Research Note

New memory-type control charts for monitoring process mean and dispersion

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Abstract. Control chart is widely used to monitor the quality of products of industrial or business processes. Max-CUSUM and Max-EWMA are based on memory-type control charts that monitor the process mean and standard deviation simultaneously. This article introduces four new control charts that monitor the process mean and dispersion simultaneously. The proposed control charting schemes are compared with the existing counterparts including Max-EWMA, Max-CUSUM, SS-EWMA, and SS-CUSUM. A case study is presented for practical considerations using a real dataset.

1. Introduction

All the manufacturing and non-manufacturing processes are bound to vary. These process variations consist of two types: natural and special cause variations. The natural variation can be background noise, which cannot be controlled, but the special cause variation is due to many artificial factors like faulty machines, operator mistakes, defective items, and so on. Statistical Process Control (SPC) is a collection of useful techniques that helps to differentiate between the two sources of variations. Control chart is a powerful tool of SPC to detect the special cause variation within a process.

Control chart is divided into two types: memoryless and memory-type charts. Shewhart-type control chart is a memoryless chart because it is based only

on the present information. Shewhart [1] introduced the fundamental control charts to monitor process behaviors in terms of location and dispersion parameters. $\bar{X}$-*chart* is extensively employed in industries to monitor process meanwhile these are extensively employed in industries to monitor process meanwhile R, S, and S^2 control charts are used for monitoring process dispersion/variability. A major limitation of Shewhart chart is that it is poor in detecting small and moderate shifts in a process parameter. Exponentially Weighted Moving Average (EWMA) and cumulative sum (CUSUM) are the major examples of memory-type control charts. They use both past and present information to detect small and moderate shifts in the process parameters. In EWMA chart, past observations are accounted for, but they are given a smaller weight as they become older; so, EWMA chart applies the utmost weight to the current observations and geometrically decreasing weight to all previous observations. CUSUM chart also gives memory using the information of past history. The basic structures of CUSUM and EWMA were presented by Page [2] and Robert [3], respectively. Some recent literatures that improve both EWMA and CUSUM include: Abbasi and Miller [4], Abbas et al. [5], Abbas et al. [6], Abbas

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et al. [7], Riaz et al. [8], Haq [9], Haq [10], Haq et al. [11], and Zaman et al. [12].

One popular approach to monitoring the process location and dispersion requires designing two different charts separately. This is time-consuming and inconvenient to use two separate charts for the process parameters. An alternative approach is in the form of Max chart developed by Chen and Cheng [13]. This chart can simultaneously monitor both process mean and standard deviation on a single chart, but it has deficiency of detecting small shifts in the process parameters. Xie [14] introduced the Max-EWMA, SS-EWMA, EWMA-Max, and EWMA-SC control charts to overcome this challenge. Following the inspiration of Xie [14] for process monitoring, Cheng and Thaga [15] and Thaga [16] developed Max-CUSUM and SS- CUSUM, respectively, which are also memory-type control charts. These charts use both present and previous information about the process; they are more effective in detecting small and moderate shifts than the max-chart.

Four new charts are proposed in this article to aid in detecting smaller shifts rather than the existing memory-type univariate control charts for monitoring both location and dispersion simultaneously. The charts are developed by replacing the statistic that monitors the process dispersion of the Max-EWMA, Max-CUSUM, SS-EWMA, and SS-CUSUM with the three parameters' logarithmic transformation to S^2, suggested by Castagliola [17].

The structure of the rest of this article is as follows: Section 2 describes different memory-type control charts that monitor both location and dispersion simultaneously. Section 3 provides the design structures of the proposed control charts of this study, and Section 4 offers comparisons of the proposals with the existing counterparts. Section 5 includes a case study for our study purposes; and finally Section 6 concludes the findings of our study.

2. Memory-type control charts for monitoring process mean and dispersion

In this section, we provide a brief description of the design structures of some useful charts considered in this study. We focus on the structures of simultaneous control charts. The commonly-used memory-type univariate control charts for simultaneous monitoring of the process mean and dispersion are: Max-EWMA, Max-CUSUM, SS-EWMA, and SS-CUSUM. The details of these control charts are given below one by one.

2.1. Max-EWMA control chart
Max-EWMA control chart is a memory-type chart introduced by Xie [14]. It is better at detecting small and moderate shifts than at detecting max-chart.

Let $X_i = X_{i1}, \cdots, X_{in}$, $i = 1, 2, 3, \cdots$, represent a sequence of samples of size n. The samples are independent and identical. They are normally distributed with μ_0 and σ_0^2 as location and variability parameters, respectively. Based on this sample information, we have:

$$\bar{X}_i = (X_{i1} + \cdots + X_{in})/n,$$

and:

$$S_i^2 = \frac{\sum_{j=1}^{n} (X_{ij} - \bar{X})^2}{n - 1},$$

denoting mean and variance, respectively, for the ith sample. These statistics are the minimum variance unbiased estimator of μ_0 and σ_0^2, respectively. Two transformed statistics from $\bar{X}_i$ and $\bar{S}_i^2$ are defined below:

$$Z_i = \sqrt{n} \frac{(\bar{X} - \mu_0)}{\sigma_0}, \tag{1}$$

$$Y_i = \Phi^{-1} \left\{ H \left[\frac{(n-1)S_i^2}{\sigma_0^2}; n - 1 \right] \right\}. \tag{2}$$

In Eq. (2), $\Phi^{-1}(.)$ represents inverse standard normal cumulative distribution function and $H(., n - 1)$ is a chi-square distribution function with $n - 1$ degrees of freedom. Functions Z_i and Y_i are independent statistics and follow the standard normal distribution. The two EWMA statistics derived from Z_i and Y_i are given below:

$$U_i = (1 - \lambda)U_{i-1} + \lambda Z_i, \tag{3}$$

$$V_i = (1 - \lambda)V_{i-1} + \lambda Y_i, \tag{4}$$

$$M_i = \max(|U_i|, |V_i|), \tag{5}$$

where $0 < \lambda \leq 1$, and the initial values for U_i and V_i are set to be U_0 and V_0, respectively. The highest absolute value (M_i) of both statistics U_i and V_i is calculated, and it is compared with the Upper Control Limit (UCL) given below in Eq. (6b):

$$\text{UCL} = E(M_i) + L\sqrt{\text{Var}(M_i)}. \tag{6a}$$

The expected value and variance of M_i are derived through the numerical computation and are given as $E(M_i) = 1.12379$ and $\text{Var}(M_i) = 0.363381$. Therefore:

$$\text{UCL} = \sqrt{\frac{\lambda}{(2 - \lambda)}} (1.128379 + L\,0.602811). \tag{6b}$$

The ARL results of Max-EWMA control chart, through simulation, for various shifts in the process mean (a) and dispersion (b) for different sample size (n) are provided in Table 1. Max-EWMA chart is very sensitive to small shifts in the process mean and dispersion when the sample size increases and sensitivity parameter λ decreases in value.

Table 1. ARL values for Max-EWMA control charting schemes at $ARL_0 = 250$.

