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Research Note

Approximate analytical solution of the time-fractional Camassa-Holm, modified Camassa-Holm, and Degasperis-Procesi equations by homotopy perturbation method

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Abstract. In this paper, the approximate analytical solutions of Camassa-Holm, modified Camassa-Holm, and Degasperis-Procesi equations with fractional time derivative are obtained with the help of approximate analytical method of nonlinear problem called the Homotopy Perturbation Method (HPM). By using initial condition, the explicit solution of the equation has been derived which demonstrates the effectiveness, validity, potentiality, and reliability of the method in reality. Comparing the methodology with the exact solution shows that the present approach is very effective and powerful. The numerical calculations are carried out when the initial condition is in the form of exponential and transcendental functions; the results are depicted through graphs.

1. Introduction

In the last three decades, a great attention has been devoted to the study of the fractional calculus and their copious applications in the area of life science, physical science and the engineering science. The fractional calculus are also used in many fields, such as chemical physics, optics, electrical networks, solitary waves, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion and signal processing, and so on, which can be successfully derived by linear or nonlinear fractional order differential equations. Many definitions and various properties of fractional calculus are available in many books, such as [1-3].

Mathematicians have formed a theory of linear operators, which contains within its scope a considerable domain of analysis. This type of work should include within its limits a large area of mathematics, which is readily tacit from the fact that the assumption of linearity in operational processes lead to most applications of analysis to the problems of the natural world. It is for this reason that a theory of linear operators, in contrast to a theory of nonlinear operators, is comparatively easy to develop. The latter is overwhelmed by many difficulties. We know that few algorithms, which can be applied, and the powerful

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existence theorems of the linear case must be replaced too often by those of special applications.

But despite the difficulties of the general physical problem, there exists need for a methodical dealing for nonlinear equations. Nature, with scant regard for the desires of the mathematician, often seems to delight in converting her mysteries in terms of nonlinear systems of equations. The theories of elasticity, ecology, and fluid dynamics are especially rich in such systems. The mathematician, however, with his/her rich store of linear algorithms, must usually attack these mysteries from the viewpoint of linear operators. These problems thus become those of reducing the equations through various analytical devices to a linear system. In this absence, he/she must then try to approximate the solution by some asymptotic process which brings it within the scope of functions which have been defined and studied by linear methods.

In this paper, we consider the nonlinear time-fractional Camassa-Holm equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+2\kappa\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^2\partial t}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3},$$
$$t>0,\ x\in R,\quad 0<\alpha\leq 1, \tag{1}$$

the nonlinear time-fractional modified Camassa-Holm equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^3 u}{\partial x^2\partial t}+3u^2\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3},$$
$$t>0,\ x\in R,\quad 0<\alpha\leq 1, \tag{2}$$

and, the nonlinear time-fractional Degasperis-Procesi equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^3 u}{\partial x^2\partial t}+4u^2\frac{\partial u}{\partial x}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3},$$
$$t>0,\ x\in R,\quad 0<\alpha\leq 1, \tag{3}$$

with initial condition:

$$u(x,0)=f(x), \tag{4}$$

in dimensionless space-time variables (x,t). These are models for the unidirectional propagation of two-dimensional shallow water waves over a flat bottom. Eq. (1) can also be written in standard form as:

$$\frac{\partial m}{\partial t}+2\kappa\frac{\partial u}{\partial x}+u\frac{\partial m}{\partial x}+2m\frac{\partial u}{\partial x}=0. \tag{5}$$

In Eqs. (1) and (5), $u(x,t)$ represents the horizontal component of the fluid velocity, or the free surface of water; $m=u-D_{xx}u$ is the momentum variable cf. [4] (see also [5]); and the solitary waves of Eq. (1) for $\kappa>0$ are smooth solitons [6,7], while in the limiting case, $\kappa=0$, they are peaked solitons (peakons) [1] which have to be understood as weak solutions [8,9]. The peakons are stable wave patterns [10,11]. Camassa-Holm equation also models axially symmetric waves in hyperelastic rods [12] and was first derived as an abstract bi-Hamiltonian equation [13]. Moreover, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group of the line [14,15]. Some solutions exist for all times, while others have a finite life-span, modelling wave breaking [16]. The solitary waves of the Camassa-Holm equation are stable solitons [10,17] with a peak at their crest.

The HPM is the new approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He [18-22] and was successfully applied to solve nonlinear equations and claimed that the approximations obtained were valid not only for small parameters but for very large parameters. Fractional diffusion equation with absorbent term and external force by Das and Gupta [23], space-time fractional advection dispersion equation by Yildirim and Koçak [24], boundary value problems by He [25], integro-differential equation by El-Shahed [26], modified Camassa-Holm and Degasperis-Procesi equations by Zhang et al. [27], fractional linear and nonlinear Schrödinger equation [28,29], analytical study of Navier-Stokes equation [30], nonlinear dispersive equation with time fractional derivative by Koçak et al. [31], nonlinear fractional predator-prey model [32], multi-order time fractional differential equation [33], and use of fractional differential equation in fluid mechanics [34] by HPM etc. have been studied in recent times. Recently, Gupta et al. [35,36] solved the heat transfer problem and fractional Benny-Lin equation using traditional technique homotopy perturbation method. The basic difference of this method with the other perturbation techniques is that it does not require small parameters in the equation, which overcomes the limitations of traditional perturbation techniques.

2. Preliminaries and notations

In this section, we give some definitions and properties of the fractional calculus [1], which are used further in this paper.

Definition 1. A real function $f(t)$, $t>0$, is said to be in the space C_μ, $\mu\in\mathcal{R}$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu, n\in N$.

