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Some improved interactive aggregation operators under interval-valued intuitionistic fuzzy environment and their application to decision making process

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Abstract. The objective of this manuscript is to present an improved aggregator operator by taking into account the effect of an unknown degree (hesitancy degree) in an Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) environment. For this, firstly, the shortcomings, of the existing operators are addressed and, then, some improved operational laws on IVIFSs have been introduced. Based on these laws, aggregation operators, namely, an Interval-Valued Intuitionistic Fuzzy Hamacher Interactive Weighted Averaging (IVIFHIWA), Ordered Weighted Averaging (IVIFHIOWA), and Hybrid Weighted Averaging (IVIFHIHWA), have been proposed. Various properties related to these operators are also investigated. Furthermore, based on these operators, an approach to deal with Multi-Criteria Decision Making (MCDM) problem is developed. Finally, a practical example is provided to illustrate the decision making process.

1. Introduction

With the growing complexities of the systems day by day, it is difficult for the decision maker to take a decision within a reasonable time by using uncertain, imprecise and vague information. Intuitionistic Fuzzy Sets (IFS) [1] theory is one of the most permissible theories to handle the uncertainties and impreciseness in the data in comparison to the crisp or probability theory [2-6]. But, in some situations, it is difficult to give the preference of an object in terms of a point value and, therefore, it is convenient to express the decision makers information/preferences in the form of interval values, hence called Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) [7]. Nowadays, decision making is one of the most significant and omnipresent human activities in business, service, manufacturing, selection of products, etc. It is understandable that the different criteria in the decision problem are likely to play different roles in reaching a final decision; thus, the primary objective in the phase of decision making is the information aggregation process. For this, Yager [8] proposed the Ordered Weighted Average (OWA) operator by giving some weights to all the inputs according to their ranking positions. Based on this pioneering work, many extensions have appeared. As IVIFSs are much easier to handle the fuzzy decision information up to the desired degree of accuracy, some researchers have applied the interval-valued intuitionistic fuzzy set theory to the field of decision making. For instance, Xu and Chen [9], and Xu [10] developed some arithmetic and geometric aggregation operators, namely, interval-valued intuitionistic fuzzy weighted averaging operator and geometric operator, respectively, for aggregating the interval-valued intuitionistic

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fuzzy information. Furthermore, Xu and Chen [11], and Wei and Wang [12], respectively, developed ordered weighted and hybrid weighted geometric aggregator operators in the interval-valued intuitionistic fuzzy environment. Wang and Liu [13,14] investigated these aggregation operators by using Einstein operations. Later on, Wang and Liu [15,16] extended these operators from IFSs to the IVIFSs environments. Wei and Zhao [17] proposed induced hesitant and Zhao et al. [18] developed some hesitant triangular aggregation operators under interval-valued Einstein operations. Apart from that, various researchers have paid more attention to decision-making process for aggregating the different alternatives using different aggregation operators [19-34] and their corresponding references.

Almost all the above studies are reliable under the restriction that the grade of membership or non-membership of Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs) is non-zero. For instance, consider two IFSs or IVIFSs, A and B, such that either $\mu_A = 0$ and $\mu_B \neq 0$ or $\nu_A = 0$ and $\nu_B \neq 0$; then, based on an aggregated operator proposed by Xu [3], Wang and Liu [13,15], Zhao [18], Zhang [23] and Liu [25]. The overall aggregated grade of either membership or non-membership values is zero, respectively, for geometric or an averaging aggregated operator. In other words, we can say that the effects of the other grades of either membership or non-membership on a corresponding geometric or an averaging aggregator operator do not play any significant role during the aggregation process. Furthermore, it has been concluded from the above aggregation process that the grades of overall membership (non-membership) functions are independent of their corresponding grades of non-membership (membership) functions. Thus, changing any values in the grades does not affect the overall aggregation process. Therefore, the corresponding results are undesirable and get an unreasonable preference order of the alternatives. Hence, there is a need to modify the existing operations by properly considering the degrees of membership functions.

