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Inventory of complementary products with stock-dependent demand under vendor-managed inventory with consignment policy

M. Hemmati, S.M.T. Fatemi Ghomi*, and M.S. Sajadieh

Department of Industrial Engineering, Amirkabir University of Technology, 424 Hafez Avenue, Tehran, P.O. Box 1591634311, Iran.

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Abstract. This paper proposes an integrated two-stage model, which consists of one vendor and one buyer for two complementary products. The vendor produces two types of products and delivers them to the buyer in distinct batches. Buyer stocks items in the warehouse and on the shelf. The demand for each product is sensitive to stock levels of both products. A vendor-managed inventory with consignment stock policy is considered. The number of shipments and replenishment lot sizes are jointly determined as decision variables in such a way that total profit is maximized. The numerical study shows that as complementation rate increases, the quantity of transfers and demand of both products increase. Hence, ignoring the complementation between products leads to the loss of some customers.

1. Introduction

In the today's competitive market, individual optimization is not profitable; hence, sharing information between supply chain members has become essential. Coordination can decrease supply chain costs and increase sales volume [1]. Supply chain coordination makes management more efficient to encounter real life uncertainty [2]. Literature review of the joint optimization of the vendor and buyer costs was first started by Goyal [3]. He assumed that the vendor had infinite production rate in presence of lot-for-lot policy. Banerjee [4] developed the model by assuming a finite rate. Goyal [5] generalized the model by relaxing lot-for-lot assumption in which the shipment was delayed until the entire batch was produced. Developing this stream, Jokar and Sajadieh [6] proposed a coordinated two-level model in which the demand was dependent on selling price. Kim et al. [7] developed a three-echelon Joint Economic Lot Sizing (JELS) model for multi-product problem in which manufacturer produced products on the single facility. Sajadieh et al. [8] considered a two-stage supply chain and developed a JELS model with stochastic lead times and shortage in which the manufacturer delivered items to the buyer in equal lots. Ben-Daya et al. [9] and Glock [10] presented a comprehensive review of the JELS problems.

Several authors considered the effect of different parameters on the demand, for instance, stock [11], price [6,12], sales teams' initiatives [13,14], and marketing effort [15]. Most of the managers have recognized the effect of amount of items on the shelf on customers' demand. In other words, facing large quantities of items leads the customers to buy more. Teng and Chang [16] studied an economic production quantity

*. *Corresponding author. Tel.: +98 21 64545381; Fax: +98 21 66954569*
E-mail addresses: m_hemmati@aut.ac.ir (M. Hemmati); fatemi@aut.ac.ir (S.M.T. Fatemi Ghomi); sajadieh@aut.ac.ir (M.S. Sajadieh)

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model for deteriorating items in which demand was sensitive to stock and price. They also considered capacity constraint of shelf. Goyal and Chang [17] proposed an inventory model for a single item with stock-dependent demand and determined transferring and ordering lot sizes. The space limitation of buyer's shelf was considered and the profit of the buyer was maximized. Sajadieh et al. [11] proposed a coordinated model in which the demand of customers was positively sensitive to the amount of items displayed on the shelf. They showed that the gains from the coordinated model were greater when demand was more sensitive to inventory level. Duan et al. [18] proposed inventory models for deteriorating items with and without backlogging, where demand was sensitive to stock and backlogging was sensitive to the demand backlogged and waiting time. Yang et al. [19] applied three different coordination policies to a two-level model for a single item with stock-sensitive demand.

The basic JELS models have been extended in many different directions. Some researchers considered JELS models with Vendor-Managed Inventory and Consignment Stock (VMI-CS) policy. Consignment Stock (CS) policy is an agreement in which vendor stores items in the buyer's warehouse, but the items are owned by the vendor and the buyer does not pay any money until the items are sold. Braglia and Zavanella [20] was the first who considered a JELS model under VMI-CS agreement. Yi and Sarker [21] studied a coordinated model under CS agreement. They considered controllable lead time and capacity constraint and solved their model using hybrid metaheuristic algorithms. Zanoni and Jaber [22] developed a JELS model under VMI-CS policy in which demand was sensitive to stock. They considered a minimum inventory level for the items on the shelf. Wang and Lee [23] corrected the cost function of Zanoni and Jaber [22] and showed the properties of the corrected model. Giri and Bardhan [24] studied a two-level supply chain for single product under CS agreement in which the demand was sensitive to stock. They considered buyer's space limitation and showed its negative effect on the total cost. Hariga and Al-Ahmari [25] proposed a model with simultaneous consideration of space allocation and lot-sizing in which demand was sensitive to inventory level. They used VMI-CS agreement for a single item and showed that it was more profitable for all supply chain members. Giri et al. [26] studied a JELS model under consignment agreement. They considered vendor's space limitation as a controllable variable.

Cárdenas-Barrón and Sana [27] studied a two-level model with promotional effort-dependent demand considering multi items and delayed payment. Ghosh et al. [28] studied a multi-item problem for deteriorating items with stock-sensitive demand under space constraint. Some authors considered multi-item models in the case of complementary products. When the items are complementary, the buyer who wants to buy one product may be motivated to buy another product. These products can be used together. Therefore, the demands for these products positively correlate with each other. For example, demand for printers makes demand for ink cartridges. Some other examples of complementary products are tooth brush and tooth paste, computer and its software, etc. Yue et al. [29] studied two complementary products considering bundling strategy and obtained optimal pricing decisions under three different cases. Yan and Bandyopadhyay [30] studied bundling of complementary products in which the demand was sensitive to the prices of both products. Wei et al. [31] considered two complementary products in two-stage supply chain under different pricing models with price-dependent demand. Taleizadeh and Charmchi [32] proposed a two-level model under cooperative advertising for two complementary products in which demand was sensitive to price. There are some papers that have studied the effect of stock level of products on their demand. Maity and Maiti [33] developed a multi-item model for deteriorating items with stock-sensitive demand. They considered complementary products in which demands of products had linearly positive effect on each other. In addition, they considered negative effect of demands for substitutable products. Sana [34] developed an inventory model for substitutable products with stock, price, and salesmen's effort under inflation and time value of money.

