$

The above equations 33-36 can be solved either directly or using the R (optim and maxLik functions), SAS (PROC NLMIXED) and Ox program (sub-routine Max BFGS) or using non-linear optimization approaches such as the quasi-Newton procedure.

6. Simulation Study

In this section, the behavior of the MLEs of the NFKBIII parameters is assessed with respect to sample size n. The steps for simulation to assess the behavior are as follows. Generate 10000 samples of sizes n from the NFKBIII distribution using the inverse cdf method. Calculate the MLEs for 10000 samples, say $(\hat{a},\hat{\mathrm{b}},\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\kappa})$ for i=1, 2,..., 10000 from non-linear optimization technique with constraint matching to range of parameters. (0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5) and (1, 6, 2, 0.8, 1.5, 1.75) are taken as the true parameter values $(a,\mathrm{b},\alpha,\beta,\gamma,\kappa)$. Calculate the means, biases and mean squared errors (MSE) of MLEs.

For this purpose, we have chosen various arbitrarily parameters and n=50,100,150 sample sizes. All codes are written in R and the results are summarized in Table 3. The results clearly show that when the sample size n increases, the estimated MSE decrease and estimated biases drop to zero. MSE of estimated parameters increases, as shape parameter rises. This reveals that MLEs for NFKBIII distribution are reliable.

7. APPLICATIONS

The potentiality and utility of use of NFKBIII distribution is established by applying it to two data sets: failure times of devices (Aarset, [35]) data and maximum annual flood discharges. The NFKBIII distribution is compared with KMBIII, NKBIII, KBIII, NIKL, KIL, modified Burr XII (MBXII), Burr XII (BXII), modified Burr III (MBIII), Burr III (BIII), Weibull and inverse Weibull distributions. R-package is applied to compute goodness of fit criteria such as “Cramer-von Mises (W*), Anderson Darling (A*), Kolmogorov- Smirnov statistics with p-values [KS(p-values], Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC)" and estimate of likelihood ratio statistics ($-\ell$) values for times to failures of 50 components and maximum annual flood discharges. Chen and Balakrishnan [36] described in detail about the statistics W* and A*.

A model is best model for which the values for goodness of fit criteria are smaller. The MLEs for unknown parameters and goodness of fit criteria values for the NFKBIII, KMBIII, NKBIII, KBIII, NIKL, KIL, modified Burr XII (MBXII), Burr XII (BXII), modified Burr III (MBIII), Burr III (BIII), Weibull and inverse Weibull models are computed.

7.1 Times to Failure: The times to failures of 50 components (Aarset,[35]) are: 0.10, 0.20, 1, 1, 1,1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67,67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. The Aarset data set is recognized as bathtub shaped.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W*, A*, KS (p-values) are summarized in table 4. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC and $-\ell$ are written in Table 5.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull and inverse Weibull distributions because the values of all criteria are smaller for the NFKBIII distribution. We can identify that the NFKBIII distribution is closer fit to empirical data (**Fig. 3**).

7.2 Maximum Annual Flood Discharges: The data for 47 years of the North Saskachevan River (Edmonton) about maximum annual flood discharges (1000 ft^3/sec) are: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W*, A*, KS (p-values) are summarized in table 6. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC and $-\ell$ are written in Table 7.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull and inverse Weibull distributions as the values of all criteria are smaller for the NFKBIII distribution. We can identify that the NFKBIII distribution is closer fit to empirical data (**Fig. 4**).

8. CONCLUDING REMARKS

We have derived the NFKBIII distribution from the T-X family technique, transformation and compounding mixture of distributions. The NFKBIII density has arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shapes. The hazard rate function for the NFKBIII distribution has various shapes such as increasing, decreasing, increasing-decreasing-increasing and bathtub. Different statistical properties such as quantile function, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, moments for residual life functions and reliability measures have derived. Two characterizations of the NFKBIII distribution have studied. The maximum likelihood estimates (MLE) for unknown parameters of NFKBIII distribution have computed. A simulation study has accomplished to evaluate the behavior of the maximum likelihood estimators. The potentiality and utility of the NFKBIII distribution has demonstrated via its applications to times to failures of 50 devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution is tested via different goodness of fit criteria. The goodness of fit statistics has shown that the NFKBIII distribution is best fit model. We have displayed that the NFKBIII distribution is empirically best for lifetime applications.