						a				
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	3.22	3.22	3.21	3.22	3.17	2.98	2.56	2	2
	0.5	5.9	5.91	5.78	4.85	3.73	3.03	2.55	2.07	2
	0.75	18.53	15.61	8.49	5.24	3.81	3.05	2.54	2.16	2.02
$n = 5$	1	250.68	24.88	8.82	5.32	3.85	3.07	2.57	2.23	2.04
$L = 2.785$	1.25	17.7	13.2	7.88	5.27	3.86	3.1	2.6	2.27	2.06
$\lambda = 0.10$	1.5	7.35	6.94	5.87	4.66	3.76	3.08	2.62	2.3	2.06
	1.75	4.8	4.66	4.32	3.87	3.38	2.95	2.57	2.27	2.05
	2	3.61	3.59	3.45	3.23	2.96	2.69	2.44	2.21	2.02
	0.25	2.7	2.7	2.69	2.7	2.68	2.26	2	2	1.97
	0.75	25.29	20.23	8.46	4.64	3.2	2.5	2.11	1.93	1.74
$n = 5$	1	250.99	31.53	8.65	4.71	3.31	2.56	2.15	1.9	1.68
$L = 3.04$	1.25	17.9	12.84	7.12	4.55	3.29	2.59	2.17	1.88	1.67
$\lambda = 0.2$	1.5	6.59	6.08	5.01	3.94	3.1	2.56	2.16	1.87	1.68
	1.75	4.04	3.95	3.55	3.18	2.78	2.38	2.09	1.85	1.66
	2	3.02	2.97	2.83	2.63	2.42	2.15	1.97	1.78	1.61
	0.25	4.29	4.29	4.3	4.28	3.89	3.12	2.79	2.05	2
	0.5	11.34	11.21	10.11	6.48	4.36	3.29	2.68	2.21	2.02
	0.75	67.43	51.27	14.95	6.89	4.45	3.35	2.7	2.3	2.06
$\lambda = 0.2$	1	252.17	51.57	13.52	6.83	4.51	3.41	2.77	2.35	2.1
$n = 3$	1.25	29.95	19.79	10.49	6.35	4.45	3.47	2.8	2.39	2.12
$L = 3.04$	1.5	10.43	9.31	7.26	5.39	4.19	3.34	2.8	2.42	2.12
	1.75	6.14	5.89	5.24	4.39	3.69	3.17	2.71	2.39	2.11
	2	4.49	4.3	3.99	3.63	3.23	2.88	2.56	2.29	2.06
	0.25	3.2	3.21	3.2	3.21	3.06	2.83	2.06	2	2
	0.5	7.04	7.12	6.68	4.98	3.58	2.8	2.24	2.02	1.97
	0.75	37.6	29.93	10.73	5.45	3.7	2.83	2.32	2.05	1.9
$\lambda = 0.2$	1	246.68	39.13	10.46	5.51	3.77	2.88	2.38	2.06	1.86
$n = 4$	1.25	22.59	15.62	8.4	5.26	3.73	2.93	2.4	2.08	1.86
$L = 3.04$	1.5	7.91	7.23	5.8	4.51	3.51	2.86	2.42	2.09	1.85
	1.75	4.82	4.65	4.15	3.66	3.14	2.67	2.35	2.07	1.84
	2	3.52	3.44	3.27	3	2.72	2.44	2.19	1.99	1.79

2.2. Max-CUSUM chart

The design structure of Max-CUSUM control chart was developed by Cheng and Thaga [15]. The statistics of Eqs. (1) and (2) are integrated in CUSUM statistics to monitor the process mean and variance as they are given in Eqs. (7) to (10):

$$C_i^+ = \max[0, Z_i - k + C_{i-1}^+], \tag{7}$$

$$C_i^- = \max[0, -Z_i - k + C_{i-1}^-], \tag{8}$$

$$S_i^+ = \max[0, Y_i - k + S_{i-1}^+], \tag{9}$$

$$S_i^- = \max[0, -Y_i - k + S_{i-1}^-], \tag{10}$$

where $C_0 = S_0 = 0$ are the starting points. Reference value k is any real number greater than zero (i.e., $k > 0$). We set the value of k to be 0.5 throughout this article.

Since Z_i and Y_i are both normally distributed, we can combine the statistics, C_i^+, C_i^-, S_i^+, and S_i^-,

Table 2. ARL values for Max-CUSUM control charting schemes at $\text{ARL}_0 = 250$ and $k = 0.50$.

	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	4.59	4.61	4.61	4.57	4.23	3.52	3	2.5	2.01
	0.5	10.41	10.35	9.44	6.63	4.68	3.57	2.97	2.5	2.11
	0.75	57.61	41.96	13.44	7.03	4.78	3.64	2.97	2.53	2.21
$h = 5.07$	1	250.86	48.03	13.28	7.11	4.84	3.7	3.02	2.59	2.27
$n = 3$	1.25	29.89	20.63	10.84	6.77	4.79	3.71	3.07	2.61	2.3
	1.5	10.87	9.81	7.75	5.84	4.55	3.66	3.04	2.64	2.32
	1.75	6.48	6.23	5.65	4.8	4.05	3.44	2.98	2.61	2.32
	2	4.75	4.66	4.35	3.99	3.55	3.16	2.81	2.51	2.29
	0.25	3.48	3.49	3.48	3.48	3.4	2.98	2.55	2	2
	0.5	7.08	7.06	6.76	5.27	3.89	3.06	2.53	2.09	2
	0.75	31.25	24.8	10.18	5.71	3.98	3.1	2.56	2.19	2.01
$h = 5.05$	1	251.87	36.65	10.32	5.74	4.05	3.13	2.58	2.25	2.02
$n = 4$	1.25	22.2	15.82	8.84	5.61	4.06	3.17	2.63	2.27	2.03
	1.5	8.3	7.65	6.19	4.82	3.83	3.13	2.64	2.29	2.04
	1.75	5.12	4.99	4.54	4	3.43	2.96	2.57	2.27	2.04
	2	3.8	3.72	3.56	3.33	3	2.7	2.42	2.19	1.98
	0.25	2.96	2.96	2.95	2.96	2.91	2.79	2.03	2	2
	0.5	5.63	5.63	5.5	4.51	3.42	2.75	2.17	2.01	1.99
	0.75	21.45	17.95	8.46	4.96	3.5	2.74	2.26	2.04	1.93
$h = 5.05$	1	249.09	30.01	8.77	5.02	3.54	2.78	2.32	2.06	1.88
$n = 5$	1.25	18.1	13.25	7.55	4.91	3.57	2.83	2.37	2.08	1.86
	1.5	6.9	6.43	5.37	4.29	3.42	2.79	2.39	2.09	1.84
	1.75	4.34	4.25	3.93	3.5	3.06	2.63	2.31	2.04	1.83
	2	3.26	3.24	3.08	2.89	2.64	2.42	2.18	1.96	1.8

and develop a new statistic that determines the highest value of the four above-mentioned statistics. This new statistic is represented by N_i defined as:

$$N_i = \max(C_i^+, C_i^-, S_i^+, S_i^-). \qquad (11)$$

Statistic N_i is always positive and it has only the upper control limit, say h. Whenever N_i exceeds control limit h, then we say that the process is in an out-of-control state; otherwise, it is in a good state. The ARL results of Max-CUSUM control charting schemes through simulation approach are provided in Table 2. Max-CUSUM chart is sensitive to small shifts in the process mean and dispersion when the sample size increases.

2.3. SS-EWMA chart

SS-EWMA was introduced by Xie [14]. Based on the statistics defined in Eqs. (3) and (4), SS-EWMA statistic is constructed and given below in the form of Eq. (12):

$$SS1_i = U_i^2 + V_i^2 \qquad \text{for} \quad i = 1, 2. \qquad (12)$$

$SS1_i$ follows a chi-square distribution when it is divided by $\sigma_{U_i}^2$, since $\frac{U_i}{\sigma_{U_i}}$ and $\frac{V_i}{\sigma_{V_i}}$ are independent and identical standard normal distribution. Based on the information of Eq. (13), the expected value and the variance of $SS1_i$ are given by:

$$\frac{SS1_i}{\sigma_{U_i}^2} = \frac{U_i^2}{\sigma_{U_i}^2} + \frac{V_i^2}{\sigma_{U_i}^2} \sim \chi_2^2, \qquad (13)$$

$$E(SS1_i) = 2\sigma_{U_i}^2, \qquad (14)$$

$$\text{Var}(SS1_i) = 4\sigma_{U_i}^4. \qquad (15)$$

Since $SS1_i$ is always positive, it has only the UCL, given by:

$$\text{UCL} = E(SS1_i) + L\sqrt{\text{Var}(SS1_i)}. \qquad (16)$$

Since $\sigma_{U_i}^2 = \frac{\lambda}{2-\lambda}$ for the steady case and by the

Table 3. ARL values for SS-EWMA control charting schemes at $ARL_0 = 250$.