Definition 2. The Riemann-Liouville fractional integral operator (J_t^α) of order $\alpha\geq 0$ of a function $f\in C_\mu\geq -1$ is defined as:

$$J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}f(\xi)d(\xi),\quad \alpha>0, t>0,$$

$$J_t^0 f(t) = f(t), \tag{6}$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator J_t^α, which we will need here, are as follows:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$:

1. $J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t)$,
2. $J_t^\alpha J_t^\alpha f(t) = J_t^\beta J_t^\alpha f(t)$,
3. $J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D_t^α) of $f(t)$, in the Caputo sense is defined as:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \tag{7}$$

for $n-1 < \alpha < n, n \in N, t > 0, f \in C_{-1}^n$.

The following are two basic properties of the Caputo fractional derivative [1,2]:

1. Let $f \in C_{-1}^n, n \in N$, then $D_t^\alpha f$, $0 \leq \alpha \leq n$ is well defined and $D_t^\alpha f \in C_{-1}$.
2. Let $n-1 \leq \alpha \leq n$, $n \in N$ and $f \in C_\mu^n$, $\mu \geq -1$, then:

$$\left(J_t^\alpha D_t^\alpha\right) f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{8}$$

3. Basic idea of HPM

To illustrate the basic ideas of this method [20-25], we consider the following non-linear functional equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{9}$$

with the following boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{10}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. The operator A can be decomposed into two operators L and N, where L is linear, and N is nonlinear operator. Eq. (9) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{11}$$

Using the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow \mathcal{R}$, which satisfies:

$$\mathcal{H}(v,p) \equiv (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0,$$
$$p \in [0,1], \quad r \in \Omega, \tag{12}$$

or:

$$\mathcal{H}(U,p) \equiv L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{13}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (9), which satisfies the boundary conditions. Obviously, from Eqs. (12) and (13), we will have:

$$\mathcal{H}(v,0) \equiv L(v) - L(u_0) = 0, \tag{14}$$

$$\mathcal{H}(v,1) \equiv A(v) - f(r) = 0, \tag{15}$$

the changing values of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopics.

According to HPM, we can first use the embedding parameter pas a small parameter, and assume that the solution of Eqs. (12) and (13) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{16}$$

Setting $p = 1$ results in the approximation to the solution of Eq. (9):

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \ldots \tag{17}$$

The combination of the perturbation method and the homotopy method is called the Homotopy Perturbation Method (HPM), which has eliminated limitations of the traditional perturbation techniques.

4. Application of HPM

4.1. Application of HPM to time-fractional Camassa-Holm equation

The standard form of the time-fractional Camassa-Holm equation (1) in an operator form is given by:

$$D_t^\alpha u + 2\kappa D_x u - D_{xxt} u + 3u D_x u = 2 D_x u D_{xx} u + u D_{xxx} u, \quad 0 < \alpha \leq 1, \tag{18}$$

with initial condition:

$$u(x,0) = f(x). \tag{19}$$

According to the HPM, we construct the following homotopy:

$$D_t^\alpha u = p\big[-2\kappa D_x u + D_{xxt} u - 3u D_x u + 2 D_x u D_{xx} u + u D_{xxx} u\big], \tag{20}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Now, applying the classical perturbation technique, we can assume that the solution of Eq. (18) can be expressed as a power series in p as given below:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + ... \quad (21)$$

When $p \to 1$, Eq. (20) corresponds to Eq. (18) and Eq. (21) becomes the approximate solution of Eq. (18), that is, of Eq. (1). Substituting Eq. (21) in Eq. (20) and comparing the alike powers of p, we obtain the following set of linear differential equations:

$$p^0 : D_t^\alpha u_0 = 0, \quad (22)$$

$$p^1 : D_t^\alpha u_1 = \Big[-2\kappa D_x u_0 + D_{xxt}u_0 - 3u_0 D_x u_0 + 2D_x u_0 D_{xx}u_0 + u_0 D_{xxx}u_0\Big], \quad (23)$$

$$p^2 : D_t^\alpha u_2 = \Big[-2\kappa D_x u_1 + D_{xxt}u_1 - 3u_0 D_x u_1 - 3u_1 D_x u_0 + 2D_x u_0 D_{xx}u_1 + 2D_x u_1 D_{xx}u_0 + u_0 D_{xxx}u_1 + u_1 D_{xxx}u_0\Big], \quad (24)$$

and so on.

The method is based on applying the operator J_t^α (the inverse operator of Caputo derivative D_t^α) on both sides of Eqs. (22)-(24); so we obtain:

$$u_0(x,t) = f(x), \quad (25)$$

$$u_1(x,t) = \Big[-2\kappa f'(x) - 3f(x)f'(x) + 2f'(x)f^{(2)}(x) + f(x)f^{(3)}(x)\Big]\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (26)$$

$$\begin{aligned} u_2(x,t) = &-\frac{1}{2}\Big[9f'(x)f^{(2)}(x) + 2\kappa f^{(3)}(x) + 3f(x)f^{(3)}(x) - 7f^{(2)}(x)f^{(3)}(x) \\ &- 4f'(x)f^{(4)}(x) - f(x)f^{(5)}(x)\Big]\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ &+ \Big[12\kappa(f'(x))^2 + 18f(x)(f'(x))^2 + 4\kappa^2 f^{(2)}(x) + 12\kappa f(x)f^{(2)}(x) \\ &+ 9(f(x))^2 f^{(2)}(x) - 30(f'(x))^2 f^{(2)}(x) - 8\kappa(f^{(2)}(x))^2 - 21f(x)(f^{(2)}(x))^2 \\ &+ 4(f^{(2)}(x))^3 - 12\kappa f'(x)f^{(3)}(x) - 33f(x)f'(x)f^{(3)}(x) + 22f'(x)f^{(2)}(x) \\ &f^{(3)}(x) + 8f(x)(f^{(3)}(x))^2 - 4\kappa f(x)f^{(4)}(x) - 6(f(x))^2 f^{(4)}(x) + 8(f'(x))^2 f^{(4)}(x) \\ &+ 13f(x)f^{(2)}(x)f^{(4)}(x) + 7f(x)f'(x)f^{(5)}(x) + (f(x))^2 f^{(6)}(x)\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \quad (27)$$

Proceeding in this manner, the rest of the components, $u_n(x,t)$, can be obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series:

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \quad (28)$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$.