Stavrulaki [35] proposed a model for two substitutable products with stock-sensitive and stochastic demand. Two heuristic solution procedures were developed and it was concluded that higher inventory level would lead to more sale. Maity and Maiti [36] developed a multi-deteriorating-item model for complementary and substitutable products. The deteriorating rate was assumed to be constant or stock-dependent. The demand was sensitive to stock and warehouse had limited capacity. Both steady-state environment and transient-state environment were considered. Krommyda et al. [37] considered an inventory model for two substitutable items in which demand of each item was dependent on its stock level and stock level of the other items. They assumed that, in a stock-out situation, the substitutable item could satisfy a particular fraction of demand.

In most of the works in which JELS with VMI-CS agreement has been studied, only one product is considered and none of them consider the relation between products. However, in the real world, the items are not displayed individually and they can affect each other's demand. Under VMI-CS agreement, vendor owns the items on buyer's side. In this policy, when the items are stored in the buyer's warehouse, vendor incurs capital part of holding cost and buyer is only responsible for the physical part of holding cost. Thus, determination of proper order quantity and number of shipments can significantly affect the vendor and buyer costs. Demands of the complementary products can affect each other. The stock level of some products affects not only their own demand, but also the demand of their complementary products. Therefore, neglecting the relation between products under VMI-CS agreement can impose additional costs to the supply chain.

The current paper deals with a coordinated model consisting of a vendor and a buyer. The vendor produces two complementary products and transfers them to the buyer under VMI-CS agreement. Some transferred items are displayed on the shelf and the rest of them are stocked in the buyer's warehouse. The demand of each product is dependent on stock level of both products. The optimal quantity and number of lots transferred from vendor's warehouse to buyer's warehouse and from buyer's warehouse to the shelf are determined. The objective is to determine variables such that total profit of the system is maximized.

The rest of the paper is organized as follows. Section 2 defines the problem and describes the notation and assumptions used throughout the paper. Section 3 presents the mathematical model. Section 4 gives a solution algorithm to find the optimal solution. Section 5 introduces some numerical examples and provides the sensitivity analysis. Finally, Section 6 is devoted to the conclusions and future researches.

2. Assumptions and notation

The following assumptions and notation are used to develop the proposed model.

2.1. Assumptions

1. There are single vendor, single buyer, and two complementary products;
2. The demand of the product i, where $i = 1, 2$, is linearly dependent on stock level $I_i(t)$ of two products. The demand functions are given by $D_1 = a_1 + b_1 I_1(t) + b_3 I_2(t)$ and $D_2 = a_2 + b_2 I_2(t) + b_3 I_1(t)$, where $a_i > 0$ and $0 < b_i < 1$. b_1 and b_2 are sensitivity of each product's demand to its own stock level, while b_3 is sensitivity of product's demand to the stock level of its complementary product;
3. The inventory is continuously reviewed. For each product, the vendor delivers order quantity in n_{vi} equal shipments, where n_{vi} is integer and Q_i is the size of each shipment. The buyer transfers each batch to shelf in n_{bi} equal lot sizes of q_i, i.e., $Q_i = n_{bi} q_i$, where n_{bi} is integer. The items are transferred to the shelf when inventory level of shelf reaches zero;
4. Shortages at each level are not allowed. Thus, production rate for each product is greater than its demand;
5. Time horizon is infinite and lead time is zero in any level;
6. Capacity of shelf is limited.

2.2. Notation

P_i	The vendor's constant production rate for product $i(P_i > D_i)$, $i = 1, 2$
Q_i	Buyer's order quantity of product i
q_i	Size of each lot transferred to the shelf for product i
S_i	Fixed cost of transferring items from buyer's warehouse to the shelf for product i
u_i	The net unit selling price of product i (net price charged by the buyer to the customers)
A_{vi}	Vendor's setup cost of product i
A_{bi}	Buyer's ordering cost of product i
h_{vi}	Vendor's unit holding cost per unit time for product i, which consists of physical and financial components $h_{vi} = h_{vi}^{fin} + h_{vi}^{phy}$
h_{wi}	Unit holding cost per unit time at the warehouse of the buyer for product i, which consists of physical and financial components $h_{wi} = h_{wi}^{phy} + h_{wi}^{fin}$
h_{ni}	Unit holding cost per unit time at the warehouse of the buyer for product i under VMI-CS policy, which consists of the physical component of the buyer's warehouse and vendor's financial component, $h_{ni} = h_{vi}^{fin} + h_{wi}^{phy}$
h_{di}	Unit holding cost per unit time at the shelf of the buyer for product i under VMI-CS policy
C_{di}	Capacity of buyer's shelf for product i
T_{vi}	Cycle time of vendor's warehouse for product i
T_{wi}	Cycle time of buyer's warehouse for product i
T_{di}	Cycle time of buyer's shelf for product i

3. Model formulation

Consider a single-vendor single-buyer supply chain of two complementary products under VMI-CS policy. Demand for each complementary product depends linearly on its own stock level and the stock level of the other product. According to Figure 1, the vendor produces both products and delivers items in n_{vi} equal-sized batches to the buyer. The buyer transfers n_{bi} equal batches of size q_i from its warehouse to the shelf. The capacity of the shelf is limited.

The inventory levels of products 1 and 2 are respectively as follows:

$$\frac{d}{dt}I_1(t) = -a_1 - b_1 I_1(t) - b_3 I_2(t), \tag{1}$$

$$\frac{d}{dt}I_2(t) = -a_2 - b_2 I_2(t) - b_3 I_1(t). \tag{2}$$

The above system of differential equations is solved with initial conditions $I_1(0) = q_1$, and $I_2(0) = q_2$. Expanding exponential function in Maclaurin series, keeping the first two terms, and neglecting the rest, as they have small quantities (see Appendix A), the inventory levels of both products are obtained:

$$I_1(t) = (-b_1 q_1 - b_3 q_2 - a_1)t + q_1, \qquad 0 \le t \le T_{d1}, \tag{3}$$

$$I_2(t) = (-b_2 q_2 - b_3 q_1 - a_2)t + q_2, \qquad 0 \le t \le T_{d2}. \tag{4}$$