						a				
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	3.47	3.38	3.17	2.97	2.78	2.28	2.01	2	2
	0.5	6.41	5.9	4.88	4.01	3.31	2.89	2.46	2.07	2
	0.75	19.98	13.24	7.51	5.08	3.84	3.13	2.66	2.24	2.04
$\lambda = 0.1$	1	252.32	25.67	9.14	5.56	4.06	3.23	2.7	2.34	2.1
$n = 5$	1.25	17.26	12.28	7.41	5.09	3.88	3.17	2.66	2.34	2.11
$L = 3.6$	1.5	7.24	6.62	5.37	4.33	3.52	2.95	2.58	2.29	2.09
	1.75	4.75	4.54	4.08	3.58	3.13	2.73	2.43	2.2	2.02
	2	3.58	3.5	3.3	3.06	2.76	2.5	2.28	2.09	1.93
	0.25	4.75	4.53	4.04	3.49	3.03	2.71	2.21	2	2
	0.5	13.18	10.93	7.32	5.15	3.86	3.13	2.63	2.22	2.03
	0.75	83.33	38.27	12.93	6.66	4.49	3.43	2.8	2.38	2.12
$\lambda = 0.2$	1	247.78	51.3	13.92	7.03	4.67	3.53	2.89	2.45	2.17
$n = 3$	1.25	28.08	18.32	10.09	6.16	4.41	3.47	2.83	2.44	2.16
$L = 4.067$	1.5	9.87	8.87	6.75	5.08	3.98	3.23	2.76	2.4	2.13
	1.75	5.84	5.55	4.88	4.14	3.48	2.98	2.57	2.29	2.06
	2	4.18	4.1	3.83	3.43	3.06	2.7	2.43	2.16	1.98
	0.25	3.46	3.36	3.1	2.82	2.45	2.06	2	2	1.97
	0.5	7.96	6.94	5.26	3.96	3.16	2.62	2.18	2.01	1.98
	0.75	43.93	22.49	9.1	5.22	3.69	2.89	2.39	2.1	1.96
$\lambda = 0.2$	1	246.36	39.45	10.8	5.69	3.9	3	2.48	2.15	1.94
$n = 4$	1.25	21.42	14.19	7.87	5.1	3.72	2.94	2.46	2.14	1.9
$L = 4.067$	1.5	7.58	6.83	5.36	4.14	3.33	2.76	2.35	2.06	1.85
	1.75	4.62	4.39	3.93	3.36	2.91	2.51	2.22	1.97	1.78
	2	3.39	3.27	3.1	2.82	2.55	2.28	2.07	1.88	1.71

substitution of Eqs. (14) and (15) into Eq. (16), the UCL approaches asymptotically to:

$$UCL = 2\sigma_{U_i}^2(1 + L), \qquad (17)$$

$$UCL = \frac{2\lambda}{2 - \lambda}(1 + L). \qquad (18)$$

The ARL results of SS-EWMA control charting schemes are provided in Table 3. SS-EWMA chart is very sensitive to small shifts at both process mean and dispersion when the sample size increases and also when the value of λ decreases.

2.4. SS-CUSUM chart
Thaga [16] introduced the structure of SS-CUSUM chart, and it is based on the statistics given in Eqs. (7)-(10). Let M_i and V_i be the maximum values of CUSUM statistics that monitor process mean and standard deviation, respectively:

$$M_i = \max(C_i^+, C_i^-), \qquad (19)$$

$$V_i = \max(S_i^+, S_i^-). \qquad (20)$$

SS-CUSUM is the sum of square of the maximum values of CUSUM statistics that monitor both process mean and standard deviation, and it is given by:

$$SS2_i = M_i^2 + V_i^2. \qquad (21)$$

$SS2_i$ is plotted against the control limit, say h. The ARL results of SS-CUSUM control charting schemes for different sample sizes are provided in Table 4. SS-CUSUM chart is highly sensitive to small shifts at both process mean and dispersion when the sample size increases.

3. The proposed control charts

In this section, we propose four new memory-type control charts that monitor both process mean and

Table 4. ARL values for SS-CUSUM control charting schemes at $ARL_0 = 250$ for $k = 0.50$.

						a				
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	4.77	4.77	4.57	4.05	3.49	3.01	2.77	2.19	2
	0.5	10.82	10.61	8.44	5.94	4.37	3.48	2.94	2.52	2.13
	0.75	61.21	40.17	12.99	6.96	4.81	3.7	3.03	2.59	2.26
$n = 3$	1	248.62	48.25	13.25	7.15	4.9	3.77	3.1	2.63	2.32
$h = 28.03$	1.25	27.85	19.22	10.7	6.58	4.74	3.74	3.06	2.64	2.34
	1.5	10.19	9.24	7.25	5.55	4.34	3.52	3.01	2.61	2.32
	1.75	6.14	5.87	5.24	4.51	3.82	3.27	2.83	2.52	2.26
	2	4.44	4.38	4.08	3.72	3.33	2.96	2.66	2.38	2.16
	0.25	3.61	3.61	3.48	3.16	2.88	2.5	2.04	2	2
	0.5	7.4	7.21	6.07	4.59	3.59	2.96	2.46	2.08	2
	0.75	32.31	23.68	9.67	5.62	4	3.14	2.6	2.23	2.03
$n = 4$	1	248.93	36.44	10.52	5.88	4.13	3.2	2.65	2.29	2.07
$h = 27.9$	1.25	21.47	14.81	8.45	5.49	4.02	3.18	2.65	2.3	2.07
	1.5	7.87	7.25	5.83	4.55	3.65	3.02	2.57	2.25	2.02
	1.75	4.88	4.7	4.27	3.68	3.2	2.75	2.42	2.17	1.96
	2	3.63	3.55	3.34	3.07	2.79	2.52	2.27	2.07	1.88
	0.25	3.06	3.05	2.96	2.77	2.39	2.03	2	2	1.98
	0.5	5.83	5.75	4.94	3.86	3.11	2.6	2.12	2	1.99
	0.75	22.22	16.89	7.95	4.84	3.5	2.79	2.31	2.05	1.95
$n = 5$	1	250.48	29.45	8.81	5.07	3.62	2.86	2.37	2.11	1.92
$h = 27.9$	1.25	17.32	12.52	7.17	4.74	3.54	2.82	2.38	2.09	1.89
	1.5	6.58	6.05	5	3.99	3.2	2.68	2.32	2.04	1.84
	1.75	4.18	4.05	3.67	3.21	2.82	2.47	2.18	1.97	1.77
	2	3.13	3.08	2.92	2.7	2.46	2.23	2.02	1.84	1.69

variance simultaneously. The proposed charts are based on the usual mean standardization of $\bar{X}$ for the location and a three-parameter logarithmic transformation to S^2 (cf., [17]) for the dispersion statistic. This section is divided into four parts for the four proposals discussed below.

It is to be mentioned that we have used the Monte Carlo simulation approach to evaluate ARL measures provided for this study. We have performed these simulations by developing a code in R language and executing it a reasonable number of times, say 10^4 times (for a relevant discussion on the number of simulations needed in control charting studies, one may see [18]).

3.1. Sum of square exponentially weighted moving average variance control chart

Sum of Square Exponentially Weighted Moving Average Variance (SS-EWMAVAR) is a new control chart that monitors both process mean and variance simultaneously. It is constructed by replacing the statistic that monitors the process dispersion in SS-EWMA control chart with the three-parameter logarithmic transformation to S^2 [17].

Suppose that $X_{ij} \sim N(\mu, \sigma^2)$ for $i = 1, 2, 3, \cdots$

and $j = 1, 2, 3, \cdots, n$, where n is the sample size. Let $\bar{X}_i = (X_{i1} + \cdots + X_{in})/n$ and $S_i^2 = \frac{\sum_{j=1}^{n}(X_{ij} - \bar{X})^2}{n-1}$ be the sample mean and variance of the distribution, respectively. These two statistics are the minimum variance unbiased estimators of μ_0 and σ_0^2, respectively. $\bar{X}_i$ is standardized into normal distribution and given by:

$$G_i = \sqrt{n}\frac{(\bar{X}_i - \mu_0)}{\sigma_0} \sim N(0,1).$$

Castagliola [17] used the three-parameter logarithmic transformation in order to monitor the process variance, which is given below by:

$$T_i = a + b\ln(S_i^2 + c), \tag{22}$$

where a, b, and, c are constants greater than zero and are defined below by:

$$b = B(n), \qquad c = C(n)\sigma_0^2,$$

$$a = A(n) - 2B(n)\ln(\sigma_0),$$

where $A(n)$, $B(n)$, and $C(n)$ are functions that depend on the value of sample size (n). Table 5 reproduces

Table 5. Values of $A(n)$, $B(n)$, $C(n)$, $\mu_T(n)$ and $\sigma_T(n)$ counterparts.

n	$A(n)$	$B(n)$	$C(n)$	$\mu_T(n)$	$\sigma_T(n)$
3	−0.6627	1.8136	0.6777	0.02472	0.9165
4	−0.7882	2.1089	0.6261	0.01266	0.9502
5	−0.8969	2.3647	0.5979	0.00748	0.967
6	−0.994	2.5941	0.5801	0.00485	0.9765
7	−1.0827	2.8042	0.5678	0.00335	0.9825
8	−1.1647	2.9992	0.5588	0.00243	0.9864
9	−1.2413	3.182	0.5519	0.00182	0.9892
10	−1.3135	3.3548	0.5465	0.00141	0.9912
11	−1.382	3.5189	0.5421	0.00112	0.9927
12	−1.4473	3.6757	0.5384	0.0009	0.9938
13	−1.5097	3.826	0.5354	0.00074	0.9947
14	−1.5697	3.9705	0.5327	0.00062	0.9955
15	−1.6275	4.11	0.5305	0.00052	0.996

the values of $A(n)$, $B(n)$, $C(n)$, $\mu_T(n)$, and $\sigma_T(n)$ for sample sizes 3 to 15.