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [37].

4.2. Application of HPM to time-fractional modified Camassa-Holm equation

The standard form of the time-fractional modified Camassa-Holm equation (2) in an operator form is given by:

$$D_t^\alpha - D_{xxt}u + 3u^2 D_x u = 2D_x u D_{xx}u + uD_{xxx}u, \quad 0 < \alpha \leq 1, \quad (29)$$

with initial condition:

$$u(x,0) = f(x). \quad (30)$$

According to the HPM, we construct the following homotopy:

$$D_t^\alpha u = p\Big[D_{xxt}u - 3u^2 D_x u + 2D_x u D_{xx}u + uD_{xxx}u\Big], \quad (31)$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Now, applying the classical perturbation technique, we can assume that the solution of Eq. (29) can be expressed as a power series in p as given below:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + ... \quad (32)$$

When $p \to 1$, Eq. (31) corresponds to Eq. (29) and Eq. (32) becomes the approximate solution of Eq. (29), that is, of Eq. (2). Substituting Eq. (32) in Eq. (31) and comparing the alike powers of p, we obtain the following set of linear differential equations:

$$p^0 : D_t^\alpha u_0 = 0, \tag{33}$$

$$p^1 : D_t^\alpha u_1 = \Big[D_{xxt}u_0 - 3u_0^2 D_x u_0 + 2D_x u_0 D_{xx}u_0 + u_0 D_{xxx}u_0\Big], \tag{34}$$

$$p^2 : D_t^\alpha u_2 = \Big[D_{xxt}u_1 - 3\Big(u_0^2 D_x u_1 + 2u_0 u_1 D_x u_0\Big) + 2\Big(D_x u_0 D_{xx}u_1 + D_x u_1 D_{xx}u_0\Big) + \Big(u_0 D_{xxx}u_1 + u_1 D_{xxx}u_0\Big)\Big], \tag{35}$$

and so on.

The method is based on applying the operator J_t^α (the inverse operator of Caputo derivative D_t^α) on both sides of Eqs. (33)-(35); so we obtain:

$$u_0(x,t) = f(x), \tag{36}$$

$$u_1(x,t) = \Big[f(x)f^{(3)}(x) - 3(f(x))^2 f'(x) + 2f'(x)f^{(2)}(x)\Big]\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{37}$$

$$\begin{aligned} u_2(x,t) = &\Big[f(x)f^{(2)}(x) + 4f'(x)f^{(4)}(x) + 7f^{(2)}(x)f^{(3)}(x) - 3(f(x))^2 f^{(3)}(x) \\ &- 6(f'(x))^3 - 18f(x)f'(x)f^{(2)}(x)\Big]\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ &+ \Big[36(f(x))^3(f'(x))^2 - 12(f'(x))^4 + 9(f(x))^4 f^{(2)}(x) - 96f(x)(f'(x))^2 f^{(2)}(x) \\ &- 30(f(x))^2(f^{(2)}(x))^2 + 4(f^{(2)}(x))^3 - 48(f(x))^2 f'(x)f^{(3)}(x) + 22f'(x)f^{(2)}(x) \\ &f^{(3)}(x) + 8f(x)(f^{(3)}(x))^2 - 6(f(x))^3 f^{(4)}(x) + 8(f'(x))^2 f^{(4)}(x) + 13f(x)f^{(2)}(x)f^{(4)}(x) \\ &+ 7f(x)f'(x)f^{(5)}(x) + (f(x))^2 f^{(6)}(x)\Big] \\ &\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{38}$$

Proceeding in this manner, the rest of the components, $u_n(x,t)$, can be obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series:

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \tag{39}$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$.

The above series solutions generally converge very rapidly.

4.3. Application of HPM to time-fractional modified Degasperis-Procesi equation

The standard form of the time-fractional modified Degasperis-Procesi equation (3) in an operator form is given by:

$$D_t^\alpha u - D_{xxt}u + 4u^2 D_x u = 3D_x u D_{xx}u + uD_{xxx}u, \quad 0 < \alpha \leq 1, \tag{40}$$

with initial condition:

$$u(x,0) = f(x). \tag{41}$$

According to the HPM, we construct the following homotopy:

$$D_t^\alpha u = p\Big[D_{xxt}u - 4u^2 D_x u + 3D_x u D_{xx}u + uD_{xxx}u\Big], \tag{42}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). Now, applying the classical perturbation technique, we can assume that the solution of Eq. (40) can be expressed as a power series in p as given below:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + p^4 u_4 + ... \tag{43}$$

When $p \to 1$, Eq. (42) corresponds to Eq. (40), and Eq. (43) becomes the approximate solution of Eq. (40), that is, of Eq. (3). Substituting Eq. (43) in Eq. (42) and comparing the alike powers of p, we obtain the following set of linear differential equations:

$$p^0 : D_t^\alpha u_0 = 0, \tag{44}$$

$$p^1 : D_t^\alpha u_1 = [D_{xxt}u_0 - 4u_0^2 D_x u_0 + 3D_x u_0 D_{xx}u_0 + u_0 D_{xxx}u_0], \tag{45}$$

$$p^2: D_t^\alpha u_2 = [D_{xxt}u_1 - 4(u_0^2 D_x u_1 + 2u_0 u_1 D_x u_0) + 3(D_x u_0 D_{xx} u_1 + D_x u_1 D_{xx} u_0) + (u_0 D_{xxx} u_1 + u_1 D_{xxx} u_0)], \quad (46)$$

and so on.