The inventory level at the end of T_{di} is zero; hence, $I(T_{di}) = 0$. By solving these equations, T_{di} is obtained as:

$$T_{d1} = \frac{q_1}{b_1 q_1 + b_3 q_2 + a_1}, \tag{5}$$

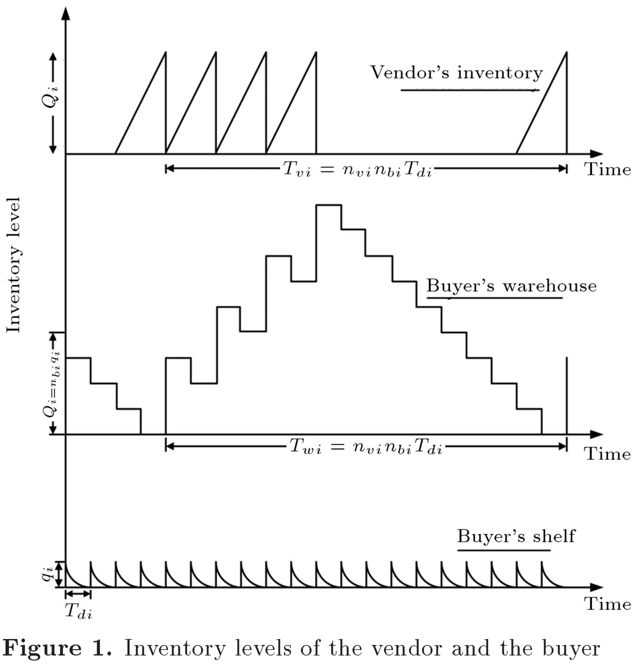

Figure 1. Inventory levels of the vendor and the buyer for product i.

$$T_{d2} = \frac{q_2}{b_2 q_2 + b_3 q_1 + a_2}. \tag{6}$$

The total cost of the supply chain consists of buyer's total cost and vendor's total cost. For each product, the components of the buyer's total cost are as follows:

Ordering cost:

$$\frac{A_{bi}}{T_{wi}} = \frac{A_{bi}}{n_{bi} n_{vi} T_{di}}. \tag{7}$$

Holding cost at the buyer's warehouse:

$$\frac{q_i h_{ni}((n_{bi} n_{vi} - 1)T_{di} - \frac{(n_{vi}-1)n_{bi} q_i}{P_i})}{2T_{di}}. \tag{8}$$

The average inventory of the buyer's shelf:

$$\int_0^{T_{di}} I_i dt = \frac{q_i T_{di}}{2}. \tag{9}$$

Buyer's holding cost at shelf:

$$\frac{h_{di} q_i}{2}. \tag{10}$$

The cost of transferring items from buyer's warehouse to the shelf:

$$\frac{S_i}{T_{di}}. \tag{11}$$

The components of the vendor's total cost for each product are:

Vendor's set-up cost:

$$\frac{A_{vi}}{T_{vi}} = \frac{A_{vi}}{n_{vi} n_{bi} T_{di}}. \tag{12}$$

Vendor's holding cost:

$$\frac{h_{vi} n_{bi} q_i^2}{2P_i T_{di}}. \tag{13}$$

Therefore, the system's total profit can be obtained as follows:

$$\begin{aligned} TP(q_1, q_2, n_{b1}, n_{b2}, n_{v1}, n_{v2}) &= \frac{u_1 q_1}{T_{d1}} + \frac{u_2 q_2}{T_{d2}} \\ &- \frac{A_{v1}}{n_{b1} n_{v1} T_{d1}} - \frac{A_{v2}}{n_{b2} n_{v2} T_{d2}} - \frac{A_{b1}}{n_{b1} T_{d1}} \\ &- \frac{A_{b2}}{n_{b2} T_{d2}} - \frac{S_1}{T_{d1}} - \frac{S_2}{T_{d2}} - \frac{h_{v1} n_{b1} q_1^2}{2P_1 T_{d1}} \\ &- \frac{h_{v2} n_{b2} q_2^2}{2P_2 T_{d2}} - \frac{h_{d1} q_1}{2} - \frac{h_{d2} q_2}{2} \\ &- \frac{q_1 h_{n1}((n_{b1} n_{v1} - 1)T_{d1} - \frac{(n_{v1}-1)n_{b1} q_1}{P_1})}{2T_{d1}} \\ &- \frac{q_2 h_{n2}((n_{b2} n_{v2} - 1)T_{d2} - \frac{(n_{v2}-1)n_{b2} q_2}{P_2})}{2T_{d2}}. \end{aligned} \tag{14}$$

Substituting Eqs. (5) and (6) into Eq. (14) gives:

$$TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2}) = u_1(b_1q_1+b_3q_2+a_1) + u_2(b_2q_2+b_3q_1+a_2) - \frac{A_{v1}\,(b_1q_1+b_3q_2+a_1)}{n_{b1}n_{v1}q_1} - \frac{A_{v2}\,(b_2q_2+b_3q_1+a_2)}{n_{b2}n_{v2}q_2} - \frac{h_{v1}n_{b1}q_1(b_1q_1+b_3q_2+a_1)}{2P_1} - \frac{h_{v2}n_{b2}q_2(b_2q_2+b_3q_1+a_2)}{2P_2} - \frac{A_{b1}(b_1q_1+b_3q_2+a_1)}{n_{b1}q_1} - \frac{A_{b2}(b_2q_2+b_3q_1+a_2)}{n_{b2}q_2} - \frac{h_{n1}}{2}\Big(\frac{(n_{b1}n_{v1}-1)q_1}{b_1q_1+b_3q_2+a_1} - \frac{(n_{v1}-1)n_{b1}q_1}{P_1}\Big)(b_1q_1+b_3q_2+a_1) - \frac{h_{n2}}{2}\Big(\frac{(n_{b2}n_{v2}-1)q_2}{b_2q_2+b_3q_1+a_2} - \frac{(n_{v2}-1)n_{b2}q_2}{P_2}\Big)(b_2q_2+b_3q_1+a_2) - \frac{S_1(b_1q_1+b_3q_2+a_1)}{q_1} - \frac{S_2(b_2q_2+b_3q_1+a_2)}{q_2} - \frac{h_{d1}q_1(b_1q_1+b_3q_2+a_1)}{2b_1q_1+2b_3q_2+2a_1} - \frac{h_{d2}q_2(b_2q_2+b_3q_1+a_2)}{2b_2q_2+2b_3q_1+2a_2}. \quad (15)$$

It is desired to find the optimal solution to the following problem:

Maximize $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$

Subject to $1 \le q_i \le C_{di}$

n_{bi}, n_{vi} integer.