In order to monitor the process variance, we will use distribution T_i and standardize it as normal so as to have the same distribution with G_i, $F_i = \frac{(T_i - \mu_T(n))}{\sigma_T(n)} \sim N(0,1)$.

Now, G_i and F_i are transformed into EWMA statistics as follows:

$$U_i = (1-\lambda)U_{i-1} + \lambda G_i, \tag{23}$$

$$V_i = (1-\lambda)V_{i-1} + \lambda F_i. \tag{24}$$

The initial values of U_i and V_i are set to be U_0 and V_0, respectively, and $0 < \lambda \leq 1$. Based on the statistics in Eqs. (23) and (24), SS-EWMA statistic is constructed in Eq. (25):

$$\text{SSEW}_i = U_i^2 + V_i^2 \qquad \text{for} \quad i = 1,2,3,\cdots \tag{25}$$

It has been proved in Section 2.3 that SSEW_i follows a chi-square distribution; their expected value and the variance are given by $E(\text{SSEW}_i) = 2\sigma_{U_i}^2$ and $\text{Var}(\text{SSEW}_i) = 4\sigma_{U_i}^4$, respectively.

Since SSEW_i is always positive, it has only the Upper Control Limit (UCL). Now, UCL is given by:

$$\text{UCL} = E(\text{SSEW}_i) + L\sqrt{\text{Var}(\text{SSEW}_i)}. \tag{26}$$

By the substitution of $E(\text{SSEW}_i)$ and $V(\text{SSEW}_i)$ in Eq. (26), we can simplify UCL as: $\text{UCL} = 2\sigma_{U_i}^2(1+L)$. Since $\sigma_{U_i}^2 = \frac{\lambda}{2-\lambda}$ for the steady case, UCL approaches asymptotically to:

$$\text{UCL} = \frac{2\lambda}{2-\lambda}(1+L). \tag{27}$$

ARL results of SS-EWMAVAR control chart, through simulation, for various shifts in the process mean (a) and dispersion (b) are provided in Table 6. We observed that SS-EWMAVAR chart is very sensitive to small shifts in the process dispersion when the value of λ increases.

3.2. Maximum exponentially weighted moving average variance control chart

Maximum Exponentially Weighted Moving Average Variance (Max-EWMAVAR) is a new control chart for monitoring both process mean and variance simultaneously. It is constructed by replacing the statistic that monitors the process dispersion in Max-EWMA control chart with the three-parameter logarithmic transformation to S^2.

Under the same assumption in Section 3.1, the statistics of Eqs. (23) and (24) can be combined in another form as:

$$\text{MEW}_i = \max(|U_i|, |V_i|).$$

MEW_i is plotted against the Upper Control Limit (UCL) derived below:

$$\text{UCL} = E(\text{MEW}_i) + L\sqrt{\text{Var}(\text{MEW}_i)}.$$

Therefore, $\text{UCL} = \sigma_{U_i}(1.128379 + L\,0.602811)$:

$$\text{UCL} = \sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{(2-\lambda)}}(1.128379 + L\,0.602811). \tag{28}$$

UCL will approach steady value when i gets larger. Consequently:

$$\text{UCL} = \sqrt{\frac{\lambda}{(2-\lambda)}}(1.128379 + L\,0.602811). \tag{29}$$

ARL results of Max-EWMAVAR control chart, through simulation, for various shifts in process mean (a) and dispersion (b) are provided in Table 7 for varying λ. In addition, Table 8 provides ARL values at $\lambda = 0.2$ for some selective n. The result of its sensitive time-varying control limit is given in Table 9 for various sample size (n). We observed that MAX-EWMAVAR chart is very sensitive to small shifts for the process dispersion when the value of λ decreases.

3.3. Sum of square of cumulative sum variance control chart

Sum of square cumulative sum variance (SS-CUSUMVAR) is a new control chart for monitoring both process mean and variance simultaneously. SS-CUSUMVAR statistic is constructed by replacing the statistic that monitors the process dispersion in SS-CUSUM control chart with the three-parameter logarithmic transformation to S^2.

Table 6. ARL values for SS-EWMAVAR control charting schemes at $\text{ARL}_0 = 250$.

		a								
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	4.73	4.36	4.01	3.4	3	2.93	2.07	2	2
	0.5	7.05	6.42	5.18	4.14	3.37	2.95	2.49	2.07	2
	0.75	20.67	13.29	7.56	5.06	3.82	3.12	2.65	2.24	2.04
$L = 3.55$	1	250.57	25.29	9.05	5.53	4.04	3.22	2.7	2.32	2.1
$\lambda = 0.10$	1.25	16.37	11.87	7.35	5.07	3.88	3.13	2.65	2.31	2.1
	1.5	7.12	6.47	5.28	4.23	3.49	2.94	2.56	2.26	2.07
	1.75	4.68	4.48	4.05	3.54	3.08	2.71	2.44	2.2	2
	2	3.58	3.5	3.3	3.05	2.75	2.5	2.28	2.09	1.94
	0.25	4.03	3.91	3.18	2.99	2.37	2	2	2	1.94
	0.5	6.79	5.97	4.54	3.47	2.83	2.27	2.02	1.99	1.87
	0.75	35.12	16.48	7.28	4.42	3.19	2.54	2.14	1.98	1.82
$L = 3.99$	1	250.41	30.99	8.74	4.83	3.38	2.64	2.21	1.97	1.77
$\lambda = 0.20$	1.25	15.75	11.16	6.5	4.31	3.23	2.58	2.17	1.92	1.71
	1.5	6.1	5.53	4.43	3.52	2.87	2.42	2.1	1.84	1.66
	1.75	3.84	3.67	3.34	2.9	2.52	2.21	1.97	1.77	1.58
	2	2.89	2.85	2.69	2.48	2.22	2.02	1.82	1.67	1.54
	0.25	3.87	3.52	3.04	2.67	2.02	2	2	1.7	1.05
	0.5	8.06	6.6	4.5	3.25	2.52	2.08	1.96	1.73	1.33
	0.75	83.44	26.21	8.11	4.28	2.93	2.3	1.97	1.73	1.45
$L = 4.15$	1	250.16	38.91	9.41	4.68	3.13	2.39	1.99	1.71	1.47
$\lambda = 0.30$	1.25	15.88	11.2	6.33	4.02	2.94	2.31	1.93	1.65	1.45
	1.5	5.72	5.13	4.08	3.21	2.58	2.16	1.84	1.61	1.43
	1.75	3.47	3.29	2.99	2.57	2.23	1.95	1.72	1.55	1.38
	2	2.54	2.49	2.36	2.18	1.95	1.76	1.59	1.46	1.35
	0.25	4.99	4.19	3.13	2.27	2	1.92	1.25	1	1
	0.5	66.4	20.67	6.43	3.4	2.35	1.93	1.56	1.18	1.03
	1	248.39	55.81	12.99	4.98	2.94	2.19	1.69	1.38	1.17
$L = 4.24$	1.25	17.54	12.04	6.44	3.95	2.74	2.04	1.67	1.41	1.23
$\lambda = 0.50$	1.5	5.34	5	3.86	2.98	2.26	1.92	1.59	1.36	1.22
	1.75	3.04	2.94	2.55	2.28	1.97	1.7	1.5	1.34	1.22
	2	2.22	2.22	2.03	1.86	1.64	1.56	1.4	1.27	1.19

Following the inspiration of Section 3.1, statistics G_i and F_i are defined in the same way. The two statistics are integrated into CUSUM statistics to monitor the process mean and variance as given in Eqs. (30)-(33):

$$\text{SSC}_i^+ = \max[0, G_i - k + \text{SSC}_{i-1}^+], \qquad (30)$$

$$\text{SSC}_i^- = \max[0, -G_i - k + \text{SSC}_{i-1}^-], \qquad (31)$$

$$\text{SSS}_i^+ = \max[0, F_i - k + \text{SSS}_{i-1}^+], \qquad (32)$$

$$\text{SSS}_i^- = \max[0, -F_i - k + \text{SSS}_{i-1}^-], \qquad (33)$$

where $\text{SSC}_0 = \text{SSS}_0 = 0$ are the starting points.