Thus, the objective of this manuscript is to present some series of averaging aggregation operators in an IVIFSs environment. For it, a new operational law on different IVIFNs has been proposed by taking the interaction between the pair of membership and non-membership functions. Based on these new operational laws, weighted aggregated operators, namely Interval-Valued Intuitionistic Fuzzy (IVIF) Hamacher Interactive Weighted Aggregation (IVIFHIWA), IVIF Hamacher Interactive Ordered Weighted Aggregation (IVIFHIOWA), and IVIF Hamacher Interactive Hybrid Weighted Aggregation (IVIFHIHWA), have been proposed by properly handling the shortcoming of the existing operators. The main significance of these operators is that the influence of the degree of non-membership function is less than that of the membership functions and, hence, these operators, are more optimistic than the others existing in the literature, especially when one of the non-membership degrees is zero. Furthermore, these operators have been tested on the problem of MCDM, where the most desirable alternative is computed under the set of different criteria. Finally, the computed results are compared with the results of the existing operators for showing the optimistic nature of the operation.

The rest of the manuscript is organized as follows. In Section 2, some basic definitions related to IVIFSs and their corresponding aggregation operators are summarized along with their shortcomings. In Section 3, some new operational laws are defined and, then, we develop some averaging aggregation operators, namely, IVIFHIWA, IVIFHIOWA, and IVIFHIHWA. Desirable properties corresponding to these operators, such as idempotency, boundedness, commutativity, homogeneity, etc., are also discussed in this section. In section 4, a method based on the proposed operators for solving MCDM problems, where individual assessment is provided as IVIFNs, is presented. An illustrative example has been provided related to MCDM problem, and comparison of the results with the existing methods is given in Section 5. Finally, some concrete conclusion of the paper has been summarized in section 6.

2. Preliminaries

2.1. Intuitionistic and interval-valued intuitionistic fuzzy sets

An Intuitionistic Fuzzy Set (IFS) A [1] in a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is given by:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where $\mu_A(x)$ and $\nu_A(x)$ are respectively the grades of membership and non-membership of an element x with the conditions that $0 \leq \mu_A(x), \nu_A(x) \leq 1$ and $\mu_A(x) + \nu_A(x) \leq 1$, while an interval-valued intuitionistic fuzzy set (IVIFSs) is defined as [7]:

$$A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle \mid x \in X\},$$

where $\mu_A^U(x) + \nu_A^U(x) \leq 1, 0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1, 0 \leq \nu_A^L(x) \leq \nu_A^U(x) \leq 1$. Clearly, for every $x \in X$, if:

$$\mu_A(x) = \mu_A^L(x) = \mu_A^U(x), \quad \nu_A(x) = \nu_A^L(x) = \nu_A^U(x),$$

then, IVIFS is reduced to an IFS.

In order to compare the two Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs), Xu [3] defined the score and accuracy function as $S(\alpha) = \frac{a+b-c-d}{2}$ and $H(\alpha) = \frac{a+b+c+d}{2}$, respectively, for an IVIFN $\alpha = \langle [a, b], [c, d]\rangle$. Later on, Wang et al. [35] introduced two new functions called the membership uncertainty index, $G(\alpha) = b + d - a - c$, and the hesitation uncertainty index, $T(\alpha) = b + c - a - d$, for comparing the two distinct IVIFNs. Based on these functions, a prioritized comparison method for any two IVIFNs $\alpha = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\beta = \langle [a_2, b_2], [c_2, d_2] \rangle$ is defined as:

1. If $S(\alpha) > S(\beta)$ then $\alpha \succ \beta$;
2. If $S(\alpha) < S(\beta)$ then $\alpha \prec \beta$;
3. If $S(\alpha) = S(\beta)$ then:
   - i. If $H(\alpha) > H(\beta)$ then $\alpha \succ \beta$;
   - ii. If $H(\alpha) < H(\beta)$ then $\alpha \prec \beta$;
   - iii. If $H(\alpha) = H(\beta)$, then:
     - a. If $T(\alpha) > T(\beta)$, then $\alpha \succ \beta$;
     - b. If $T(\alpha) < T(\beta)$, then $\alpha \prec \beta$;
     - c. If $T(\alpha) = T(\beta)$, then:
       - If $G(\alpha) > G(\beta)$, then $\alpha \succ \beta$;
       - If $G(\alpha) < G(\beta)$, then $\alpha \prec \beta$;
       - If $G(\alpha) = G(\beta)$, then $\alpha = \beta$.