REFERENCES

1. Burr, I.W. “Cumulative frequency distributions”, *Ann. Math. Stat.*, 13, pp.215-232 (1942).
2. Mielke, P.W. “Another family of distributions for describing and analyzing precipitation data”, *J. Appl. Meterol.*, 12, pp.275-280 (1973).
3. Kleiber, C. and Kotz, S. “*Statistical size distribution in economics and actuarial sciences”*. New York, John Wiley & Sons, (2003).
4. Gove, J.H., Ducey, M.J., Leak, W.B. and Zhang, L. “Rotated sigmoid structures in managed uneven-aged northern hardwork stands: a look at the Burr Type III distribution”, *Foresty*, 81(2), pp. 161-176 (2008).
5. Shao, Q., Chen, Y.D. and Zhang, L. “An extension of three-parameter Burr III distribution for low-flow frequency analysis”, *Comput. Stat. Data Anal*., 52, pp.1304-1314 (2008).
6. Benjamin, S.M., Humberto, V.H. and Arnold, B.C. “Use of the Dagum Distribution for Modeling Tropospheric Ozone levels”, *J. Env. Stat.*, 5(6) pp. 1-11(2013).
7. Ali, A. Husnain, S. A., and Ahmad, M. “Modified Burr III distribution, Properties and Applications”, *Pak. J. Statist.*, 31(6), pp.697-708 (2015).
8. Gomes, A. E., da-Silva, C. Q., and Cordeiro, G. M. “Two extended Burr models: Theory and practice”, *Communications in Statistics-Theory and Methods*, *44*(8), pp.1706-1734(2015).
9. Sinner, C., Dominicy, Y., Ley, C., Trufin, J., and Weber, P. “An Interpolating Family of Size Distributions”, (2016)." *arXiv preprint arXiv: 2016; 1606.04430*.
10. Moisheer A. S. A. “A mixture of two Burr Type III distributions: Identifiability and estimation under type II censoring”. *Mathematical Problems in Engineering, (*2016).
11. Olobatuyi, K. I., Asiribo, O. E., and Dawodu, G. A., “A New Class of Generalized Burr III Distribution for Lifetime Data” (2017). https://arxiv.org/pdf/1701.00403
12. Cordeiro G.M., Gomes, A. E., da-Silva, C. Q., Ortega E.M.M. “A useful extension of the Burr III distribution”, *Journal of Statistical Distributions and Applications”*, 4(1), pp.24 (2017).
13. Jamal, F., Nasir, M. A., Tahir, M. H., & Montazeri, N. H. “The odd Burr-III family of distributions”, *Journal of Statistics Applications and Probability*, *6*(1), *pp.*105-122 (2017).
14. Nasir, A., Bakouch, H. S., & Jamal, F. “Kumaraswamy odd Burr G family of distributions with applications to reliability data”, *Studia Scientiarum Mathematicarum Hungarica*, *55*(1), pp.94-114 (2018).
15. Kehinde, O., Osebi, A., & Ganiyu, D. “A New Class of Generalized Burr III Distribution for Lifetime Data”, *International Journal of Statistical Distributions and Applications*, *4*(1), pp. 6 (2018).
16. Marshall, A. W. and Olkin, I. “A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families”, *Biometrika*, 84(3), pp. 641-652 (1997).
17. Cordeiro, G. M. and Castro, M. “A new family of generalized distributions”, *Journal of statistical computation and simulation*, 81(7), pp. 883-898 (2011).
18. Alizadeh, M., Cordeiro, G. M., Nascimento, A. D., Lima, M. D. C. S., and Ortega, E. M. “Odd-Burr generalized family of distributions with some applications”, *Journal of Statistical Computation and Simulation*, 87(2), pp.367-389 (2017).
19. Cordeiro, G. M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E. M. M. and Altun, E. “The generalized odd log-logistic family of distributions: properties, regression models and applications”, *Journal of Statistical Computation and Simulation*, 87(5), pp.908-932 (2017).
20. Haghbin, H., Ozel, G., Alizadeh, M. and Hamedani, G. G. “A new generalized odd log-logistic family of distributions”, *Communications in Statistics-Theory and Methods*, 46(20), pp. 9897-9920 (2017).
21. Korkmaz, M. Ç., & Genç, A. İ. “Two-sided generalized exponential distribution”, *Communications in Statistics-Theory and Methods*, 44(23), pp. 5049-5070 (2015).
22. Korkmaz, M. Ç. and Genç, A. I. “A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution”, *Communications in Statistics - Simulation and Computation*, 46 (2), pp.1441-1460 (2017).
23. Cordeiro, G. M., Yousof, H. M., Ramires, T. G. and Ortega, E. M. M. “The Burr XII system of densities: properties, regression model and applications”, *Journal of Statistical Computation and Simulation, 88(3)*, pp. 432-456 (2018).
24. Alizadeh, M., Korkmaz, M. Ç., Almamy, J. A., & Ahmed, A. A. E. “Another odd log-logistic logarithmic class of continuous distributions”, *Journal of Statisticians: Statistics and Actuarial Sciences*, 11(2), pp.55-72 (2018).