Let MC_i and VS_i be the maximum values of CUSUM statistics that monitor process mean and variance, respectively:

$$MC_i = \max(\text{SSC}_i^+, \text{SSC}_i^-), \qquad (34)$$

$$VS_i = \max(\text{SSS}_i^+, \text{SSS}_i^-). \qquad (35)$$

SS-CUSUMVAR is the sum of square of the maximum values of CUSUM statistics that monitor both process mean and variance and it is given by variance, respectively:

$$\text{SS3}_i = MC_i^2 + VS_i^2. \qquad (36)$$

Table 7. ARL values for Max-EWMAVAR control charting scheme when $n = 5$ at $ARL_0 = 250$.

	b	a								
		0	**0.25**	**0.5**	**0.75**	**1**	**1.25**	**1.5**	**1.75**	**2**
	0.25	4.13	4.13	4.13	4.13	3.82	3	2.55	2	2
	0.5	6.47	6.51	6.32	5.03	3.75	3.02	2.52	2.06	2
	0.75	18.91	16.07	8.53	5.22	3.78	3.03	2.53	2.16	2.02
$L = 2.77$	1	250.29	24.54	8.73	5.26	3.84	3.07	2.55	2.24	2.03
$\lambda = 0.1$	1.25	14.06	10.99	7.08	4.98	3.78	3.06	2.58	2.25	2.05
	1.5	5.65	5.39	4.76	4.04	3.42	2.93	2.53	2.26	2.03
	1.75	3.69	3.62	3.42	3.22	2.94	2.63	2.38	2.17	2
	2	2.82	2.79	2.75	2.63	2.49	2.34	2.19	2.04	1.89
	0.25	3.659	3.66	3.67	3.65	3.03	2.33	2	2	1.95
	0.5	6.051	6.09	5.8	4.36	3.14	2.41	2.04	1.97	1.8
	0.75	29.09	22.5	8.55	4.6	3.17	2.47	2.09	1.92	1.71
$L = 2.99$	1	251.18	30.54	8.43	4.62	3.23	2.53	2.13	1.89	1.67
$\lambda = 0.2$	1.25	15.51	11.4	6.59	4.35	3.22	2.56	2.14	1.86	1.65
	1.5	5.53	5.14	4.39	3.62	2.93	2.45	2.11	1.85	1.65
	1.75	3.46	3.4	3.15	2.83	2.54	2.25	1.99	1.8	1.61
	2	2.59	2.57	2.49	2.34	2.18	2	1.84	1.68	1.56
	0.25	3.251	3.26	3.25	3.24	2.89	2.03	2	1.85	1.1
	0.5	6.649	6.72	6.27	4.25	2.87	2.17	1.95	1.68	1.27
	0.75	59.94	41.7	10.1	4.51	2.92	2.25	1.91	1.64	1.34
$L = 3.06$	1	251.4	38.3	9	4.46	2.98	2.29	1.89	1.63	1.38
$\lambda = 0.3$	1.25	16.22	11.9	6.51	4.12	2.96	2.31	1.9	1.61	1.41
	1.5	5.429	5	4.17	3.36	2.65	2.2	1.87	1.62	1.43
	1.75	3.276	3.19	2.94	2.58	2.29	2.01	1.76	1.58	1.42
	2	2.388	2.38	2.28	2.13	1.96	1.79	1.64	1.5	1.38
	1	250.5	89.6	21.2	7.01	3.34	2.06	1.49	1.22	1.08
$L = 3.12$	1.25	21.88	16.7	9.11	4.83	3.01	2.04	1.54	1.28	1.13
$\lambda = 0.8$	1.5	6.205	5.62	4.39	3.31	2.41	1.89	1.54	1.31	1.18
	1.75	3.175	3.08	2.74	2.34	1.98	1.68	1.47	1.3	1.19
	2	2.131	2.11	2	1.83	1.67	1.5	1.37	1.25	1.17

The control limit (h) of $SS3_i$ is derived through simulation. Statistic $SS3_i$ is plotted against h. For out-of-control signals, we compare the statistic with the corresponding control limit and receive the signals if points fall outside the limit, i.e. $SS3_i > h$.

3.4. Maximum cumulative sum variance control chart

Maximum cumulative sum variance (Max-CUSUM-VAR) is a new control chart that monitors both process mean and variance simultaneously. It is based on the assumption in Section 3.3. The statistics of Eqs. (34) and (35) can be combined in another form as:

$$MCVAR_i = \max(MC_i, VS_i). \tag{37}$$

Since statistic $MCVAR_i$ is always positive, it has

the only upper control limit h. Whenever $MCVAR_i$ exceeds control limit h, then we say that the process is in an out-of-control state; otherwise, it is statistically in-control.

4. Comparative analysis

In this section, we will compare the performance of the proposed charts with their counterparts including max-chart, Max-CUSUM, Max-EWMA, SS-CUSUM, and SS-EWMA control charts. For different amounts of shifts in process mean (a) and dispersion (b), ARL values of the proposed charts and the other competing charts are provided in Tables 1-11. These results are based on 10^4 Monte Carlo simulations, at each run, for our study purposes. We have created Figures 1-6

Table 8. ARL values for Max-EWMAVAR control charting scheme for $\lambda = 0.2$ at $ARL_0 = 250$.

						a				
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	6.37	6.37	6.38	6.11	4.28	3.15	2.77	2.04	2
	0.5	15.04	15.14	12.63	6.91	4.36	3.27	2.66	2.19	2.02
	0.75	194.05	88.97	15.85	6.89	4.44	3.33	2.69	2.29	2.06
$n = 3$	1	256.08	50.26	13.43	6.73	4.48	3.38	2.77	2.35	2.08
$L = 3.02$	1.25	24.56	17.6	10.13	6.17	4.39	3.42	2.78	2.39	2.1
	1.5	9.26	8.51	6.79	5.21	4.09	3.29	2.78	2.4	2.12
	1.75	5.63	5.45	4.86	4.23	3.62	3.09	2.66	2.35	2.09
	2	4.14	4.09	3.85	3.51	3.17	2.81	2.52	2.26	2.04
	0.25	5.41	5.4	5.4	5.32	4.18	3.1	2.69	2.03	2
	0.5	10.63	10.66	9.77	6.44	4.27	3.2	2.62	2.17	2.01
	0.75	64.24	47.25	14.34	6.63	4.32	3.25	2.65	2.26	2.04
$n = 4$	1	249.27	48.04	13	6.56	4.39	3.31	2.71	2.32	2.06
$L = 3.68$	1.25	27.94	18.67	10	6.2	4.34	3.35	2.74	2.35	2.07
	1.5	9.97	8.95	6.93	5.17	4.06	3.27	2.74	2.35	2.07
	1.75	5.9	5.66	5.02	4.25	3.61	3.05	2.65	2.32	2.06
	2	4.29	4.2	3.89	3.55	3.18	2.82	2.52	2.26	2.04

Table 9. ARL values for time-varying Max-EWMAVAR control charting scheme for $\lambda = 0.2$ at $ARL_0 = 250$.

						a				
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	5.82	5.83	5.83	5.52	3.54	2.37	1.93	1.38	1.02
	0.5	14.87	14.98	12.33	6.31	3.6	2.44	1.85	1.45	1.16
	0.75	201.33	90.77	15.29	6.13	3.62	2.49	1.87	1.5	1.25
$n = 3$	1	258.08	49.15	12.41	5.81	3.59	2.52	1.95	1.57	1.33
$L = 3.034$	1.25	22.46	15.69	8.68	5.04	3.4	2.5	1.93	1.6	1.4
	1.5	7.48	6.8	5.32	3.93	3.01	2.33	1.92	1.63	1.42
	1.75	4.15	4.01	3.57	3.05	2.52	2.11	1.81	1.6	1.43
	2	2.87	2.83	2.66	2.4	2.15	1.9	1.7	1.53	1.4
	0.25	4.79	4.79	4.8	4.7	3.42	2.3	1.89	1.31	1.01
	0.5	10.31	10.33	9.39	5.82	3.5	2.37	1.8	1.41	1.14
	0.75	65.34	47.81	13.73	5.88	3.49	2.41	1.84	1.48	1.23
$n = 4$	1	252.78	46.86	11.95	5.62	3.5	2.46	1.88	1.53	1.31
$L = 3.7$	1.25	25.5	16.62	8.53	4.99	3.34	2.44	1.91	1.58	1.38
	1.5	8.11	7.2	5.42	3.93	2.98	2.33	1.9	1.6	1.4
	1.75	4.36	4.14	3.68	3.08	2.54	2.1	1.81	1.58	1.42
	2	3	2.88	2.69	2.42	2.17	1.9	1.71	1.55	1.41
	0.25	2.9221	2.9227	2.9166	2.9056	2.1741	1.7056	1.041	1	1
	0.5	5.3571	5.4056	5.0906	3.534	2.2981	1.6298	1.1968	1.0256	1.0014
	0.75	28.505	21.799	7.7832	3.7448	2.302	1.6674	1.2964	1.0931	1.0203
$n = 5$	1	248.06	29.297	7.4438	3.7105	2.3807	1.7319	1.3683	1.17	1.0627
$L = 2.992$	1.25	14.414	10.39	5.7319	3.4203	2.3373	1.7474	1.4202	1.2226	1.1073
	1.5	4.9141	4.4422	3.5428	2.7191	2.1149	1.7225	1.4279	1.2589	1.1368
	1.75	2.7985	2.6746	2.4348	2.1242	1.8047	1.5673	1.388	1.2618	1.1485
	2	1.9793	1.9561	1.8515	1.7418	1.5625	1.4376	1.3099	1.2253	1.1528

Table 10. ARL values for SS-CUSUMVAR charting schemes at $ARL_0 = 250$, $h = 27.66$, and $k = 0.5$.