The method is based on applying the operator J_t^α (the inverse operator of Caputo derivative D_t^α) on both sides of Eqs. (44)-(46), then we obtain:

$$u_0(x,t) = f(x), \quad (47)$$

$$u_1(x,t) = \left[f(x)f^{(3)}(x) - 4(f(x))^2 f'(x) + 3f'(x)f^{(2)}(x)\right]\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (48)$$

$$u_2(x,t) = \Bigg[f(x)f^{(5)}(x) + 5f'(x)f^{(4)}(x) + 10f^{(2)}(x)f^{(3)}(x) - 4(f(x))^2 f^{(3)}(x) - 24f(x)f'(x)f^{(2)}(x) - 8(f'(x))^3\Bigg]\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \Bigg[(f(x))^2 f^{(6)}(x) + 9f(x)f'(x)f^{(5)}(x) + 18f(x)f^{(2)}(x)f^{(4)}(x) + 15(f'(x))^2 f^{(4)}(x) - 8(f(x))^3(f^{(4)}(x)) + 11f(x) (f^{(3)}(x))^2 + 45f'(x)f^{(2)}(x)f^{(3)} - 72(f(x))^2 f'(x)f^{(3)}(x) + 9(f^{(2)}(x))^3 - 48(f(x))^2(f^{(2)}(x))^2 - 168f(x)(f'(x))^2 f^{(2)}(x) + 16(f(x))^4 f^{(2)}(x) - 24(f'(x))^4 + 64(f(x))^3(f'(x))^2\Bigg]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (49)$$

Proceeding in this manner, the rest of the components, $u_n(x,t)$, can be obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series:

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \quad (50)$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$.

The above series solutions generally converge very rapidly.

5. Numerical results and discussion

In this section, numerical results of the displacement $u(x,t)$ for different time-fractional Brownian motions, $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and for the standard motion, $\alpha = 1$, are calculated for various values of t and x at $\kappa = 0.005$ and $c = 0.01$. In Section 4.1, the initial condition is taken as $u(x,0) = f(x) = (\kappa + c)e^{-|x|} - \kappa$ as a particular case for showing the nature of the displacement. The numerical results of $u(x,t)$ for various values of t, x and α are depicted through Figure 1(a)-(d). In Section 4.2, the initial condition is considered as $u(x,0) = f(x) = (\kappa + c)e^{-|x|} - \kappa$ as a particular case for viewing the nature of the displacement. The numerical outcomes of $u(x,t)$ for various values of t, x and α are depicted through Figure 2(a)-(d). And in Section 4.3, the initial condition is predicted as $u(x,0) = f(x) = (\kappa+c)e^{-|x|} - \kappa$ as a particular case for presenting the nature of the displacement. The numerical effects of $u(x,t)$ for various values of t, x and α are illustrated through Figure 3(a)-(d). According to numerical solutions, we make a comparison between the approximate solution and its exact solution [38] in Figures 4 and 5, 6 and 7, and 8 and 9.

In order to illustrate that the approximate solution is efficient and accurate, we will give explicit values of the parameters x, κ, c, and t. Then, we calculate the three particular exact solutions and make a comparison between them. Also the graphics of their surfaces are plotted in Figures 4, 6 and 8 with the given values of the parameters as well as the profiles of them, given in Figures 5, 7, and 9.

According to numerical solutions from Figures 4, 6, and 8, we can see, at the same time t, that the values of the approximates solutions and the exact solutions are quite close. One can also see that when the value of x increases, the approximate solutions are more and more close to the exact solutions. From Figures 4 and 5, 6 and 7 and 8 and 9, we can also see that their surface graphics and profiles are almost the same. That is to say that the solution obtained by HPM is efficient and accurate. It is also suggested that HPM is a powerful method for solving differential equation with fully nonlinear dispersion terms.

6. Conclusion

In this article, an approximate analytical method has been used to solve the fractional CH, mCH, and Degasperis-Procesi equations. This method is very effective, convenient, and efficient and avoids the appearance of ill-conditioned matrices, complicated integrals, and infinite series. This technique does not require a small parameter in an equation. In this method, according to the homotopy technique, a homotopy with an imbedding parameter $p \in [0,1]$ is constructed, and