25. Yousof, H. M., Altun, E., Ramires, T. G., Alizadeh, M. And Rasekhi, M. "A new family of distributions with properties, regression models and applications", *Journal of Statistics and Management Systems,* 21(1), pp. 163-188 (2018).
26. Korkmaz, M. Ç., Cordeiro, G. M., Yousof, H. M., Pescim, R. R., Afify, A. Z., & Nadarajah, S. "The Weibull Marshall–Olkin family: Regression model and application to censored data", *Communications in Statistics-Theory and Methods, (Accepted)*. (2019). DOI: 10.1080/03610926.2018.1490430.
27. Gurvich, M. R., Dibenedetto, A. T., & Ranade, S. V. "A new statistical distribution for characterizing the random strength of brittle materials", *Journal of Materials Science*, *32*(10), pp. 2559-2564 (1997).
28. Alzaatreh, A., Mansoor, M., Tahir, M. H., Zubair, M., & Ali, S. "The Gamma Half-Cauchy Distribution: Properties and Applications", *Hacettepe Journal of Mathematics and Statistics*, 45 (4), pp.1143-1159 (2016).
29. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. "The Weibull-G Family of Probability Distributions", *Journal of Data Science*, 12, pp.53-68 (2014).
30. Kies, J.A. "The strength of glass performance", *Naval Research Lab Report No. 5093*, Washington, D.C., (1958).
31. Kumar, C. Satheesh; Dharmaja, S. H. S. "On some properties of Kies distribution", *Metron*, 72(1),pp. 97-122 (2014).
32. Kotz S, Lai CD, Xie M. "On the Effect of Redundancy for Systems with Dependent Components", *IIE Trans*, 35, pp.1103-1110 (2003).
33. Bhattacharyya, G. K., & Johnson, R. A. "Estimation of reliability in a multicomponent stress-strength model". *Journal of the American Statistical Association*, *69*(348), pp.966-970 (1974).
34. Glänzel, W. A. *Characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory* (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, pp.75-84 (1987).
35. Aarset, M. V. "How to identify a bathtub hazard rate", *IEEE Transactions on Reliability*, 36(1), pp.106-108 (1987).
36. Chen, G. and Balakrishnan N. "A general purpose approximate goodness-of-fit test", *J. Qual. Technol*. 27, pp.154-161 (1995).