	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	6.71	6.7	6.19	5.04	4.04	3.21	2.99	2.43	2.01
	0.5	12.16	11.85	9.2	6.19	4.53	3.55	2.99	2.56	2.14
	0.75	73.53	46.09	13.38	7.06	4.85	3.71	3.04	2.6	2.26
$n = 3$	1	252.22	47.87	13.33	7.19	4.92	3.77	3.08	2.65	2.32
$h = 28$	1.25	24.36	17.79	9.96	6.48	4.69	3.67	3.07	2.63	2.33
	1.5	9.41	8.56	6.91	5.32	4.24	3.49	2.94	2.58	2.3
	1.75	5.72	5.51	5.03	4.33	3.7	3.2	2.81	2.48	2.22
	2	4.27	4.19	3.93	3.61	3.24	2.91	2.62	2.37	2.17
	0.25	5.04	5.03	4.62	3.97	3.1	2.97	2.2	2	2
	0.5	8.24	8.04	6.55	4.8	3.68	3	2.51	2.09	2
	0.75	34.52	24.64	9.84	5.65	4.01	3.15	2.6	2.23	2.03
$n = 4$	1	252.72	36.63	10.38	5.8	4.11	3.2	2.64	2.28	2.05
$h = 27.6$	1.25	19.33	14.09	8.23	5.38	3.96	3.14	2.64	2.29	2.05
	1.5	7.49	6.9	5.59	4.42	3.57	2.98	2.55	2.24	2.01
	1.75	4.73	4.58	4.15	3.64	3.15	2.75	2.42	2.17	1.97
	2	3.57	3.48	3.3	3.06	2.77	2.5	2.26	2.06	1.88
	0.25	4.05	4.05	3.97	3.14	2.99	2.28	2	2	2
	0.5	6.52	6.39	5.32	4.07	3.2	2.67	2.13	2	1.99
	0.75	23.06	17.48	8.07	4.88	3.51	2.79	2.3	2.05	1.96
$n = 5$	1	249.9	29.13	8.85	5.12	3.62	2.84	2.37	2.09	1.92
$h = 27.66$	1.25	16.41	11.98	7.03	4.75	3.5	2.8	2.37	2.08	1.88
	1.5	6.44	6.05	4.92	3.93	3.2	2.65	2.3	2.04	1.84
	1.75	4.15	4.02	3.63	3.21	2.8	2.46	2.17	1.96	1.77
	2	3.14	3.08	2.91	2.7	2.48	2.23	2.03	1.85	1.69

Table 11. ARL values for Max-CUSUMVAR control charting scheme at $ARL_0 = 250$ and $k = 0.5$.

|					a					
	b	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
	0.25	6.44	6.43	6.44	6.25	4.63	3.56	3	2.49	2.01
	0.5	11.66	11.61	10.56	6.9	4.7	3.57	2.96	2.5	2.11
	0.75	69.22	48.15	13.77	7.06	4.77	3.64	2.97	2.52	2.21
$h = 5.06$	1	248.73	47.71	13.24	7.09	4.83	3.7	3.01	2.58	2.27
$n = 3$	1.25	25.44	18.77	10.39	6.65	4.75	3.7	3.07	2.61	2.3
	1.5	9.9	9.06	7.33	5.64	4.46	3.62	3.03	2.63	2.32
	1.75	6.05	5.86	5.36	4.62	3.94	3.39	2.94	2.59	2.31
	2	4.53	4.46	4.18	3.86	3.47	3.11	2.79	2.5	2.28
	0.25	4.87	4.88	4.87	4.84	3.91	3.01	2.54	2	2
	0.5	7.93	7.9	7.53	5.51	3.91	3.06	2.51	2.08	2
	0.75	33.27	26.12	10.36	5.71	3.97	3.09	2.55	2.18	2.01
$h = 5.03$	1	252.77	36.58	10.29	5.73	4.04	3.12	2.58	2.24	2.01
$n = 4$	1.25	20.31	14.9	8.64	5.55	4.03	3.16	2.63	2.27	2.02
	1.5	7.88	7.31	5.99	4.73	3.78	3.11	2.63	2.28	2.04
	1.75	4.98	4.84	4.44	3.93	3.39	2.94	2.56	2.27	2.03
	2	3.76	3.68	3.51	3.3	2.99	2.7	2.42	2.2	1.99
	0.25	4.01	4.01	4.02	4.01	3.35	2.89	2.03	2	2
	0.5	6.28	6.29	6.1	4.73	3.44	2.75	2.17	2.01	1.98
	0.75	22.27	18.62	8.58	4.96	3.49	2.74	2.26	2.03	1.93
$h = 5.035$	1	250.21	29.25	8.77	5.04	3.56	2.78	2.32	2.05	1.88
$n = 5$	1.25	17.09	12.7	7.43	4.91	3.58	2.82	2.37	2.06	1.85
	1.5	6.7	6.38	5.3	4.23	3.39	2.78	2.37	2.08	1.86
	1.75	4.32	4.24	3.89	3.47	3.02	2.64	2.31	2.05	1.84
	2	3.28	3.22	3.09	2.9	2.69	2.42	2.19	1.99	1.8
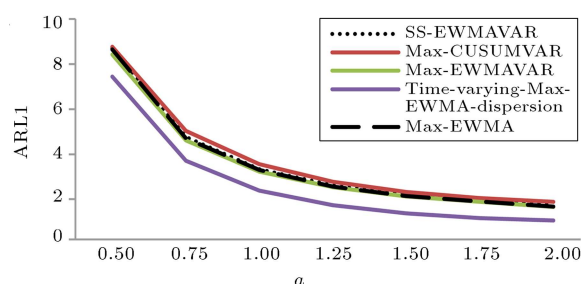
Figure 1. ARL curves for the process mean of the proposed charts versus Max-EWMA at $ARL_0 = 250$, $b = 1$, and $\lambda = 0.20$.
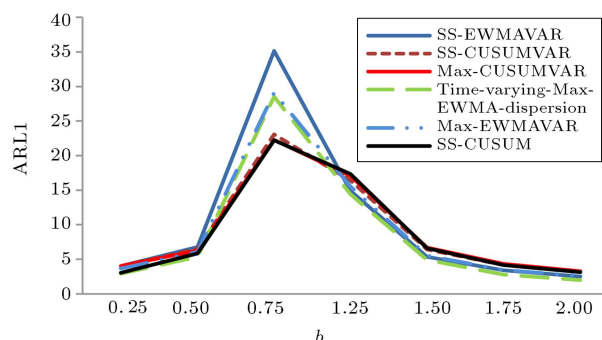

Figure 2. ARL curves for the process dispersion of the proposed charts versus Max-EWMA when $a = 0$ and $\lambda = 0.20$.
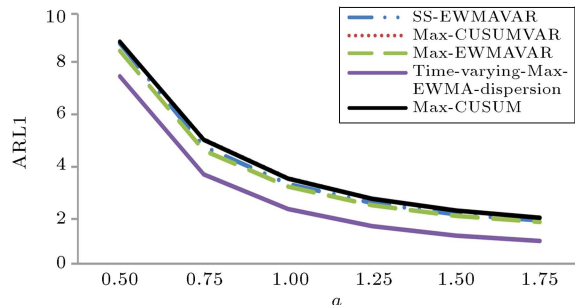