By assuming n_{vi} and n_{bi} as continuous variables, and taking the second partial derivative of $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ with respect to n_{vi} for given values of q_i and n_{bi}, Eq. (16) is obtained:

$$\frac{\partial^2 TP\,(n_{vi})}{\partial n_{vi}{}^2} = -\frac{2A_{vi}}{n_{bi}n_{vi}^3T_{di}}. \quad (16)$$

Taking the second partial derivative of $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ with respect to n_{bi} for given values of q_i and n_{vi} yields Eq. (17):

$$\frac{\partial^2 TP\,(n_{bi})}{\partial n_{bi}{}^2} = -\frac{2A_{vi}}{n_{bi}^3n_{vi}T_{di}} - \frac{2A_{bi}}{n_{bi}^3T_{di}}. \quad (17)$$

Eq. (16) is negative; thus, $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ is a concave function of n_{vi} for given n_{bi} and q_i. Eq. (17) is negative, too; hence, $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ is a concave function of n_{bi} for given values of q_i and n_{vi}. The first partial derivatives of $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ with respect to n_{vi} and n_{bi} are taken. By solving:

$$\frac{\partial TP\,(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})}{\partial n_{vi}} = 0,$$

and:

$$\frac{\partial TP\,(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})}{\partial n_{bi}} = 0.$$

Eqs. (18) and (19) are obtained as positive roots as shown in Box I.

Using Eqs. (18) and (19), the upper bounds of optimal values of n_{bi} and n_{vi} are obtained. Eq. (18) shows that there is inverse relation between n_{vi} and n_{bi}; thus, the maximum value of n_{vi} can be obtained at $n_{bi}=1$. There is not any obvious relation between n_{bi} or n_{vi} and other variables. Thus, to find the upper bounds of n_{vi} and n_{bi}, the numerators of Eqs. (18) and (19) are maximized and their denominators are minimized. Therefore, $n_{vi\,\max}$ and $n_{bi\,\max}$ are calculated as shown in Box II, where $q_{i\,\min=1}$, $q_{i\,\max}=C_{di}$, $T_{di\,\max}=\frac{C_{di}}{b_i+b_3+a_i}$ and $T_{di\,\min}=\frac{1}{b_iC_{di}+b_3C_{d(3-i)}+a_i}$.

Eq. (20) and Eq. (21) are used in the solution algorithm as upper bounds of the optimum number of shipments to the buyer's warehouse and the optimum number of transferring items from buyer's warehouse to the shelf, respectively.

To find the optimal solution to Eq. (15), the first derivative of $TP(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})$ is taken with respect to q_1 for given values of n_{bi} and n_{vi}:

$$\frac{\partial TP\,(q_1,q_2,n_{b1},n_{b2},n_{v1},n_{v2})}{\partial q_1} = q_1^3 + \frac{A_2}{A_1}q_1^2 + \frac{A_3}{A_1}, \quad (22)$$

where:

$$A_1 = \frac{b_1n_{b1}((n_{v1}-1)h_{n1}-h_{v1})}{P_1},$$

$$n_{vi} = \frac{\sqrt{2q_i h_{ni}(P_i T_{di} - q_i) A_{vi} P_i}}{q_i h_{ni}(P_i T_{di} - q_i) n_{bi}}, \tag{18}$$

$$n_{bi} = \frac{\sqrt{2P_i(h_{ni}(P_i T_{di} - q_i) n_{vi} + q_i(h_{ni} + h_{vi})) q_i (A_{bi} n_{vi} + A_{vi}) n_{vi}}}{(h_{ni}(P_i T_{di} - q_i) n_{vi} + q_i(h_{ni} + h_{vi})) q_i n_{vi}}. \tag{19}$$

Box I

$$n_{vi\max} = \left\lceil \frac{\sqrt{2q_{i\max} h_{ni}(P_i T_{di\max} - q_{i\max}) A_{vi} P_i}}{q_{i\min} h_{ni}(P_i T_{di\min} - q_{i\max})} \right\rceil, \tag{20}$$

$$n_{bi\max} = \left\lceil \frac{\sqrt{2P_i(h_{ni}(P_i T_{di\max} - 1) n_{vi} + q_{i\max}(h_{ni} + h_{vi})) q_{i\max}(A_{bi} n_{vi} + A_{vi}) n_{vi\max}}}{(h_{ni}(P_i T_{di\min} - q_{i\max}) + q_{i\min}(h_{ni} + h_{vi})) q_{i\min}} \right\rceil. \tag{21}$$

Box II

$$A_2 = \frac{1}{P_1 q_2 P_2}\Bigg(\frac{1}{2} b_3 n_{b2} P_1((n_{v2} - 1)h_{n2} - h_{v2}) q_2^2 + P_2((-\frac{1}{2} n_{v1} n_{b1} h_n + u_1 b_1 + u_2 b_3 - \frac{h_{d1}}{2} - \frac{h_{n1}}{2}) P_1 + \frac{n_{b1}}{2}((n_{v1} - 1)h_{n1} - h_{v1}) M_1) q_2 - E_2 P_1 P_2 b_3\Bigg),$$

$$A_3 = \left(\frac{A_{v1}}{n_{b1} n_{v1}} + \frac{A_{b1}}{n_{b1}} + S_1\right) M_1.$$

Considering that the signs of A_2 and A_1 are not specified, the cubic equation can have zero to three real roots. The possible roots of Eq. (22) are given in Appendix B.

Due to the complexity of the second derivative of $TP(q_1, q_2, n_{b1}, n_{b2}, n_{v1}, n_{v2})$ with respect to q_i, it is not possible to prove the convexity of $TP(q_1, q_2, n_{b1}, n_{b2}, n_{v1}, n_{v2})$ in q_i. Hence, a heuristic technique is developed to maximize total profit. Substituting $R1$, $R2$, and $R3$ into Eq. (15), single-variable equations are obtained for given values of n_{bi} and n_{vi}. Thus, the problem would be to find the optimum values for these single-variable equations.