2.2. Hamacher t-norm and t-conorm

T-norm (t) and t-cornorm (T) are used to define the union and intersection of two IFSs (IVIFSs), A and B, as follows:

$$A \cap B = \{\langle x, t(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\},$$

$$A \cup B = \{\langle x, T(\mu_A(x), \mu_B(x)), t(\nu_A(x), \nu_B(x)) \rangle \mid x \in X\}.$$

Hamacher [36] proposed a more generalized t-norm and t-conorm by defining:

$$t(x, y) = \frac{xy}{\gamma + (1-\gamma)(x + y - xy)},$$

and:

$$T(x, y) = \frac{x + y - xy - (1-\gamma)xy}{1 - (1-\gamma)xy},$$

respectively. It is clearly seen from these norms that when $\gamma = 1$, the equations are reduced to algebraic t-norm and t-cornorm, $t(x, y) = xy$ and $T(x, y) = x + y - xy$. Similarly, when $\gamma = 2$, they are respectively reduced to Einstein t-norm and t-cornorm as $t(x, y) = \frac{xy}{1+(1-x)(1-y)}$ and $T(x, y) = \frac{x+y}{1+xy}$. Based on these norms, different aggregation operators have been proposed by Liu [25] for aggregating the different IVIFNs, $\alpha_i = \langle [a_i, b_i], [c_i, d_i] \rangle, (i = 1, 2, \ldots, n)$, by using weight vector, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, of α_i such that $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$ as:

The Interval-Valued Intuitionistic Fuzzy Hamacher Weighted Averaging (IVIFHWA) operator is calculated as follows:

$$\text{IVIFHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \ldots \oplus \omega_n\alpha_n$$

$$= \Bigg\langle \Bigg[\frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i} - \prod\limits_{i=1}^{n}(1-a_i)^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-a_i)^{\omega_i}}, \frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i} - \prod\limits_{i=1}^{n}(1-b_i)^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-b_i)^{\omega_i}} \Bigg]$$

$$\Bigg[\frac{\gamma\prod\limits_{i=1}^{n}c_i^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)(1-c_i))^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}c_i^{\omega_i}}, \frac{\gamma\prod\limits_{i=1}^{n}d_i^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)(1-d_i))^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}d_i^{\omega_i}} \Bigg] \Bigg\rangle.$$

The Interval-Valued Intuitionistic Fuzzy Hamacher Ordered Weighted Averaging (IVIFHOWA) operator is calculated as follows:

$$\text{IVIFHOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= \omega_1\alpha_{\delta(1)} \oplus \omega_2\alpha_{\delta(2)} \oplus \ldots \oplus \omega_n\alpha_{\delta(n)}$$

$$= \Bigg\langle \Bigg[\frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i}}, \frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i}} \Bigg]$$

$$\Bigg[\frac{\gamma\prod\limits_{i=1}^{n}c_{\delta(i)}^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)(1-c_{\delta(i)}))^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}c_{\delta(i)}^{\omega_i}}, \frac{\gamma\prod\limits_{i=1}^{n}d_{\delta(i)}^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)(1-d_{\delta(i)}))^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}d_{\delta(i)}^{\omega_i}} \Bigg] \Bigg\rangle,$$

where $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\alpha_{\delta(i-1)} \geq \alpha_{\delta(i)}$ for all $i = 1, 2, \ldots, n$.

The Interval-Valued Intuitionistic Fuzzy Hamacher Hybrid Weighted Averaging (IVIFHHWA) operator is calculated as follows:

$$\text{IVIFHHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$=\omega_1\dot{\alpha}_{\sigma(1)}\oplus\omega_2\dot{\alpha}_{\sigma(2)}\oplus\ldots\oplus\omega_n\dot{\alpha}_{\sigma(n)}$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{n}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}},$$

$$\frac{\prod_{i=1}^{n}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}}\Bigg]$$

$$\Bigg[\frac{\gamma\prod_{i=1}^{n}\dot{c}_{\sigma(i)}^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\dot{c}_{\sigma(i)}))^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}\dot{c}_{\sigma(i)}^{\omega_i}},$$

$$\frac{\gamma\prod_{i=1}^{n}\dot{d}_{\sigma(i)}^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\dot{d}_{\sigma(i)}))^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}\dot{d}_{\sigma(i)}^{\omega_i}}\Bigg]\Bigg\rangle,$$

where $\dot{\alpha}_{\sigma(i)}$ is the ith largest weighted intuitionistic fuzzy value $\dot{\alpha}_i$ ($\dot{\alpha}_i=nw_i\alpha_i,\ i=1,2,\ldots,n$).