Figure 3. ARL curves for the process mean of the proposed charts versus Max-CUSUM at $ARL_0 = 250$, $b = 1$, and $\lambda = 0.20$.
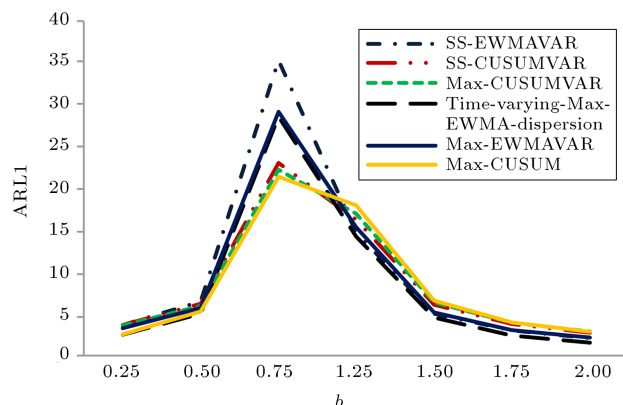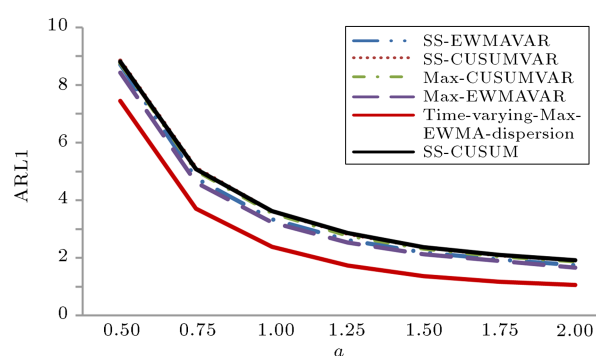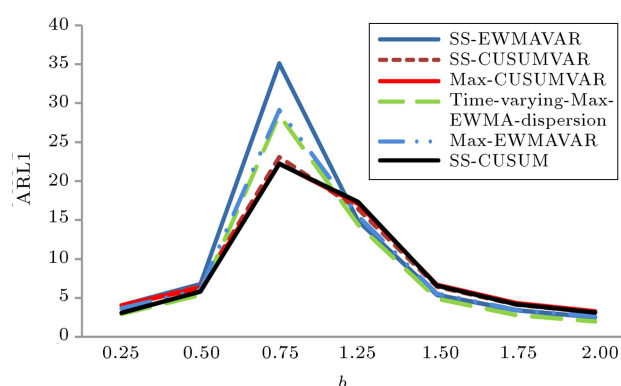

Figure 4. ARL curves for the process dispersion of the proposed charts versus Max-CUSUM when, $a = 0$ and $\lambda = 0.20$.

Figure 5. ARL curves for the process mean of the proposed charts versus SS-CUSUM at $ARL_0 = 250$, $b = 1$, and $\lambda = 0.20$.

Figure 6. ARL Curves for the process dispersion of the proposed charts versus SS-CUSUM at $a = 0$ and $\lambda = 0.20$.

to serve purpose of ease in comparison and discussion. For a comparative analysis of the proposed charts with their existing counterparts, these results show that:

- The proposed charts are better than Max-CUSUM, Max-EWMA, SS-CUSUM, and SS-EWMA in the increasing shifts of the process dispersion when $n = 3$;

- For $n = 4$, only SS-CUSUMVAR and Max-CUSUMVAR are better than their counterparts under study at the increasing shifts of the process dispersion;

- SS-EWMAVAR and Max-EWMAVAR are better than Max-EWMA and SS-EWMA in the increasing shifts of the process dispersion when $n = 5$ for all values of λ;

- SS-EWMAVAR and Max-EWMAVAR are better than Max-CUSUM and SS-CUSUM at the increasing shifts of the process dispersion when $n = 5$;

- SS-EWMAVAR is better than Max-CUSUM, SS-CUSUM, and SS-EWMA at a very small shift ($a < 0.5$) of the process mean when $n = 5$;

- Max-EWMAVAR is better than Max-CUSUM, and SS-CUSUM at small shift ($a < 0.5$) of the process

mean, but slightly better than SS-EWMA at all shifts of the mean when $n = 5$;

- SS-EWMAVAR, Max-EWMAVAR, and Max-EWMA have nearly equal values at all shifts in the process mean when $n = 5$;

- SS-CUSUMVAR and Max-CUSUMVAR similar ARL values with Max-CUSUM, Max-EWMA, SS-CUSUM, and SS-EWMA at all shifts in the process mean when $n = 5$;

- SS-CUSUMVAR is slightly better than the performance of all its existing counterparts (except max-chart) at all shifts of the process dispersion when $n = 5$;

- Max-CUSUMVAR is slightly better than the performance of all its existing counterparts in small and moderate shifts of the process dispersion when $n = 5$;

- SS-EWMAVAR, SS-CUSUMVAR, Max-CUSUMVAR, and Max-EWMAVAR are performing better than max-chart at small and moderate shifts in both parameters ($n = 5$);

- Generally, the proposed charts are poorer at detecting decreasing shifts in the process dispersion than their counterparts.

5. Illustrative example

A practical example was taken from the book of Montgomery [19] about the hard-bake process with photolithography in semiconductor manufacturing. Twenty-five samples were initially used in phase 1 with the subgroup, each of 5 sample size of wafers, and they are all in a statistical process. Another set of twenty samples of wafers are added to be monitored in phase 2. The output of the proposed charts and their counterparts are shown in Table 12 and their graphical displays are shown in Figures 7 to 12. The sample mean and variance are calculated to be 1.53184 and
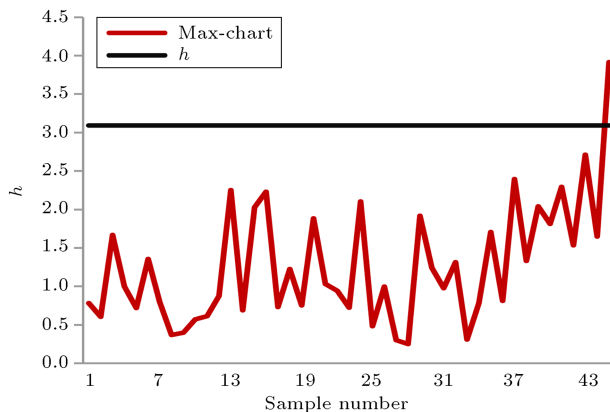
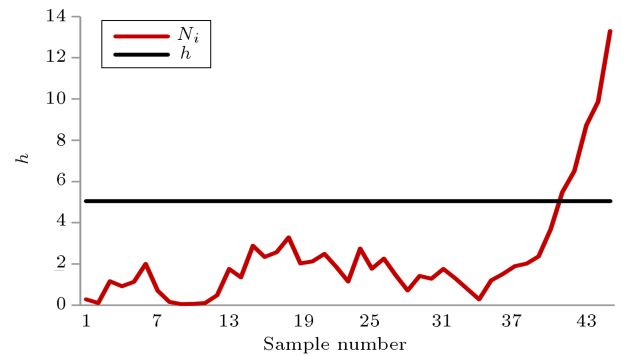**Figure 7.** Max-chart when $h = 3.09$ at $ARL_0 = 250$.

Figure 8. Max-CUSUM when $h = 5.05$, $k = 0.5$ at $ARL_0 = 250$.
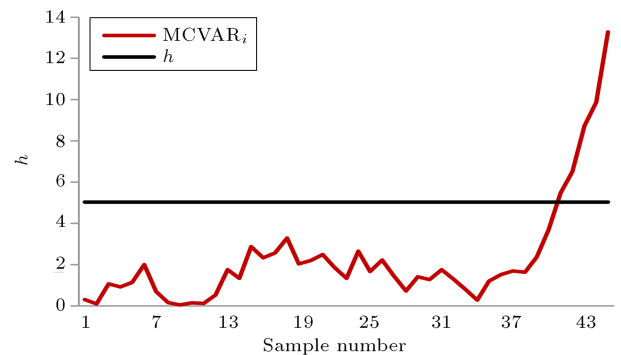
Figure 9. Max-CUSUMVAR when $h = 5.035$ at $ARL_0 = 250$.
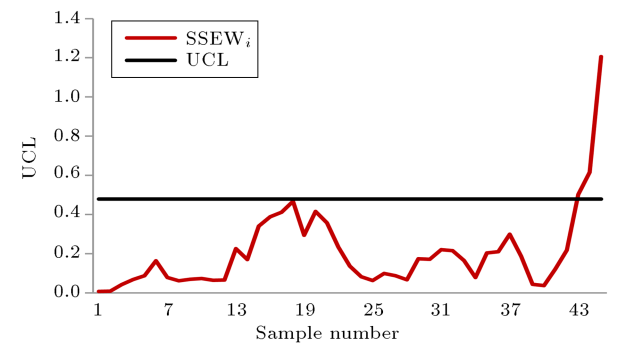
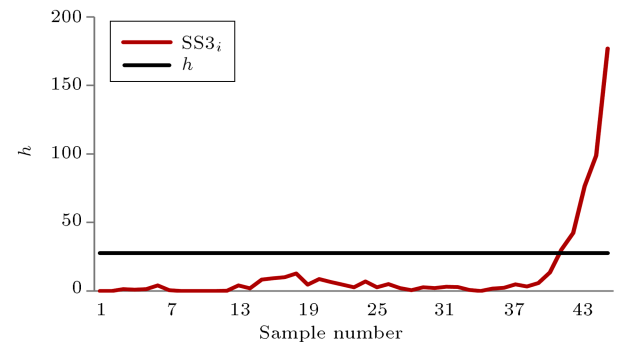**Figure 10.** SS-EWMAVAR when $L = 3.55$, and $\lambda = 0.1$ at $ARL_0 = 250$.