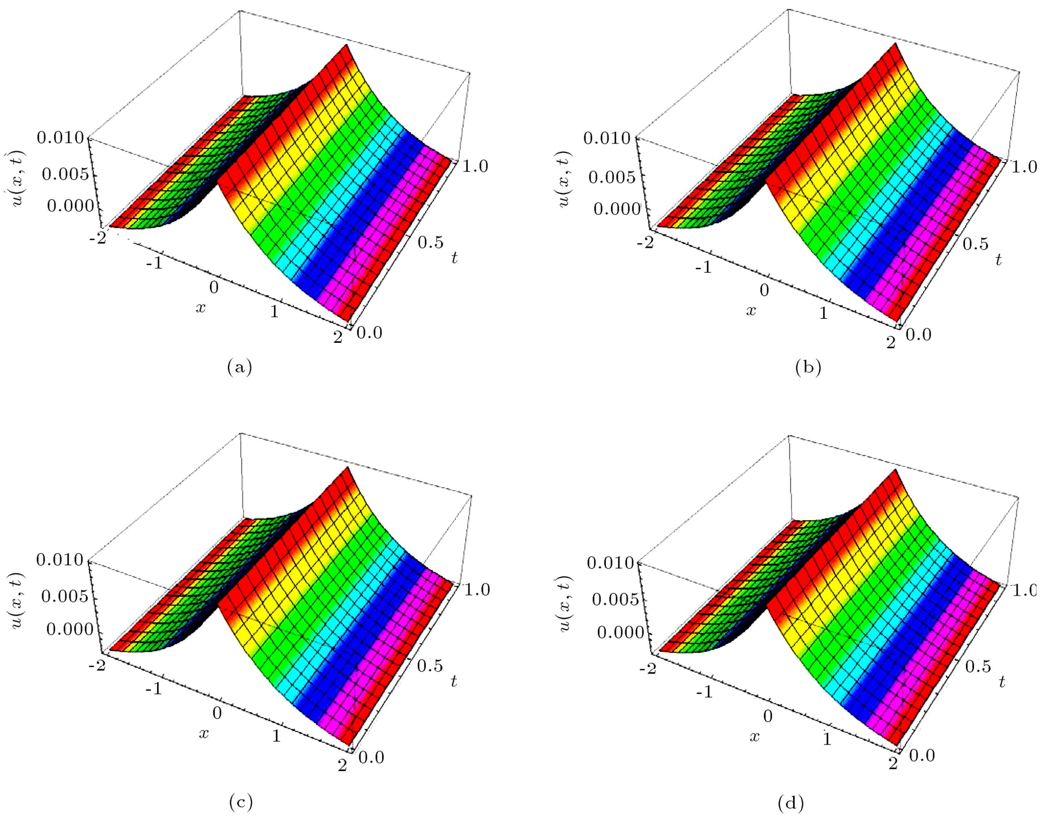


Figure 1. Plot of $u(x,t)$ with respect to x and t at (a) $\alpha = \frac{1}{4}$, (b) $\alpha = \frac{1}{2}$, (c) $\alpha = \frac{3}{4}$, and (d) $\alpha = 1$, with $\kappa = 0.005$ and $c = 0.01$ for Eq. (28).

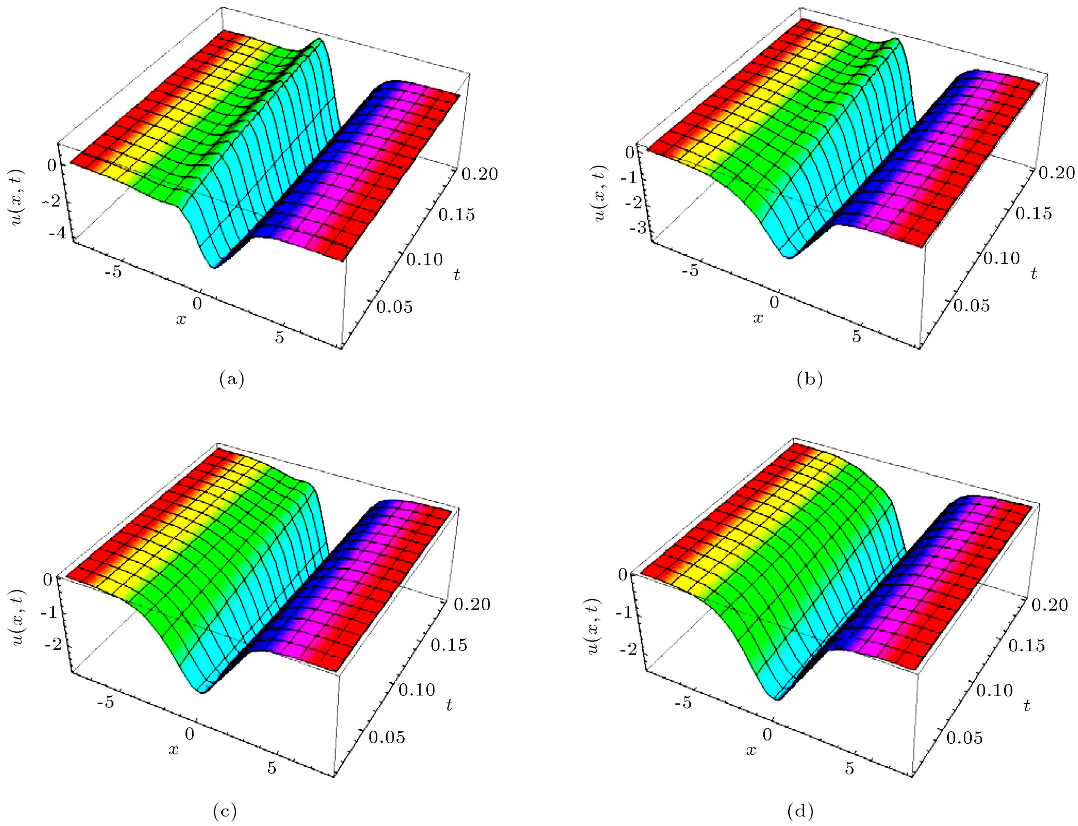


Figure 2. Plot of $u(x,t)$ with respect to x and t at (a) $\alpha = \frac{1}{4}$, (b) $\alpha = \frac{1}{2}$, (c) $\alpha = \frac{3}{4}$, and (d) $\alpha = 1$ for Eq. (39).