2.3. Shortcomings of the existing work

The following shortcomings have been observed in the operators, which prevent the existing operators from giving the sufficient information in the phase of the aggregation process.

Example 2.1.

Let $\alpha_1=\langle[0.23,0.33],[0,0]\rangle$, $\alpha_2=\langle[0.65,0.72],[0.22,0.27]\rangle$, $\alpha_3=\langle[0.31,0.35],[0.55,0.58]\rangle$, and $\alpha_4=\langle[0.17,0.23],[0.65,0.69]\rangle$ be four IVIFNs and $\omega=(0.2,0.3,0.4,0.1)^T$ be the standardized weight vector of these numbers; then, by using IVIFHWA operator, to aggregate these IVIFNs, we get the aggregated IVIFN as $\langle[0.4139,0.4834],[0,0]\rangle$ corresponding to $\gamma=1$ and $\langle[0.4010,0.4703],[0,0]\rangle$ corresponding to $\gamma=2$. From these results, it is seen that the degree of non-membership is zero and is independent of the parameter γ. Furthermore, this degree is independent of the degree of the other non-memberships (those which are nonzero in α_i's), which hence play an insignificant role during the aggregation process.

Example 2.2.

Let $\alpha_1=\langle[0.23,0.33],[0.35,0.45]\rangle$, $\alpha_2=\langle[0.45,0.55],[0.23,0.28]\rangle$, $\alpha_3=\langle[0.65,0.73],[0.17,0.21]\rangle$ and $\alpha_4=\langle[0.50,0.60],[0.20,0.30]\rangle$ be four IVIFNs and $\omega=(0.2,0.3,0.4,\ 0.1)^T$ be the standardized weight vector of these numbers. Then, based on IVIFHWA operator we get the aggregated IVIFNs as $\langle[0.5137,0.6074],[0.2186,0.2763]\rangle$ by taking $\gamma=1$ and $\langle[0.5060,0.6011],[0.2196,0.2783]\rangle$ when $\gamma=2$. On the other hand, if we replace the IVIFNs α_2 and α_3 with:

$$\beta_2=\langle[0.32,0.36],[0.23,0.28]\rangle,$$

and:

$$\beta_3=\langle[0.37,0.40],[0.17,0.21]\rangle,$$

then their corresponding aggregated IVIFNs become $\langle[0.3443,0.3995],\ [0.2186,0.2763]\rangle$ when $\gamma=1$ and $\langle[0.3422,0.3973],\ [0.2196,0.2783]\rangle$ when $\gamma=2$. Hence, it is seen that the degree of non-membership values of aggregator IVIFN becomes independent of the change of the degree of membership values. Therefore, it is inconsistent and, hence, does not give correct information to the decision maker.

Therefore, the existing operators, as proposed by Liu [25], are invalid to rank the alternative and, hence, there is a necessity to pay more attention to these issues.

3. Improved operational laws for intuitionistic fuzzy Hamacher aggregation operators

Here, we define some new operational laws for IVIFNs, which overcome the shortcomings of the existing operators as follows.

Definition 3.1.

Let $\alpha=\langle[a,b],[c,d]\rangle$ and $\alpha_i=\langle[a_i,b_i],[c_i,d_i]\rangle$ ($i=1,2$) be a collection of the IVIFNs and $\lambda>0$ be a real number; then, the new operational rules for these IVIFNs are defined as follows:

(i):

$$\alpha_1\oplus\alpha_2=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]-\prod_{i=1}^{2}(1-a_i)}{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-a_i)},$$

$$\frac{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]-\prod_{i=1}^{2}(1-b_i)}{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-b_i)}\Bigg],$$

$$\Bigg[\frac{\gamma\prod_{i=1}^{2}(1-a_i)-\gamma\prod_{i=1}^{2}\left[1-a_i-c_i\right]}{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-a_i)},$$

$$\frac{\gamma\prod_{i=1}^{2}(1-b_i)-\gamma\prod_{i=1}^{2}\left[1-b_i-d_i\right]}{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-b_i)}\Bigg]\Bigg\rangle.$$