Fig.1: Plots of pdf of NFKBIII Distribution

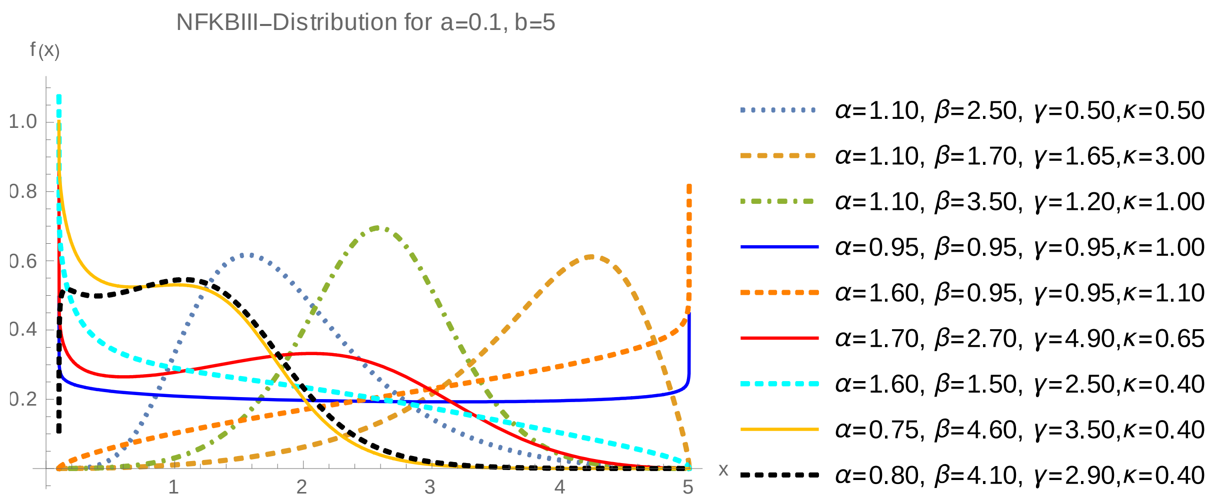


Fig. 2: Plots of hrf of NFKBIII Distribution

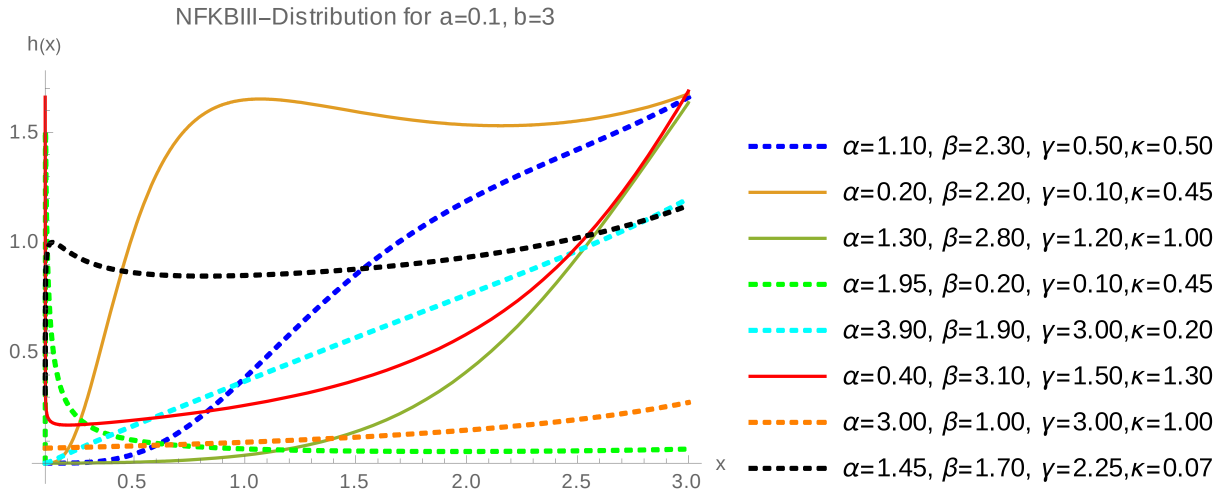


Fig. 3: Fitted pdf, cdf, survival and pp plots of the NFKBIII distribution for device failure times

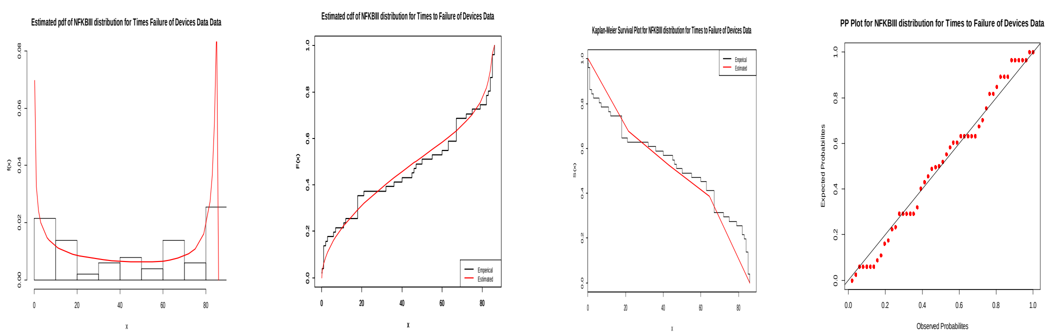
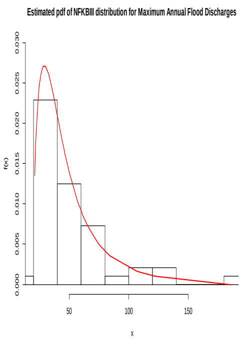
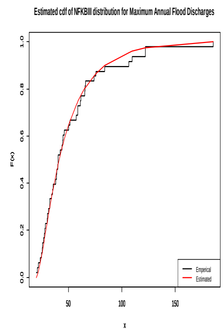
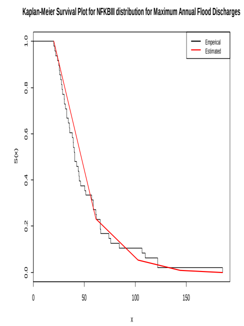