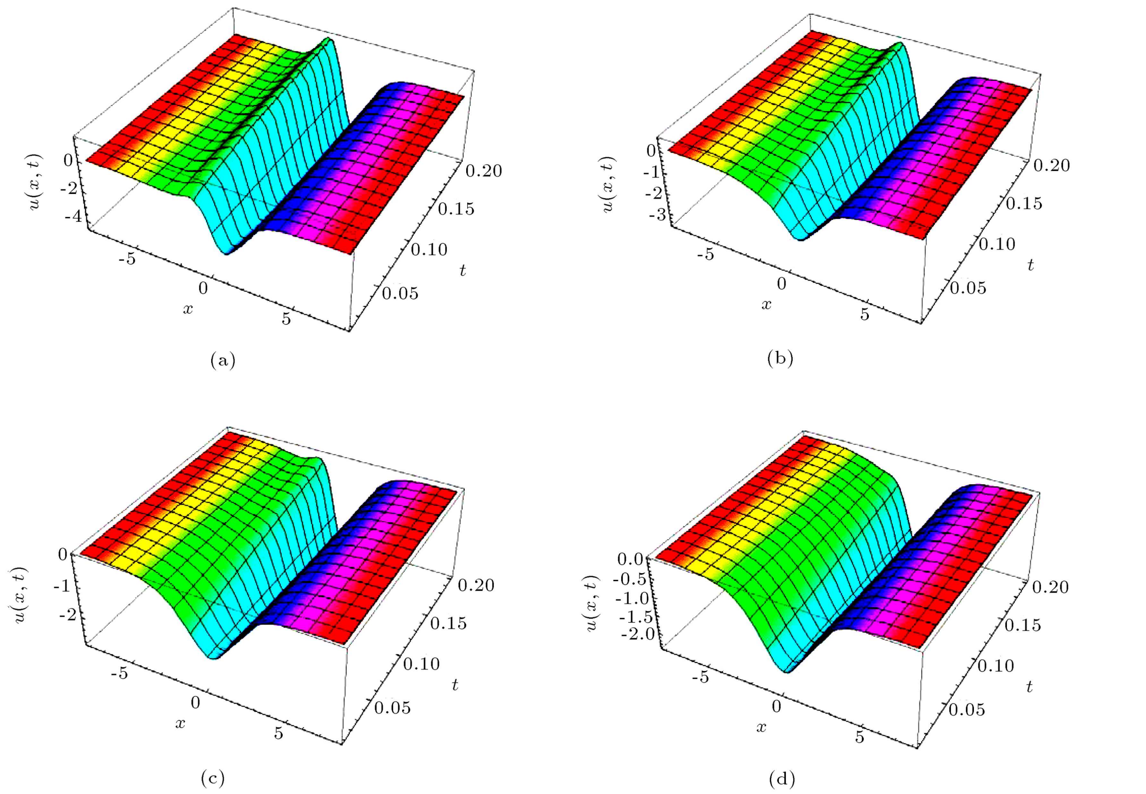


Figure 3. Plot of $u(x,t)$ with respect to x and t at (a) $\alpha=\frac{1}{4}$, (b) $\alpha=\frac{1}{2}$, (c) $\alpha=\frac{3}{4}$, and (d) $\alpha=1$ for Eq. (50).

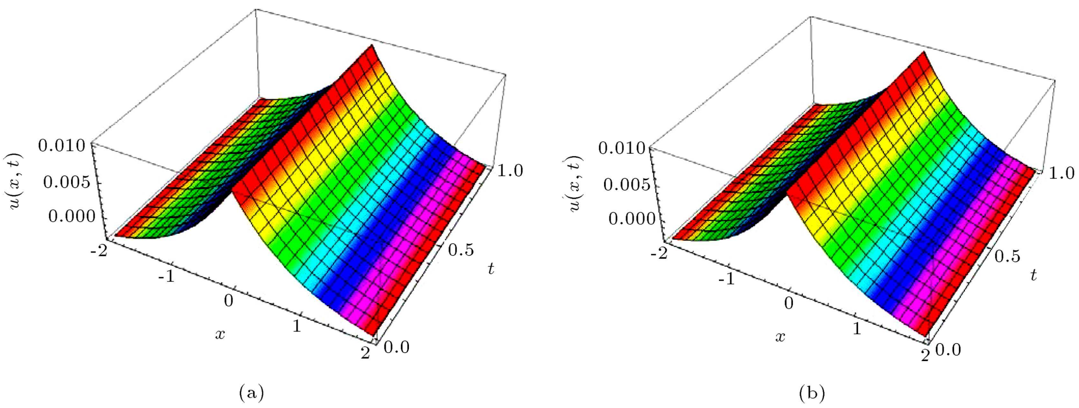
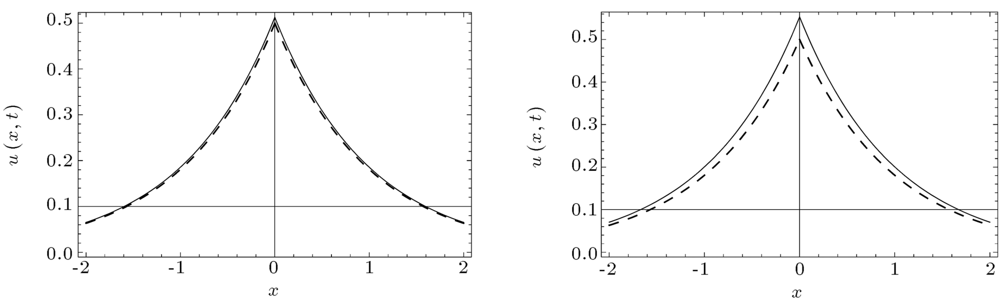
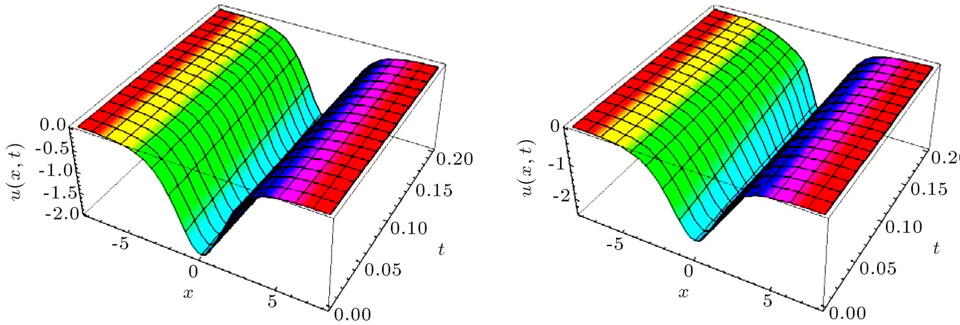


Figure 4. The surface of the solutions of CH equation: (a) The exact solution; and (b) the approximate solution obtained by HPM with $\kappa=0.005$ and $c=0.01$ for Eq. (28).