(ii):

$$\alpha_1\otimes\alpha_2=\Bigg\langle\Bigg[\frac{\gamma\prod_{i=1}^{2}(1-a_i)-\gamma\prod_{i=1}^{2}\left[1-a_i-c_i\right]}{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-a_i)},$$

$$\frac{\gamma\prod_{i=1}^{2}(1-b_i)-\gamma\prod_{i=1}^{2}\left[1-b_i-d_i\right]}{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-b_i)}\Bigg],$$

$$\Bigg[\frac{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]-\prod_{i=1}^{2}(1-a_i)}{\prod_{i=1}^{2}\left[1+(\gamma-1)a_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-a_i)},$$

$$\frac{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]-\prod_{i=1}^{2}(1-b_i)}{\prod_{i=1}^{2}\left[1+(\gamma-1)b_i\right]+(\gamma-1)\prod_{i=1}^{2}(1-b_i)}\Bigg]\Bigg\rangle.$$

(iii):

$$\lambda\alpha=\Bigg\langle\Bigg[\frac{\left[1+(\gamma-1)a\right]^{\lambda}-\left[1-a\right]^{\lambda}}{\left[1+(\gamma-1)a\right]^{\lambda}+(\gamma-1)\left[1-a\right]^{\lambda}},$$

$$\frac{\left[1+(\gamma-1)b\right]^{\lambda}-\left[1-b\right]^{\lambda}}{\left[1+(\gamma-1)b\right]^{\lambda}+(\gamma-1)\left[1-b\right]^{\lambda}}\Bigg],$$

$$\Bigg[\frac{\gamma\left[1-a\right]^{\lambda}-\gamma\left[1-a-c\right]^{\lambda}}{\left[1+(\gamma-1)a\right]^{\lambda}+(\gamma-1)\left[1-a\right]^{\lambda}},$$

$$\frac{\gamma\left[1-b\right]^{\lambda}-\gamma\left[1-b-d\right]^{\lambda}}{\left[1+(\gamma-1)b\right]^{\lambda}+(\gamma-1)\left[1-b\right]^{\lambda}}\Bigg]\Bigg\rangle.$$

(iv):

$$\alpha^{\lambda}=\Bigg\langle\Bigg[\frac{\gamma\left[1-a\right]^{\lambda}-\gamma\left[1-a-c\right]^{\lambda}}{\left[1+(\gamma-1)a\right]^{\lambda}+(\gamma-1)\left[1-a\right]^{\lambda}},$$

$$\frac{\gamma\left[1-b\right]^{\lambda}-\gamma\left[1-b-d\right]^{\lambda}}{\left[1+(\gamma-1)b\right]^{\lambda}+(\gamma-1)\left[1-b\right]^{\lambda}}\Bigg]$$

$$\Bigg[\frac{\left[1+(\gamma-1)a\right]^{\lambda}-\left[1-a\right]^{\lambda}}{\left[1+(\gamma-1)a\right]^{\lambda}+(\gamma-1)\left[1-a\right]^{\lambda}},$$

$$\frac{\left[1+(\gamma-1)b\right]^{\lambda}-\left[1-b\right]^{\lambda}}{\left[1+(\gamma-1)b\right]^{\lambda}+(\gamma-1)\left[1-b\right]^{\lambda}}\Bigg]\Bigg\rangle.$$

As it is clearly observed from the above definition, the sum of IVIFNs becomes more optimistic than the existing sum because the non-membership degree of $\alpha_1\oplus\alpha_2$ contains the pairs of membership and non-membership, i.e., $a_i\cdot c_i$ and $b_i\cdot d_i$, while membership function does not. Hence, the attitude is more inclined towards the membership function than the non-membership one; therefore, the decision is more optimistic. Now, based on these operations, averaging aggregation operators have been proposed as follows.

3.1. Interval-valued Intuitionistic fuzzy Hamacher Interactive Weighting Averaging operator

Definition 3.2.

Let $\alpha_i,(i=1,2,\ldots,n)$ be the collection of IVIFNs, and IVIFHIWA : $\Omega^n\longrightarrow\Omega$, if:

$$\text{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\omega_1\alpha_1\oplus\omega_2\alpha_2\oplus\ldots\oplus\omega_n\alpha_n,$$

where Ω is the set of IVIFNs and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of α_i with $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$; therefore, IVIFHIWA is called an interval valued intuitionistic fuzzy Hamacher interactive weighting averaging operator.