4. Algorithm

This section proposes an algorithm to obtain the optimal solution to the problem. The optimal value of Eq. (15) with six decision variables can be obtained from the following algorithm:

Step 1: Set integer variables of n_{bi} and n_{vi} equal to 1 and start with initial values of $TP^{opt} = 0$, $n_{v1}^{opt} = 0$, $n_{v2}^{opt} = 0$, $n_{b1}{}^{opt} = 0$, $n_{b2}^{opt} = 0$, $q_1^{opt} = 0$, and $q_2^{opt} = 0$;

Step 2: Set q_1 equal to Eq. (B.1); substituting it into Eq. (15) gives a function of q_2, namely, $TP(q_2)$. Take the first derivative of $TP(q_2)$ with respect to q_2 and use b_i-section method with an initial interval of $[1, C_{d2}]$ to find the optimal value of $TP(q_2)$, i.e., q_2^*;

Step 3: Substitute the point obtained from Step 2 into Eq. (B.1) to achieve q_1^*;

Step 4: Put q_1^* and q_2^* obtained in Steps 2 and 3 into Eq. (15). If $TP(q_1, q_2, n_{v1}, n_{v2}, n_{b1}, n_{b2}) > TP^{opt}$, set $T^{Popt} = TP(q_1, q_2, n_{v1}, n_{v2}, n_{b1}, n_{b2})$, $q_1^{opt} = q_1$, $q_2^{opt} = q_2$, $n_{b1}^{opt} = n_{b1}$, $n_{b2}^{opt} = n_{b2}$, $n_{v1}^{opt} = n_{v1}$, and $n_{v2}^{opt} = n_{v2}$. Repeat Steps 2-4 by setting q_1 equal to Eqs. (B.2) and (B.3) and constant values of 1 and 500;

Step 5: Set $q_1 = 1$ and $q_2 = C_{d2}$ and put them into Eq. (15). If $TP(q_1, q_2, n_{v1}, n_{v2}, n_{b1}, n_{b2}) > TP^{opt}$, set $TP^{opt} = TP(q_1, q_2, n_{v1}, n_{v2}, n_{b1}, n_{b2})$, $q_1^{opt} = q_1$, $q_2^{opt} = q_2$, $n_{b1}^{opt} = n_{b1}$, $n_{b2}^{opt} = n_{b2}$, $n_{v1}^{opt} = n_{v1}$, and $n_{v2}^{opt} = n_{v2}$. Repeat this step for $q_1 = C_{d1}$ and $q_2 = C_{d2}$, $q_1 = 1$ and $q_2 = 1$, and $q_1 = C_{d1}$ and $q_2 = 1$;

Step 6: Set $n_{b1} = n_{b1} + 1$; if $n_{b1} \leq n_{b1}^{\max}$, go back to Step 2;

Step 7: Set $n_{b2} = n_{b2} + 1$; if $n_{b2} \leq n_{b2}^{\max}$, set $n_{b1} = 1$ and then go back to Step 2;

Step 8: Set $n_{v1} = n_{v1}+1$; if $n_{v1} \leq n_{v1}^{\max}$, set $n_{b1} = 1$ and $n_{b2} = 1$ and then go back to Step 2;

Step 9: Set $n_{v2} = n_{v2}+1$; if $n_{v2} \leq n_{v2}^{\max}$, set $n_{v1} = 1$, $n_{b1} = 1$, and $n_{b2} = 1$ and then go back to Step 2.

5. Numerical study

In order to demonstrate the solution procedure numerically, an inventory system of two complementary products is studied. Referring to the existing literature, relevant data are chosen and shown in Table 1. When defined parameters change, the changes in the optimal decision values are studied. Tables 2-4 show the computational results. To represent improvement in the total profit, percentage improvement PI is defined as 100 $(TP-TP')/TP'$, where TP' indicates the model profit when the complementation rate is ignored. In other words, PI represents profitability of the model with two complementary products in comparison with the case where two independent products are concerned.

Figure 2 plots the function $TP(q_1, q_2, n_{v1}^*, n_{v2}^*, n_{b1}^*, n_{b2}^*)$, where n_{v1}^*, n_{v2}^*, n_{b1}^*, and n_{b2}^* are optimal values.

Using the proposed model in Section 3, the effect of complementation rate is studied. Table 2 shows that the complementation rate has significant effect on the total profit. To analyze the effect of sensitivity of each product to inventory level of the other product, b_3, three levels for parameters, b_1 and b_2, and nine levels for parameter b_3 are defined. Table 2 indicates that

Table 1. Parameter values in numerical analysis.

Parameters	Basic values of product 1	Basic values of product 2
a_i	400	350
b_i	0.2	0.15
b_3	0.05	0.05
S_i	25	20
A_{bi}	100	80
A_{vi}	400	300
h_{di}	20	15
h_{wi}	5	4
h_{vi}	4	2
$h_{phy,wi}$	1	1
$h_{fin,wi}$	4	3
$h_{phy,vi}$	2	1
$h_{fin,v}$	2	1
h_{ni}	3	2
C_{di}	500	500
P_i	5000	4500
u_i	30	25

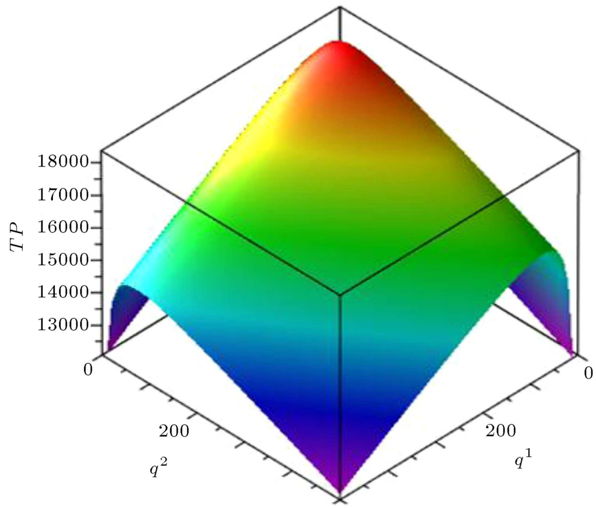

Figure 2. Net profit per unit of time as function of q_1 and q_2.

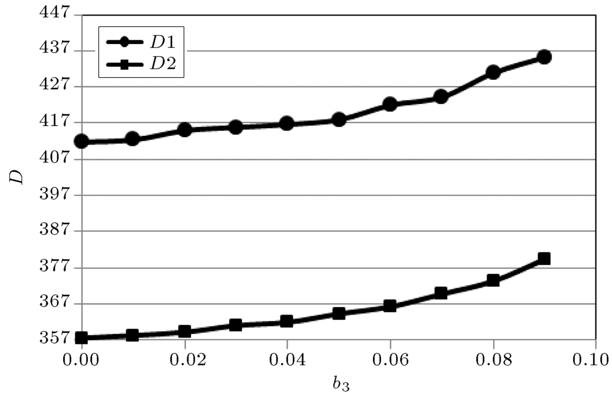

Figure 3. Effect of b_3 on demand ($b_1 = 0.2$, $b_2 = 0.15$).

when the sensitivity of each product to stock level of its complementary product b_3 increases, the quantity of batch transferred from the warehouse to the shelf of both products q_i increases. Furthermore, as b_3 increases, the number of transferring lots from buyer's warehouse to shelf of both products n_{bi} decreases. Hence, when complementation rate of two products is great, the buyer can take advantage of the economy of scale.