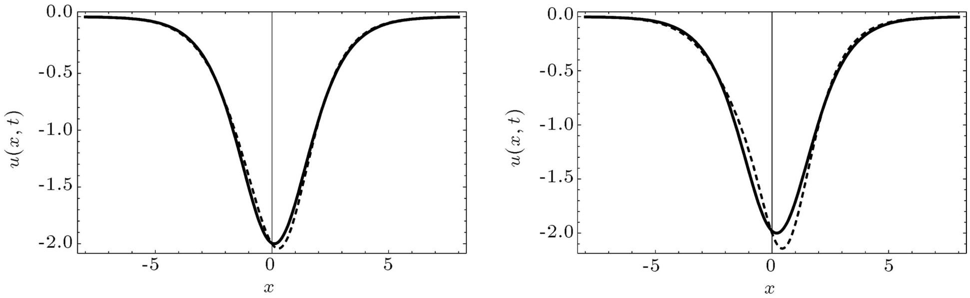
(a) $\kappa=0.005$, $c=0.5$, $t=0.05$, $\alpha=1$ (b) $\kappa=0.005, c=0.5, t=0.2, \alpha=1$

Figure 5. Two different profiles of u_{exact} (solid line) and u_{HPM} (dotted line) of CH equation when $-2\leq x\leq 2$ for Eq. (28).



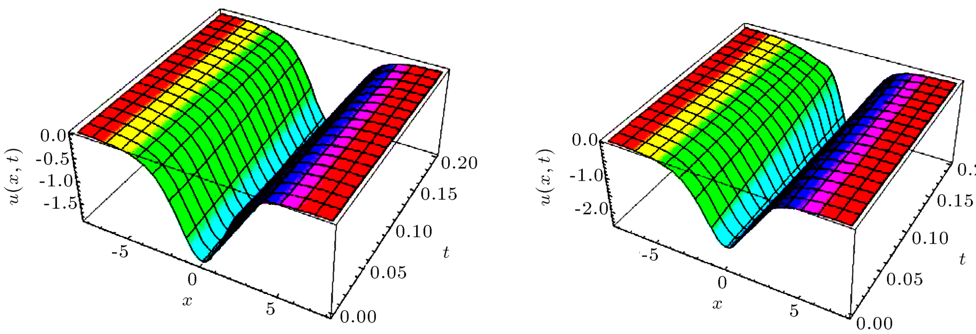
(a) (b)

Figure 6. The surface of the solutions of mCH equation: (a) The exact solution; and (b) the approximate solution obtained by HPM for Eq. (39).



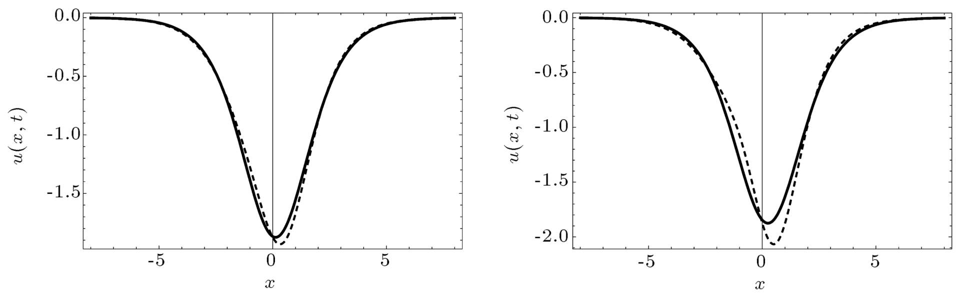
(a) $t = 0.05, \alpha = 1$ (b) $t = 0.1, \alpha = 1$

Figure 7. Two different profiles of u_{exact} (solid line) and u_{HPM} (dotted line) of mCH equation when $-8 \leq x \leq 8$ for Eq. (39).



(a) (b)

Figure 8. The surface of the solutions of mDP equation: (a) The exact solution; and (b) the approximate solution obtained by HPM for Eq. (50).



(a) $t = 0.05, \alpha = 1$ (b) $t = 0.1, \alpha = 1$

Figure 9. Two different profiles of u_{exact} (solid line) and u_{HPM} (dotted line) of mDP equation when $-8 \leq x \leq 8$ for Eq. (50).

the imbedding parameter is considered as a "small parameter", which can take the whole advantages of the traditional perturbation methods and homotopy techniques. It can also be applied in real problems, where differential equations governing the process are nonlinear and boundary conditions are complicated.

From the surface graphics of the two kinds of solutions, one can see that they are almost the same. On the other hand, comparing it with the exact solution, we find that HPM overcomes the simple solution procedure arising in calculation of exact solutions. At the same time, comparing these two methods, we find that the solution procedure is much simpler and the results are more accurate and close.