Fig. 4: Fitted pdf, cdf, survival and pp plots of the NFKBIII distribution for maximum annual flood discharges







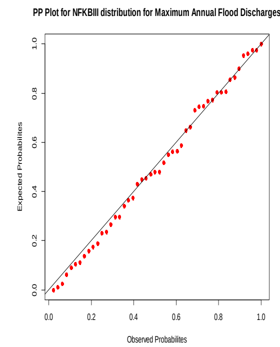


Table 1: Sub-Models of the NFKBIII Distribution

1	X	a	b	α	β	γ	κ	New Family of Kies Burr III (NFKBIII)
2	X	a	b	α	β	γ	**1**	Kies Modified Burr III (KMBIII)
3	X	a	b	α	β	1	κ	New Kies Burr III (NKBIII)
4	X	a	b	α	β	1	**1**	Kies Burr III (KBIII)
5	X	0	1	α	β	1	**1**	Reduced Kies Burr III (RKBIII)
6	X	0	1	α	β	γ	κ	Reduced New Kies Burr III (RNKBIII)
7	X	a	b	α	1	γ	κ	New Kies Modified Inverse Lomax (NKMIL)
8	X	a	b	α	1	γ	**1**	Kies Modified Inverse Lomax (KMIL)
9	X	a	b	α	1	1	**1**	Kies Inverse Lomax(KIL)
10	X	0	1	α	1	1	**1**	Reduced Kies inverse Lomax (RKIL)
11	X	a	b	α	β	γ	κ	Reduced Kies Burr III (RKBIII)
12	X	0	1	α	β	γ	κ	Reduced New Kies Burr III (RNKBIII)
13	X	a	b	α	β	$\gamma \to 0$	κ	New Inverse Kies (NIK)
14	X	0	1	α	β	$\gamma \to 0$	κ	New Reduced Inverse Kies (NRK)
15	X	a	b	α	β	$\gamma \to 0$	κ	New Modified inverse Kies (NMIK)
16	X	0	1	α	β	$\gamma \to 0$	1	Reduced New Inverse Kies(RNIK)
17	X	0	1	α	β	$\gamma \to 0$	1	Reduced Inverse Kies (RIK)
18	X	a	b	α	β	$\gamma \to 0$	1	New Modified Inverse Kies (NMIK)
19	X	a	b	α	β	$\gamma \to 0$	κ	Generalized Inverse Kies
20	$\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)$	a	b	α	β	$\gamma \to 0$	1	Kies (Kies;[30], and Kumar & Dharmaja ;[31])
21	X	a	b	α	β	$\gamma \to 0$	1	Inverse Kies
22	X	0	b	α	β	γ	κ	Modified Burr III Power (MBIII-Power)

23		a	0	α	β	γ	κ	Modified Burr III Pareto (MBIII-Pareto)
24	X	0	b	α	β	1	κ	Burr III Power (BIII-Power)
25	X	a	0	α	β	1	κ	Burr III Pareto (BIII-Pareto)
26	X	0	b	α	β	$\gamma \to 0$	κ	Inverse Weibull- Power
27	X	a	0	α	β	$\gamma \to 0$	κ	Inverse Weibull Pareto

Table 2: Median, mean, standard deviation, skewness and Kurtosis of the NFKBIII Distribution