Theorem 3.1.

Let $\alpha_i=\langle[a_i,b_i],[c_i,d_i]\rangle,(i=1,2,\ldots,n)$ be the collection of IVIFNs, then:

$$\text{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}-\prod_{i=1}^{n}(1-a_i)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}},$$

$$\frac{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}-\prod_{i=1}^{n}(1-b_i)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}}\Bigg]$$

$$\Bigg[\frac{\gamma\left\{\prod_{i=1}^{n}(1-a_i)^{\omega_i}-\prod_{i=1}^{n}(1-a_i-c_i)^{\omega_i}\right\}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}},$$

$$\frac{\gamma\left\{\prod_{i=1}^{n}(1-b_i)^{\omega_i}-\prod_{i=1}^{n}(1-b_i-d_i)^{\omega_i}\right\}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}}\Bigg]\Bigg\rangle. \quad (1)$$

Proof

When $n=1$ and $\omega_1=1$, we have:

$$\text{IVIFHIWA}(\alpha_1)=\omega_1\alpha_1=\langle[a_1,b_1],[c_1,d_1]\rangle$$

$$=\Bigg\langle\Bigg[\frac{(1+(\gamma-1)a_1)^1-(1-a_1)^1}{(1+(\gamma-1)a_1)^1+(\gamma-1)(1-a_1)^1},$$

$$\frac{(1+(\gamma-1)b_1)^1-(1-b_1)^1}{(1+(\gamma-1)b_1)^1+(\gamma-1)(1-b_1)^1}\Bigg],$$

$$\Bigg[\frac{\gamma\{(1-a_1)^1-(1-a_1-c_1)^1\}}{(1+(\gamma-1)a_1)^1+(\gamma-1)(1-a_1)^1},$$

$$\frac{\gamma\{(1-b_1)^1-(1-b_1-d_1)^1\}}{(1+(\gamma-1)b_1)^1+(\gamma-1)(1-b_1)^1}\Bigg]\Bigg\rangle.$$

Thus, Eq. (1) holds for $n=1$. Assume that Eq. (1) holds for $n=k$, i.e.:

$$\text{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_k)$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}-\prod_{i=1}^{k}(1-a_i)^{\omega_i}}{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-a_i)^{\omega_i}},$$

$$\frac{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}-\prod_{i=1}^{k}(1-b_i)^{\omega_i}}{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-b_i)^{\omega_i}}\Bigg]$$

$$\Bigg[\frac{\gamma\left\{\prod_{i=1}^{k}(1-a_i)^{\omega_i}-\prod_{i=1}^{k}(1-a_i-c_i)^{\omega_i}\right\}}{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-a_i)^{\omega_i}},$$

$$\frac{\gamma\left\{\prod_{i=1}^{k}(1-b_i)^{\omega_i}-\prod_{i=1}^{k}(1-b_i-d_i)^{\omega_i}\right\}}{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-b_i)^{\omega_i}}\Bigg]\Bigg\rangle.$$

Then, when $n=k+1$, we have:

$$\text{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_{k+1})=\bigoplus_{i=1}^{k+1}\omega_i\alpha_i$$

$$=\text{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_k)\oplus\omega_{k+1}\alpha_{k+1}$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}-\prod_{i=1}^{k}(1-a_i)^{\omega_i}}{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-a_i)^{\omega_i}},$$

$$\frac{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}-\prod_{i=1}^{k}(1-b_i)^{\omega_i}}{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-b_i)^{\omega_i}}\Bigg],$$

$$\Bigg[\frac{\gamma\left\{\prod_{i=1}^{k}(1-a_i)^{\omega_i}-\prod_{i=1}^{k}(1-a_i-c_i)^{\omega_i}\right\}}{\prod_{i=1}^{k}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-a_i)^{\omega_i}},$$

$$\frac{\gamma\left\{\prod_{i=1}^{k}(1-b_i)^{\omega_i}-\prod_{i=1}^{k}(1-b_i-d_i)^{\omega_i}\right\}}{\prod_{i=1}^{k}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k}(1-b_i)^{\omega_i}}\Bigg]\Bigg\rangle$$