It is obvious from Table 2 and Figure 3 that with increasing complementary rate, the demand of product 1 is always greater than that of product 2. This is due to the fact that demand of product 1 is more sensitive to its stock level than that of product 2 is, i.e., $b_1 > b_2$. Moreover, as complementation rate increases, the demand of both products increases, because there is positive relation between demand of a product and stock level of its complementary product. Hence, when the complementation rate increases, as a customer buys one product, he is more likely to buy its complementary product. In other words, ignoring the

Table 2. Sensitivity analysis for parameter b_3.

Parameters		Product 1				Product 2				TP	PI
	b_3	q_1	n_{v1}	n_{b1}	D_1	q_2	n_{v2}	n_{b2}	D_2		
	0.00	49.63	1	7	407.44	39.85	1	9	353.98	18083.12	0.00
	0.01	50.03	1	7	407.94	43.75	1	8	354.88	18106.65	0.13
	0.02	50.44	1	7	408.45	44.29	1	8	355.44	18130.70	0.26
	0.03	50.86	1	7	408.97	44.84	1	8	356.01	18155.01	0.40
$b_1 = 0.15$	0.04	57.73	1	6	410.66	50.06	1	7	357.31	18181.07	0.54
$b_2 = 0.1$	0.05	58.29	1	6	411.28	50.79	1	7	357.99	18208.78	0.69
	0.06	58.87	1	6	411.92	51.54	1	7	358.69	18236.86	0.85
	0.07	59.50	1	6	413.04	58.84	1	6	360.05	18266.59	1.01
	0.08	69.52	1	5	415.22	59.95	1	6	361.56	18298.14	1.19
	0.09	70.45	1	5	416.91	70.49	1	5	363.39	18332.42	1.38
	0.00	58.84	1	6	407.44	50.20	1	7	357.53	18210.62	0.00
	0.01	59.43	1	6	407.94	50.93	1	7	358.23	18238.66	0.15
	0.02	69.47	1	5	408.45	51.72	1	7	359.15	18267.81	0.31
	0.03	70.33	1	5	408.97	59.10	1	6	360.97	18300.50	0.49
$b_1 = 0.2$	0.04	71.21	1	5	410.66	60.17	1	6	361.87	18333.90	0.68
$b_2 = 0.15$	0.05	72.16	1	5	411.28	70.86	1	5	364.24	18369.22	0.87
	0.06	88.25	1	4	411.92	72.55	1	5	366.18	18409.65	1.09
	0.07	89.82	1	4	413.04	89.48	1	4	369.71	18453.65	1.33
	0.08	117.55	1	3	415.22	92.45	1	4	373.27	18501.72	1.60
	0.09	120.62	1	3	416.91	123.48	1	3	379.38	18564.00	1.94
	0.00	88.21	1	4	422.05	60.38	1	6	362.08	18373.55	0.00
	0.01	89.66	1	4	423.13	71.23	1	5	365.14	18413.56	0.22
	0.02	117.43	1	3	430.82	72.93	1	5	366.93	18455.94	0.45
	0.03	120.16	1	3	432.75	90.20	1	4	371.64	18508.03	0.73
$b_1 = 0.25$	0.04	179.69	1	2	449.72	120.02	1	3	381.19	18565.63	1.05
$b_2 = 0.2$	0.05	187.10	1	2	455.89	182.21	1	2	395.80	18644.54	1.47
	0.06	411.40	1	1	527.26	406.75	1	1	456.03	18804.58	2.35
	0.07	459.69	1	1	549.63	495.83	1	1	481.34	19036.77	3.61
	0.08	500.00	1	1	565.00	500.00	1	1	490.00	19297.31	5.03
	0.09	500.00	1	1	570.00	500.00	1	1	495.00	19561.50	6.47

complementation between two products leads to the loss of some customers. Furthermore, Table 2 shows that as sensitivity of both products to their own stock levels b_i increases, quantity transferred to the shelf q_i increases and the number of lots transferred from buyer's warehouse to the shelf n_{bi} decreases. Increase in b_i and q_i leads to increase in D_i. This is due to the fact that each product positively depends on its inventory level. Furthermore, increment of b_i leads to increase in the total profit.

It is obvious from Figure 4 that as complementation rate increases, the value of percentage improvement PI increases. Moreover, as sensitivity of each product to its stock level b_i increases, PI increases. Thus, the profit of selling both products on one retailer shelf is more than the profit of selling them in two different retailer stores separately. Furthermore, studying complementary products is more profitable when the items are more sensitive to their stock.

It is obvious from Table 3 and Figure 5 that as sensitivity of product 1 to its stock level b_1 increases, q_1 increases up to $b_1 > 0.26$, where capacity constraint of the first product is activated, and after that, by increasing b_1, q_2 is almost constant. Increase in quantity transferred from buyer's warehouse to the shelf causes higher demand; hence, the profit increases.

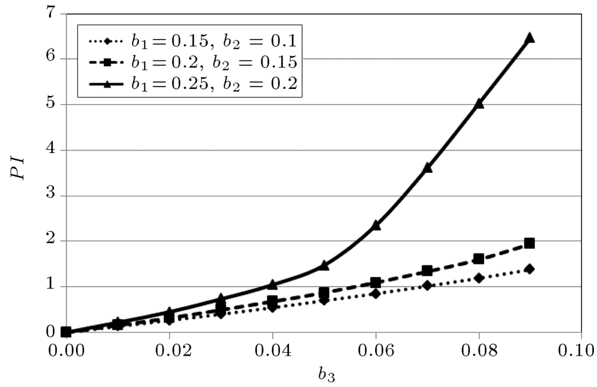

Figure 4. Effect of b_3 on profitability of the proposed model in comparison with the case that complementation rate is ignored.

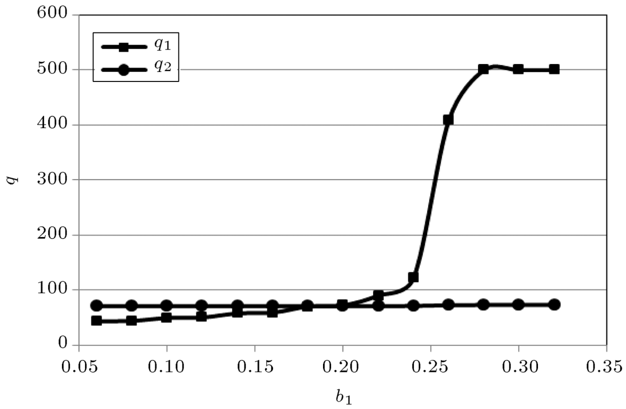

Figure 5. Effect of b_1 on q_i.