References

1. Podlubny, I., *Fractional Differential Equations.*, New York: Academic Press (1999).
2. Oldham AKB, Spanier J., *The Fractional Calculus*, New York: Academic Press (1974).
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier (2006).
4. Camassa, R. and Holm, D. "An integrable shallow water equation with peaked solitons", *Phys. Rev. Lett.*, **71**, pp. 1661-1664 (1993).
5. Johnson, R.S. "Camassa-Holm, Korteweg-de Vries and related models for water waves", *J. Fluid Mech.*, **457**, pp. 63-82 (2002).
6. Constantin, A. and Strauss, W. "Stability of the Camassa-Holm solitons", *J. Nonlinear Sci.*, **12**, pp. 415-422 (2002).
7. Johnson, R.S. "On solutions of the Camassa-Holm equation", *Proc. Roy. Soc. London A*, **459**, pp. 1687-1708 (2003).
8. Constantin, A. and Escher, J. "Global weak solutions for a shallow water equation", *Indiana Univ. Math. J.*, **47**, pp. 1527-1545 (1998).
9. Constantin, A. and Molinet, L. "Global weak solutions for a shallow water equation", *Comm. Math. Phys.*, **211**, pp. 45-61 (2000).
10. Constantin, A. and Strauss, W. "Stability of peakons", *Comm. Pure Appl. Math.*, **53**, pp. 603-610 (2000).
11. Lenells, J. "Stability of periodic peakons", *Int. Math. Res. Notices*, **10**, pp. 485-499 (2004).
12. Dai, H.H. "Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod", *Acta Mechanica*, **127**, pp. 193-207 (1998).
13. Fokas, A.S. and Fuchssteiner, B. "Symplectic structures, their Bäcklund transformation and hereditary symmetries", *Physica D*, **4**, pp. 47-66 (1981).
14. Constantin, A. "Existence of permanent and breaking waves for a shallow water equation: A geometric approach", *Ann. Inst. Fourier (Grenoble)*, **50**, pp. 321-362 (2000).
15. Misiolek, G.A. "Shallow water equation as a geodesic flow on the Bott-Virasoro group", *J. Geom. Phys.*, **24** pp. 203-208 (1998).
16. Constantin, A. and Escher, J. "Wave breaking for nonlinear nonlocal shallow water equations", *Acta Mathematica*, **181**, pp. 229-243 (1998).
17. Beals, R., Sattinger, D.H. and Szmigielski, J. "Multipeakons and a theorem of Stieltjes", *Inverse Problems*, **15**, pp. L1-L4 (1999).
18. He, J.H. "Homotopy perturbation technique", *Comput. Methods in Appld. Mech. and Engng.*, **178**, pp. 257-262 (1999).
19. He, J.H. "A coupling method of homotopy technique and perturbation technique for nonlinear problems", *Int. J. Nonlinear Mech.*, **35**, pp. 37-43 (2000).
20. He, J.H. "Periodic solutions and bifurcations of delay-differential equations", *Phys. Lett. A*, **347**, pp. 228-230 (2005).
21. He, J.H. "Application of homotopy perturbation method to nonlinear wave equations", *Chaos Solitons Fractals*, **26**, pp. 695-700 (2005).
22. He, J.H. "Limit cycle and bifurcation of nonlinear problems", *Chaos Solitons Fractals*, **26**, pp. 827-833 (2005).
23. Das, S. and Gupta, P.K. "An approximate analytical solution of the fractional diffusion equation with absorbent term and external force by Homotopy Perturbation Method", *Zeitsc für Natur*, **65a**(3), pp. 182-190 (2010).
24. Yildirim, A. and Koçak, H. "Homotopy perturbation method for solving the space-time fractional advection-dispersion equation", *Advances in Water Resources*, **32**(12), pp. 1711-1716 (2009).
25. He, J.H. "Homotopy perturbation method for solving boundary value problems", *Phys. Letter A*, **350**, pp. 87-88 (2006).
26. El-Shahed, M. "Application of He's homotopy perturbation method to Volterra's integro-differential equation", *Int. J. Nonlin. Sci. Numer. Simul.*, **6**, pp. 163-168 (2005).
27. Zhang, B.G., Li, S.Y. and Liu, Z.R. "Homotopy perturbation method for modified Camassa-Holm and Degasperis-Procesi equations", *Phys. Letter A*, **372**, pp. 1867-1872 (2008).
28. Das, S., Gupta, P.K. and Barat, S. "A note on fractional Schrodinger equation", *Nonlinear Science Letters A*, **1**(1), pp. 91-94 (2010).
29. Yildirim, A. "An algorithm for solving the fractional nonlinear Schrodinger equation by means of the homotopy perturbation method", *Int. J. Nonlin. Sci. Num. Sim.*, **10**, pp. 445-450 (2009).
30. Khan, N.A., Ara, A., Ali, S.A. and Mahmood, A. "Analytical study of Navier-stokes equation with fractional orders using He's homotopy perturbation and variational iteration methods", *Int. J. Nonlin. Sci. Num. Sim.*, **10**, pp. 1127-1134 (2009).

31. Koçak, H., Özis, T. and Yildirim, A. "Homotopy perturbation method for the nonlinear dispersive K(m,n,1) equations with fractional time derivatives", *International Journal of Numerical Methods for Heat & Fluid Flow*, **20**(2), pp. 174-185 (2010).
32. Das, S., Gupta, P.K. and Rajeev, A. "Fractional predator-prey model and its solution", *Int. J. Nonlin. Sci. Numer. Simul.*, **10**(4), pp. 873-876 (2009).
33. Golbabai, A. and Sayevand, K. "The homotopy perturbation method for multi-order time fractional differential equations", *Nonlinear Sci. Lett. A*, **1**, pp. 147-154 (2010).
34. Yildirim, A. "Analytical approach to fractional partial differential equations in fluid mechanics by means of the homotopy perturbation method", *Int. J. of Numer. Methods for Heat & Fluid Flow*, **20**(2), pp. 186-200 (2010).
35. Gupta, P.K., Singh, J. and Rai, K.N. "Numerical simulation for heat transfer in tissues during thermal therapy", *J. of Ther. Bio.*, **35**(6), pp. 295-301 (2010).
36. Gupta, P.K. "Approximate analytical solutions of fractional Benney-Lin equation by reduced differential transformation and homotopy perturbation method", *Comp. Math. with Appl.*, **61**(9), pp. 2829-2842 (2011).
37. Abbaoui, K. and Cherruault, Y. "New ideas for proving convergence of decomposition methods", *Comp. Math. with Appl.*, **29**, pp. 103-108 (1995).
38. Zhang, B.G., Liu, Z.R. and Mao, J.F. "Approximate explicit solution of Camassa-Holm equation by He's homotopy perturbation method", *J. Appl. Math. Comp.*, **31**, pp. 239-246 (2009).

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