$$\oplus\Bigg\langle\Bigg[\frac{(1+(\gamma-1)a_{k+1})^{w_{k+1}}-(1-a_{k+1})^{w_{k+1}}}{(1+(\gamma-1)a_{k+1})^{w_{k+1}}+(\gamma-1)(1-a_{k+1})^{w_{k+1}}},$$

$$\frac{(1+(\gamma-1)b_{k+1})^{w_{k+1}}-(1-b_{k+1})^{w_{k+1}}}{(1+(\gamma-1)b_{k+1})^{w_{k+1}}+(\gamma-1)(1-b_{k+1})^{w_{k+1}}}\Bigg],$$

$$\Bigg[\frac{\gamma\{(1-a_{k+1})^{w_{k+1}}-(1-a_{k+1}-c_{k+1})^{w_{k+1}}\}}{(1+(\gamma-1)a_{k+1})^{w_{k+1}}+(\gamma-1)(1-a_{k+1})^{w_{k+1}}},$$

$$\frac{\gamma\{(1-b_{k+1})^{w_{k+1}}-(1-b_{k+1}-d_{k+1})^{w_{k+1}}\}}{(1+(\gamma-1)b_{k+1})^{w_{k+1}}+(\gamma-1)(1-b_{k+1})^{w_{k+1}}}\Bigg]\Bigg\rangle$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{k+1}(1+(\gamma-1)a_i)^{\omega_i}-\prod_{i=1}^{k+1}(1-a_i)^{\omega_i}}{\prod_{i=1}^{k+1}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k+1}(1-a_i)^{\omega_i}},$$

$$\frac{\prod_{i=1}^{k+1}(1+(\gamma-1)b_i)^{\omega_i}-\prod_{i=1}^{k+1}(1-b_i)^{\omega_i}}{\prod_{i=1}^{k+1}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{k+1}(1-b_i)^{\omega_i}}\Bigg].$$

Thus results are true for $n=k+1$; hence, by the principle of mathematical induction, results are true for all $n\in\mathbb{Z}^+$.□

Example 3.1

If we apply the proposed IVIFHIWA operator to aggregate the different IVIFNs as given in Example 2.1, then we get aggregated IVIFNs, as $\langle[0.4139, 0.4834], [0.3886, 0.4545]\rangle$ when $\gamma=1$ and $\langle[0.4010, 0.4703], [0.3972, 0.4660]\rangle$ when $\gamma=2$. From this, it is seen that the degree of non-membership function is non-zero; even non-membership function of one of the IVIFNs is zero. Thus, non-membership function of IVIFNs plays a dominant role during the aggregation process by the proposed operator.

Example 3.2

If we apply the proposed IVIFHIWA operator to aggregate the different IVIFNs as given in Example 2.2, then we get aggregated IVIFN, $\langle[0.5137, 0.6074], [0.2196, 0.2807]\rangle$ when $\gamma = 1$ and $\langle[0.5060, 0.6011], [0.2231, 0.2852]\rangle$ when $\gamma = 2$. On the other hand, if we apply the proposed aggregated operator to modified IVIFNs then we get IVIFHIWA$(\alpha_1, \beta_2, \beta_3, \alpha_4) = \langle[0.3443, 0.3995], [0.2257, 0.3042]\rangle$ for $\gamma = 1$ and $\langle[0.3422, 0.3973], [0.2264, 0.3053]\rangle$ for $\gamma = 2$. Thus, the change of membership function will affect the degree of non-membership functions and is non-zero. Therefore, there is a proper interaction between the degrees of membership and non-membership functions and, hence, the results are consistent and more practical than the results of the existing operators.

Lemma 3.1 [3]

Let α_i, $\omega_i > 0$ for $i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n} \omega_i = 1$, then:

$$\prod_{i=1}^{n} \alpha_i^{\omega_i} \leq \sum_{i=1}^{n} \omega_i \alpha_i,$$

which equality holds if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n$.