Parameters $\alpha,\beta,\gamma,\kappa, a=0.1, b=5$	Median	Mean	Standard Deviation	Skewness	Kurtosis
0.5,0.5,0.5,0.5	0.4897	1.4750	1.7079	0.9765	2.3803
0.5,1.5,1.5,0.5	0.6763	1.1133	1.1156	1.2380	3.7107
1,0.5,0.5,0.5	2.1126	2.3588	1.8197	0.1678	1.4319
1,1,1,2	3.5333	3.3328	1.1766	-0.5611	2.3903
0.5,1,1,1,	1.3210	1.7305	1.4595	0.6413	2.1471
1,1,1,1	2.5460	2.5475	1.4138	0.0019	1.8006
2,1,1,1	3.3649	3.3649	1.1546	-0.5638	2.3977
1.5,1,1,1	3.1834	3.0379	1.2827	-0.3377	2.0496
1,1,1,0.5	1.6268	1.9349	1.4408	0.4984	2.0057
1.5,1.5,1.5,1	2.8765	2.8088	1.1137	-0.2179	2.2432
1.5,1.5,1.5,1.5	3.4298	3.2977	0.9797	-0.5674	2.7456
1.5,1.5,1.5,2.5	3.9849	3.8405	0.7525	-0.9578	3.7872
1.5,1.5,1.5,3	4.1375	4.0009	0.6698	-1.0776	4.2189
1.5,1.5,1.5,0.5	1.9744	2.0897	1.1797	0.3375	2.2034
2.5,1.5,1.5,0.5	2.5717	2.5902	1.0939	0.0430	2.1806
2,2,2,2	3.8261	3.7287	0.6639	-0.8429	3.8245
2.5,1.5,2.5,2.5	4.2032	4.0413	0.6924	-1.2421	4.6981
2.5,1.5,2.5,2.5	4.0501	3.9854	0.4675	-0.9543	4.6041
5,2.5,2.5,2.5	4.2659	4.2409	0.3136	-0.5782	3.6917
3.25,2.4,0.65,1.5	3.7515	3.7577	0.4326	0.0016	2.7084
4.5,2.4,0.65,1.5	3.8735	3.8788	0.3956	0.0010	2.6739
5,2.5,2.5,2.5	4.2659	4.2409	0.3136	-0.5782	3.6917
5,2.5,2.5,0.5	3.2661	3.2192	0.9057	-0.2531	2.4020
6,2,1.5,0.5	3.0029	3.0224	0.7507	0.0597	2.5103
6,2,1.5,0.5	4.3075	4.2356	0.4691	-0.8193	3.6134
5,1.5,1.5,1.5	4.2268	4.1482	0.5200	-0.8090	3.5801
5,0.5,1.5,1.5	4.9222	4.6458	0.6553	-3.0134	13.2449
5,1,1.5,1.5	4.5453	4.3729	0.5804	-1.5137	5.6080

Table 3: Means, Bias and MSEs of the NFKBIII distribution (0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5) and (1, 6, 2, 0.8, 1.5, 1.75)

Sample	Statistics	$a=0.10$	$b=4$	$\alpha=1.2$	$\beta=0.4$	$\gamma=1.1$	$\kappa=1.2$
n=50	Means	0.1002	3.9997	1.2416	0.4132	1.2034	1.2803
	Bias	2e-04	-3e-04	0.0416	0.0132	0.1034	0.0803
	MSE	0	0	0.011	0.0015	0.027	0.0215
n=100	Means	0.10	4	1.2191	0.4099	1.1843	1.2772
	Bias	0	0	0.0191	0.0099	0.0843	0.0772
	MSE	0	0	0.0046	7e-04	0.018	0.0193
n=150	Means	0.1	4	1.2092	0.4083	1.1722	1.2714
	Bias	0	0	0.0092	0.0083	0.0722	0.0714
	MSE	0	0	0.0024	4e-04	0.0133	0.0165

Sample	Statistics	$a=0.5$	$b=5$	$\alpha=1.5$	$\beta=0.5$	$\gamma=1.3$	$\kappa=1.5$
n=50	Means	0.5015	4.9989	1.5386	0.5069	1.4244	1.6339
	Bias	0.0015	-0.0011	0.0386	0.0069	0.1244	0.1339
	MSE	3e-04	0	0.0232	0.001	0.0456	0.0418
n=100	Means	0.5003	4.9997	1.519	0.5011	1.4072	1.6386
	Bias	3e-04	-3e-04	0.019	0.0011	0.1072	0.1386
	MSE	0	0	0.0095	2e-04	0.0298	0.0402

n=150	Means	0.5001	4.9999	1.5108	0.5	1.3944	1.6333
	Bias	1e-04	-1e-04	0.0108	0	0.0944	0.1333
	MSE	0	0	0.0049	0	0.0237	0.0357