Figure 11. SS-CUSUMVAR when $h = 27.66$ at $ARL_0 = 250$.

Table 12. Output of the hard-bake process for the proposed charts and their counterparts.

| S/no	N_i	M_i	$SS1_i$	$SSEW_i$	$SS3_i$	$MCVAR_i$	MaxChat	$SS2_i$	MEW_i
1	0.28	0.08	0.01	0.01	0.1	0.31	0.78	0.08	0.081
2	0.1	0.09	0.01	0.01	0.01	0.1	0.6	0.01	0.09
3	1.17	0.16	0.04	0.04	1.33	1.07	1.67	1.55	0.163
4	0.92	0.25	0.07	0.07	0.94	0.92	1	0.99	0.247
5	1.14	0.29	0.09	0.09	1.3	1.14	0.72	1.3	0.294
6	2	0.4	0.16	0.16	4	2	1.36	4	0.4
7	0.7	0.28	0.08	0.08	0.56	0.7	0.8	0.55	0.28
8	0.16	0.25	0.06	0.06	0.03	0.16	0.37	0.03	0.248
9	0.06	0.26	0.07	0.07	0	0.06	0.4	0	0.263
10	0.07	0.27	0.07	0.07	0.02	0.15	0.57	0	0.267
11	0.11	0.25	0.06	0.06	0.02	0.12	0.61	0.01	0.253
12	0.49	0.23	0.07	0.07	0.28	0.53	0.87	0.24	0.233
13	1.75	0.44	0.23	0.23	4.03	1.75	2.25	3.88	0.435
14	1.35	0.4	0.17	0.17	1.88	1.35	0.69	1.85	0.401
15	2.88	0.56	0.34	0.34	8.3	2.88	2.03	8.3	0.564
16	2.33	0.5	0.39	0.39	9.36	2.33	2.23	8.99	0.503
17	2.57	0.53	0.41	0.41	9.99	2.57	0.73	9.69	0.526
18	3.29	0.6	0.47	0.47	12.77	3.29	1.22	12.66	0.596
19	2.04	0.46	0.3	0.29	4.68	2.04	0.75	4.7	0.461
20	2.13	0.46	0.42	0.42	8.69	2.2	1.89	8.39	0.457
21	2.49	0.51	0.36	0.36	6.65	2.49	1.03	6.63	0.515
22	1.87	0.45	0.24	0.23	4.59	1.87	0.94	4.36	0.451
23	1.15	0.36	0.14	0.14	2.65	1.34	0.72	2.17	0.361
24	2.75	0.26	0.09	0.08	7.05	2.65	2.1	7.57	0.264
25	1.77	0.25	0.07	0.06	2.8	1.67	0.48	3.12	0.247
26	2.26	0.27	0.1	0.1	4.95	2.23	1	5.13	0.275
27	1.46	0.28	0.09	0.09	2.09	1.45	0.3	2.14	0.276
28	0.71	0.25	0.07	0.07	0.52	0.72	0.25	0.51	0.247
29	1.42	0.41	0.17	0.17	2.66	1.42	1.92	2.56	0.414
30	1.28	0.41	0.17	0.17	2.22	1.28	1.24	2.19	0.409
31	1.76	0.47	0.22	0.22	3.16	1.76	0.98	3.13	0.465
32	1.3	0.42	0.21	0.22	2.88	1.3	1.32	2.74	0.423
33	0.8	0.38	0.16	0.17	0.74	0.8	0.31	0.69	0.381
34	0.28	0.26	0.08	0.08	0.08	0.28	0.78	0.08	0.265
35	1.21	0.41	0.2	0.2	1.84	1.21	1.71	1.78	0.409
36	1.52	0.45	0.21	0.21	2.4	1.52	0.81	2.39	0.45
37	1.9	0.47	0.32	0.3	4.91	1.69	2.4	6.46	0.472
38	2.02	0.35	0.21	0.19	3.34	1.63	1.33	4.76	0.32
39	2.37	0.23	0.06	0.04	5.75	2.37	2.04	6.09	0.202
40	3.68	0.17	0.04	0.04	13.61	3.68	1.81	13.6	0.143
41	5.48	0.35	0.13	0.12	30.17	5.48	2.3	30.14	0.345
42	6.51	0.46	0.22	0.22	42.38	6.51	1.53	42.38	0.464
43	8.72	0.69	0.51	0.5	76.55	8.72	2.71	76.5	0.689
44	9.87	0.78	0.62	0.62	98.73	9.87	1.65	98.56	0.785
45	13.28	1.1	1.21	1.21	176.86	13.28	3.91	176.83	1.098
Control limits	5.05	0.644	0.484	0.479	27.6	5.035	3.09	27.9	0.642
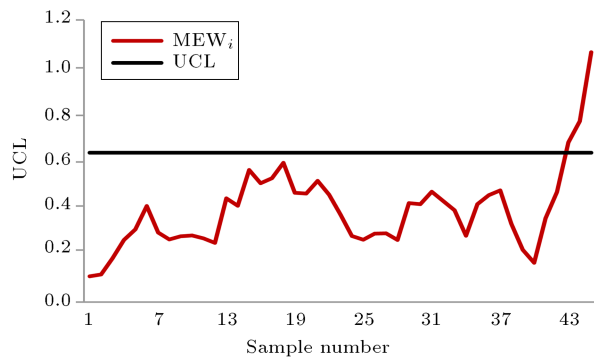
Figure 12. Max-EWMAVAR when $L = 2.77$, and $\lambda = 0.1$ at $ARL_0 = 250$.

0.1360771, respectively. The UCL of Max-EWMA, SS-EWMA, SS-EWMAVAR, and Max-EWMAVAR can be calculated from Eqs. (6b), (18), (27), and (29), respectively; we substitute the values of L, sample mean and variance in the equations. For a desired in-control $ARL_0 = 250$, we fixed the value of $k = 0.50$ and $\lambda = 0.10$ for each of the charts where necessary; the values of h and L are derived from Tables 1 to 11.

The results show that:

- There is the increased large values in the process at samples 41 to 45;

- SS-CUSUMVAR and Max-CUSUMVAR detect 5 out-of-control signals at sample points 41-45;

- SS-EWMAVAR, Max-EWMAVAR, Max-EWMA, and SS-EWMA triggers 3 out-of control points, at sample points 43-45;

- Max-chart gives only one out-of-control signal at sample point 45.

6. Summary, conclusions and recommendations

In this study, we have proposed four new control charts that monitor the process mean and dispersion simultaneously. We have investigated ARL properties of the proposed schemes and compared them with the existing counterparts including Max-EWMA, Max-CUSUM, SS-EWMA, and SS-CUSUM. The comparisons showed that the proposed schemes are really performing better for the increasing shifts in the process dispersion than the other existing schemes covered in this study. They have slightly the same ARL values in the process mean with their counterparts. The scope of this study may be extended to monitor process vector mean and dispersion simultaneously in the multivariate setups for an improved and efficient monitoring of process parameters.

Acronyms

Max-EWMA	Max-Exponentially Weighed Moving Average
Max-CUSUM	Max-Cumulative Sum
SS-EWMA	Sum of Square Exponential Weighted Moving Average
SS-CUSUM	Sum of Square Cumulative Sum
Max-EWMAVAR	Maximum Exponentially Weighted Moving Average Variance
Max-CUSUMVAR	Maximum Cumulative Sum Variance
SS-EWMAVAR	Sum of Square Exponentially Weighted Moving Average Variance
SS-CUSUMVAR	Sum of Square Cumulative Sum Variance
SPC	Statistical Process Control
CUSUM	Cumulative Sum
EWMA	Exponential Weighted Moving Average
UCL	Upper Control Limit
R	Range
S^2	Sample Variance

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