When $b_1 < 0.18$, the transferred quantity of product 1, q_1, is smaller than that of product 2, q_2, because sensitivity of product 1 to its stock level is smaller than that of product 2 ($b_1 < b_2$). Given that product 1 and product 2 are complementary, when b_1 increases, demand of both products increases; however, since complementation rate is small, the increased value is small. Table 3 indicates that demand of product 1 is always more than that of product 2 although the sensitivity of product 1 to its stock level is less than that of product 2 when $b_1 < 0.16$.

Table 4 shows that when the price of the first product increases, the transferred quantity of both products increases and their number of transfers decreases. When the ratio of u_1/u_2 is smaller than 1.2, although b_1 is greater than b_2, the transferred quantity of product 2 is more than that of product 1. For $u_1/u_2 > 1.2$, q_1 significantly increases until the capacity constraint of product 1 is activated. In this case, increase in q_1 is considerably greater than in q_2. Furthermore, as the quantity transferred to the shelf for both products increases, their demand increases. Hence, increase in u_1 leads to increase in total profit. Thus, study of complementary products with stock-dependent demand is more profitable when the items are expensive.

As noted earlier, the following managerial insights can be gained from this paper. Increasing the degree of complementation between two products leads to decrease in the number of transferring batches and increase in transferring size in both vendor and buyer levels. This fact leads to lower supply chain cost. In the other words, considering complementary products helps the supply chain components to benefit from the economy of scale. Moreover, displaying complementary products simultaneously on one shelf motivates a buyer, who wants to buy one product, to buy some other products as well. As a result, ignoring complementary relation between products causes decrease in the demand and loss of some customers. Therefore,

Table 3. Sensitivity analysis for parameter b_1.

Parameters		Product 1				Product 2				TP
b_1	b_2	q_1	n_{v1}	n_{b1}	D_1	q_2	n_{v2}	n_{b2}	D_2	
0.06	0.15	43.30	1	8	406.13	70.65	1	5	362.76	18154.47
0.08	0.15	44.00	1	8	407.05	70.66	1	5	362.80	18178.79
0.10	0.15	49.35	1	7	408.47	70.70	1	5	363.07	18204.11
0.12	0.15	50.29	1	7	409.57	70.71	1	5	363.12	18231.93
0.14	0.15	57.70	1	6	411.62	70.76	1	5	363.50	18260.91
0.16	0.15	59.05	1	6	412.99	70.77	1	5	363.57	18293.60
0.18	0.15	70.05	1	5	416.15	70.84	1	5	364.13	18329.27
0.20	0.15	72.16	1	5	417.98	70.86	1	5	364.24	18369.22
0.22	0.15	90.26	1	4	423.40	70.97	1	5	365.16	18418.68
0.24	0.15	122.24	1	3	432.90	71.14	1	5	366.78	18482.98
0.26	0.15	409.01	1	1	509.97	72.54	1	5	381.33	18634.65
0.28	0.15	500.00	1	1	543.65	73.01	1	5	385.95	18900.33
0.30	0.15	500.00	1	1	553.65	73.01	1	5	385.95	19187.83
0.32	0.15	500.00	1	1	563.65	73.01	1	5	385.95	19475.33

Table 4. Sensitivity analysis for parameter u_1.

Parameters			Product 1				Product 2				TP
u_1	u_2	u_1/u_2	q_1	n_{v1}	n_{b1}	D_1	q_2	n_{v2}	n_{b2}	D_2	
10	25	0.40	43.44	1	8	411.25	51.33	1	7	359.87	10090.76
15	25	0.60	44.67	1	8	411.85	58.43	1	6	361.00	12148.07
20	25	0.80	50.88	1	7	413.14	59.33	1	6	361.44	14212.47
25	25	1.00	59.46	1	6	414.91	60.29	1	6	362.02	16284.27
30	25	1.20	72.16	1	5	417.98	70.86	1	5	364.24	18369.22
35	25	1.40	120.11	1	3	427.65	72.47	1	5	366.88	20485.06
40	25	1.60	449.54	1	1	494.46	91.00	1	4	386.13	22795.03
45	25	1.80	500.00	1	1	504.68	93.62	1	4	389.04	25312.59
50	25	2.00	500.00	1	1	506.19	123.82	1	3	393.57	27841.16
55	25	2.20	500.00	1	1	509.35	186.91	1	2	403.04	30378.19
60	25	2.40	500.00	1	1	520.16	403.19	1	1	435.48	32951.36

considering the relation between complementary products makes increase in the total profit.

This study also shows that when items are more sensitive to stock level, selling of complementary products on one retailer shelf leads to more gain. When the sensitivity of one product to its stock level is high, not only the demand of this product, but also the demand of other products increases. Hence, considering the relation between complementary products is more crucial when their demand is sensitive to stock.

6. Conclusions

This paper proposed an integrated model for a two-stage supply chain under vendor-managed inventory with consignment stock agreement. The contribution of this paper to the existing literature of consignment stocking policy was considering complementary products. Two complementary products were studied and the demand for each product was influenced not only by its stock level, but also by stock level of the other product. Both of the products were delivered from a vendor to the buyer in equal sizes. The buyer stocked items in the warehouse and on the shelf. Joint total profit of the vendor and the buyer was maximized. A solution algorithm was proposed to find the optimal transferred quantities and numbers of shipments for both products.

Numerical results showed that increase in sensitivity of each product to inventory level of its complementary product would cause increase in the quantity of transfers and decrease in the number of shipments. Hence, the supply chain components could benefit from advantages of the economy of scale. Furthermore, considering complementary products could motivate customers to buy more and lead to greater demand. Thus, increase in complementation rate led to increase in total system profit. The paper also studied the effect of sensitivity of each product to its stock level. The results indicated that when the sensitivity of one product to its inventory level increased, the transferred quantity of both products and, consequently, their demand increased. Analyzing the price of complementary products showed that as the price of one product increased, the demand of both products and the total profit increased. Thus, study of complementary products can be more profitable when the items are expensive.