Sample	Statistics	$a = 1.0$	$b = 6$	$\alpha = 2.0$	$\beta = 0.8$	$\gamma = 1.5$	$\kappa = 1.75$
n=50	Means	1.0619	5.84	2.0538	0.8496	1.7006	1.8917
	Bias	0.0619	-0.16	0.0538	0.0496	0.2006	0.1417
	MSE	0.0119	0.0908	0.0365	0.0125	0.0835	0.0652
n=100	Means	1.0177	5.9591	2.0144	0.8182	1.6366	1.8777
	Bias	0.0177	-0.0409	0.0144	0.0182	0.1366	0.1277
	MSE	0.0026	0.0218	0.0205	0.0041	0.0586	0.0542
n=150	Means	1.0073	5.9861	2.0003	0.81	1.6132	1.8685
	Bias	0.0073	-0.0139	3e-04	0.01	0.1132	0.1185
	MSE	8e-04	0.0066	0.0116	0.0018	0.0417	0.0426

Table 4: MLEs and their standard errors (in parentheses) for times to failure of devices

Model	α	β	γ	κ	a	b	W*	A*	KS (p-value)
NFKBIII	115364.0 (379.4410)	4.956983 (0.3467780)	4889524 (16199.92)	4.722609 (0.7202355)	0.10	86	**0.0454792**	**0.414671**	**0.076 (0.9445)**
KMBIII	1.5122204 (0.9932218)	0.6942384 (0.2090516)	2.2315263 (2.7777485)	----------	0.10	86	0.05448629	0.4984068	0.0777 (0.9343)
NKBIII	2.9425405 (2.26453756)	0.587427 (0.08992104)	---------	0.0244289 (0.53930075)	0.10	86	0.05219844	0.4703996	0.0778 (0.9332)
KBIII	1.0520786 (0.1653031)	0.5805595 (0.0747764)	------	1	0.10	86	0.06114472	0.5322372	0.0744 (0.9531)
NKIL	0.7396103 (0.5740525)	1	------	1.0210726 (0.8272429)	0.10	86	0.07033046	0.5912904	0.2278 (0.0137)
KIL	0.7542468 (0.108866)	1	1	1	0.10	86	0.07026344	0.5906328	0.228 (0.01359)
MBXII	171.999510 (202.1714396)	5.275727 (0.9105317)	4243.399938 (5424.6188490)	------	------	------	1.298802	6.86492	0.3558 (6.342e-06)
BXII	0.2454656	1.2600795	1	------	------	------	1.0933	5.850392	0.3336

	(0.06939703)	(0.32079056)							(2.941e-05)
MBIII	455699.1 (23876.41)	3.224871 (0.08486702)	1959580 (13762.39)	------	------	------	0.3964946	2.474767	0.1614 (0.1478)
BIII	4.1810540 (0.63742201)	0.5766612 (0.05248543)	1	------	------	------	0.9456985	5.177504	0.2656 (0.001724)
Weibull	0.0272128 (0.39009785)	0.9476152 (0.04439031)	------	------	------	------	0.4949391	3.001556	0.1933 (0.04769)
Inverse Weibull	2.6499805 (0.39009785)	0.4634121 (0.04439031)	------	------	------	------	1.039875	5.565583	0.2856 (0.0005731)

Table 5: Goodness-of-fit statistics for times to failure of devices

Model	AIC	CAIC	BIC	HQIC	$-\ell$
NFKBIII	**403.3062**	**404.2365**	**410.791**	**406.1348**	**197.6531**
KMBIII	408.9108	409.4563	414.5244	411.0322	201.4554
NKBIII	407.7712	408.3166	413.3848	409.8926	200.8856
KBIII	407.3655	407.6322	411.1079	408.7798	201.6828
NIKL	427.3011	427.5677	431.0435	428.7153	211.6505
KIL	425.3018	425.3887	427.173	426.0089	211.6509
MBXII	577.3329	577.8546	583.069	579.5172	285.6664
BXII	548.6714	548.9267	552.4954	550.1276	272.3357
MBIII	478.7943	479.316	484.5304	480.9786	236.3972
BIII	525.2932	525.5485	529.1172	526.7494	260.6466
Weibull	485.9593	486.2146	489.7833	487.4155	240.9796
Inverse Weibull	533.973	534.2283	537.797	535.4292	264.9865

Table 6: MLEs and their standard errors (in parentheses) for maximum annual flood discharges