The current paper can be extended in the several directions. The proposed model considered two products. It can be extended to any number of products with different complementation rates. Study of substitutable products is another possible extension. The proposed model can also be developed for deteriorating items and other demand functions, e.g., stock- and price-sensitive demands and stochastic demand. Furthermore, multi-vendor multi-buyer is recommended as another topic for future research.

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Appendix A

Table A.1 shows that the percentage error of applying Maclaurin series is small. Therefore, the third term and higher terms can be neglected.

Table A.1. Absolute percentage error for neglecting the terms higher than 2 in Maclaurin series.

α	Exp(α)	$1+\alpha$	Percentage error
–0.01	0.99	0.99	0.01
–0.02	0.98	0.98	0.02
–0.03	0.97	0.97	0.05
–0.04	0.96	0.96	0.08
–0.05	0.95	0.95	0.13
–0.06	0.94	0.94	0.19
–0.07	0.93	0.93	0.26
–0.08	0.92	0.92	0.34
–0.09	0.91	0.91	0.43
–0.1	0.90	0.9	0.53
–0.11	0.90	0.89	0.65
–0.12	0.89	0.88	0.78
–0.13	0.88	0.87	0.92
–0.14	0.87	0.86	1.08

Appendix B

The possible roots of Eq. (22) are:

$$\begin{aligned}
R_1 = -\frac{1}{2}&(((-((L_1q_2+L_2+L_3/q_2)/A_1)^6\\
&+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1))^2)^{1/2}/27)\\
&-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-(L_4q_2+L_5)/2A_1)^{1/3}\\
&-\frac{1}{2}(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1)\\
&-((-((L_1q_2+L_2+L_3/q_2)/A_1)^6\\
&+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1))^2)^{1/2}/27))^{1/3}\\
&+\frac{1}{2}(-3((((-((L_1q_2+L_2+L_3/q_2)/A_1)^6\\
&+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1))^2)^{1/2}/27)\\
&-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-(L_4q_2+L_5)/2A_1)^{1/3}\\
&-(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1)\\
&-((-((L_1q_2+L_2+L_3/q_2)/A_1)^6\\
&+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3\\
&-((L_4q_2+L_5)/2A_1))^2)^{1/2}/27))^{1/3})^2)^{1/2}\\
&-(L_1q_2+L_2+L_3/q_2)/3A_1, \qquad \text{(B.1)}
\end{aligned}$$

$$L_1 = \frac{(\frac{1}{2}b_3 n_{b2} P_1((n_{v2}-1)h_{n2}-h_{v2}) + \frac{1}{2}P_2 n_{b1}((n_{v1}-1)h_{n1}-h_{v1})b_3)}{P_1 P_2}$$

$$L_2 = \frac{(-\frac{n_{v1}n_{b1}h_{n1}}{2} + u_1 b_1 + u_2 b_3 - \frac{h_{d1}}{2} + \frac{h_{n1}}{2})P_1 + \frac{n_{b1}((n_{v1}-1)h_{n1}-h_{v1})a_1}{2}}{P_1}$$

$$L_3 = -b_3(\frac{A_{v2}}{n_{b2}n_{v2}} + \frac{A_{b2}}{n_{b2}} + S_2)$$

$$L_4 = (\frac{A_{v1}}{n_{b1}n_{v1}} + \frac{A_{b1}}{n_{b1}} + S_1)b_3$$

$$L_5 = (\frac{A_{v1}}{n_{b1}n_{v1}} + \frac{A_{b1}}{n_{b1}} + S_1)a_1.$$

Box B.I

$$\begin{aligned} R_2 =& (\frac{1}{27}(-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2} \\ &-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^{1/3} \\ &+(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1 \\ &-((-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2})/27)^{1/3} \\ &-(L_1q_2+L_2+L_3/q_2)/3A_1, \end{aligned} \tag{B.2}$$

$$\begin{aligned} R_3 =& -\frac{1}{2}(((-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2}/27) \\ &-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^{1/3} \\ &-\frac{1}{2}(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1 \\ &-(-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2}/27)^{1/3} \\ &-\frac{1}{2}(-3(((-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2}/27 \\ &-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^{1/3} \\ &-(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1 \\ &-(-((L_1q_2+L_2+L_3/q_2)/A_1)^6 \\ &+729(-((L_1q_2+L_2+L_3/q_2)/3A_1)^3 \\ &-(L_4q_2+L_5)/2A_1)^2)^{1/2}/27)^{1/3})^2)^{1/2} \\ &-(L_1q_2+L_2+L_3/q_2)/3A_1, \end{aligned} \tag{B.3}$$

L_1 to L_5 are defined in Box B.I. Expressions (B.1), (B.2), and (B.3) can be real or complex.

Biographies

Mahya Hemmati received her BS degree in Industrial Engineering from Isfahan University of Technology, Isfahan, Iran, in 2014 and her MS degree in Industrial Engineering from Amirkabir University of Technology, Tehran, Iran, in 2016. Her research interests include mathematical optimization, supply chain management, inventory management, and operations research.

Seyyed Mohammad Taghi Fatemi Ghomi was born in Ghom, Iran, on March 11, 1952. He received his BS degree in Industrial Engineering from Sharif University of Technology, Tehran, in 1973 and the PhD degree in Industrial Engineering from University of Bradford, England, in 1980. He worked as planning and control expert in the group of construction and cement industries, a group in the Organization of National Industries of Iran, during the years 1980-1983. Also, he founded the Department of Industrial Training in the aforementioned organization in 1981. He joined Amirkabir University of Technology, Tehran, Iran, as a faculty member in 1983. He is the author and coauthor of more than 360 technical papers and the author of 6 books on the topics in the area of industrial engineering. His research and teaching interests are in stochastic activity networks, production planning, scheduling, queueing theory, statistical quality control, and time series analysis and forecasting. He is currently professor in the Department of Industrial Engineering at Amirkabir University of Technology, Tehran, Iran, where he was recognized as one of the best researchers of the years 2004 and 2006. He was also recognized as one of the best professors of Iran in the year 2010 by the Ministry of Science and Technology, and in the year 2014 by Amirkabir University of Technology, Tehran, Iran.

Mohsen Sheikh Sajadieh received his PhD degree in Industrial Engineering, in 2009, from Sharif University of Technology, Tehran, Iran. He is now assistant professor at Amirkabir University of Technology. He is the author and coauthor of more than 30 technical papers and the author of 5 books on the topics in the area of industrial engineering. His research area is focused on supply chain. He is also interested in inventory control, stochastic modeling, and mathematical optimization.