Model	α	β	γ	κ	a	b	W*	A*	K-S (p-value)
NFKBIII	0.30254078 (1.6306527)	1.70483835 (0.4027486)	0.35210310 (1.9110147)	0.02101155 (2.8260695)	19.885	185.560	0.01576999	0.125909	0.0571 (0.9982)
KMBIII	0.005113609 (0.002923934)	8.889992999 (NAN)	0.133291960 (0.041725229)	1	19.885	185.560	0.611651	3.719855	0.2657 (0.003024)
NKBIII	0.005549908 (0.003225763)	67.49297076 (40.584567979)	1	1.619899847 (0.042015909)	19.885	185.560	0.3781306	2.386018	0.3031 (0.0004277)
KBIII	0.1659637 (0.05820467)	3.0726999 (0.99621453)	1	1	19.885	185.560	0.1799505	1.224586	0.2379 (0.01094)
NKIL	0.7983516990 (0.1614468)	1	1	0.0000000001 (0.2410621)	19.885	185.560	0.01863201	0.1529237	0.239 (0.01046)
KIL	0.4679514 (0.06899535)	1	1	1	19.885	185.560	0.03613052	0.2757191	0.2982 (0.0005609)
MBXII	0.0113894 (0.01673093)	124.9109988 (760.63939777)	5.4099658 (37.86181308)	------	------	------	0.05350551	0.3691381	0.5461 (7.386e-13)
BXII	0.07560631 (0.3409857)	3.48218254 (15.7016695)	1	------	------	------	0.05340923	0.368504	0.5449 (8.376e-13)

MBIII	6107.715659 (12522.06)	2.447060 (0.5375678)	1.738394 (1616.620)	------	------	------	0.01920378	0.1367274	0.0701 (0.9725)
BIII	6106.072865 (6901.9841287)	2.447011 (0.3285967)	1	------	------	------	0.01920334	0.1367219	0.0701 (0.9724)
Weibull	0.002567956 (0.000497175)	1.489705016 (0.054448946)	------	------	------	------	0.2385166	1.511058	0.1981 (0.04618)
Inverse Weibull	6098.970551 (6310.531373)	2.446713 (0.301231)	------	------	------	------	0.0192021	0.1367111	0.0701 (0.9725)

Table 7: Goodness-of-fit statistics for maximum annual flood discharges

Model	AIC	CAIC	BIC	HQIC	$-\ell$
NFKBIII	**407.7306**	**408.7062**	**415.0452**	**410.4707**	**199.8653**
KMBIII	519.4343	520.0057	524.9202	521.4894	256.7171
NKBIII	434.2349	434.8064	439.7209	436.29	214.1175
KBIII	423.3242	423.6033	426.9815	424.6943	209.6621
NKIL	423.6303	423.9093	427.2875	425.0003	209.8151
KIL	437.3848	437.3848	437.4757	439.2134	217.6924
MBXII	595.1127	595.6582	600.7263	597.2341	294.5564
BXII	592.772	593.0386	596.5144	594.1862	294.386
MBIII	436.2281	436.7736	441.8417	438.3495	215.1141
BIII	434.2277	434.4944	437.9701	435.642	215.1139
Weibull	458.1291	458.3958	461.8715	459.5434	227.0646
Inverse Weibull	434.2272	434.4938	437.9696	435.6414	215.1136

Biographies

Fiaz Ahmad Bhatti is currently working as Assistant Professor at Govt. Postgraduate College, Satellite Town, Gujranwala; Pakistan. He completed his MSc from Quad-i- Azam University, Islamabad, Pakistan. He completed his MPhil from Allama Iqbal Open University, Islamabad; Pakistan. He received the PhD degree in Statistics from National College of Business Administration and Economics, Lahore; Pakistan. He has published 12 research articles. His research interests include distribution theory, reliability analysis and Multivariate Analysis.

Dr. Munir Ahmad is currently working as Rector of National College of Business Administration and Economics, Lahore; Pakistan. He is chief editor of Pakistan journal of Statistics. He received the PhD degree in Statistics from Iowa State University, USA. He has published over 150 research articles. His research interests include distribution theory, reliability analysis, rank set sampling, quality control and statistical inference.

*.Corresponding author, Tel.: +92 3338106672

Email address: fiazahmad72@gmail.com (Fiaz Ahmad Bhatti).