Corollary 3.1

The IVIFHWA and IVIFHIWA operators have the following relation:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IVIFHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where α_i $(i = 1, 2, \ldots, n)$ is a collection of IVIFNs. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of α_i such that $\omega_i > 0$, $i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n} \omega_i = 1$

Proof

Let IVIFHIWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle[a_\alpha^p, b_\alpha^p], [c_\alpha^p, d_\alpha^p]\rangle = \alpha^p$ and IVIFHWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle[a_\alpha, b_\alpha], [c_\alpha, d_\alpha]\rangle = \alpha$. Since:

$$\prod_{i=1}^{n}(1 + (\gamma - 1)a_i)^{\omega_i} + (\gamma - 1)\prod_{i=1}^{n}(1 - a_i)^{\omega_i} \leq \sum_{i=1}^{n} \omega_i(1 + (\gamma - 1)a_i) + (\gamma - 1)\sum_{i=1}^{n} \omega_i(1 - a_i) = \gamma,$$

$$\frac{\gamma\{\prod\limits_{i=1}^{n}(1 - a_i)^{\omega_i} - \prod\limits_{i=1}^{n}(1 - a_i - c_i)^{\omega_i}\}}{\prod\limits_{i=1}^{n}(1 + (\gamma - 1)a_i)^{\omega_i} + (\gamma - 1)\prod\limits_{i=1}^{n}(1 - a_i)^{\omega_i}} \geq \prod_{i=1}^{n}(1 - a_i)^{\omega_i} - \prod_{i=1}^{n}(1 - a_i - c_i)^{\omega_i}$$

$$\geq \frac{\gamma \prod\limits_{i=1}^{n} c_i^{\omega_i}}{\prod\limits_{i=1}^{n}(1 + (\gamma - 1)(1 - c_i))^{\omega_i} + (\gamma - 1)\prod\limits_{i=1}^{n} c_i^{\omega_i}}.$$

Thus $a_\alpha^p = a_\alpha$ and $c_\alpha^p \geq c_\alpha$, where equality holds if and only if $a_1 = a_2 = \ldots = a_n$ and $c_1 = c_2 = \ldots = c_n$. Similarly, $b_\alpha^p = b_\alpha$ and $d_\alpha^p \geq d_\alpha$. Thus:

$$S(\alpha^p) = \frac{a_\alpha^p + b_\alpha^p - c_\alpha^p - d_\alpha^p}{2} \leq \frac{a_\alpha + b_\alpha - c_\alpha - d_\alpha}{2} = S(\alpha).$$

If $S(\alpha^p) < S(\alpha)$, then for every ω, we have:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) < \text{IVIFHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

If $S(\alpha^p) = S(\alpha)$, i.e.: $\frac{a_\alpha^p + b_\alpha^p - c_\alpha^p - d_\alpha^p}{2} = \frac{a_\alpha + b_\alpha - c_\alpha - d_\alpha}{2}$ then, by the condition $c_\alpha^p \geq c_\alpha$ and $d_\alpha^p \geq d_\alpha$, we have $a_\alpha^p = a_\alpha$, $b_\alpha^p = b_\alpha$, $c_\alpha^p = c_\alpha$ and $d_\alpha^p = d_\alpha$; thus, the accuracy function is: $H(\alpha^p) = \frac{a_\alpha^p + b_\alpha^p + c_\alpha^p + d_\alpha^p}{2} = \frac{a_\alpha + b_\alpha + c_\alpha + d_\alpha}{2} = H(\alpha)$. Therefore, in this case, from the definition of score function, it follows that:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{IVIFHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Hence:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IVIFHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where that equality holds if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n$.

Therefore, it has been concluded from Corollary 3.1 that the proposed IVIFHIWA operator shows the decision maker's attitude more optimistically than the existing IVIFHWA operator [25] during the aggregation process.

Theorem 3.2

If $\alpha_i = \langle[a_i, b_i], [c_i, d_i]\rangle$, $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs, then the aggregated value by IVIFHIWA operator is also an IVIFN; i.e.:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in IVIFN.$$

Proof

Since $\alpha_i = \langle[a_i, b_i], [c_i, d_i]\rangle$ is an IVIFN, for $i = 1, 2, \ldots, n$, by definition of IVIFN, we have:

$$0 \leq a_i, b_i, c_i, d_i \leq 1 \quad \text{and} \quad b_i + d_i \leq 1$$

IVIFHIWA$(\alpha_1, \ldots, \alpha_n) = \langle[a_\alpha^p, b_\alpha^p], [c_\alpha^p, d_\alpha^p]\rangle$ as:

$$\frac{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)(1-a_i)^{\omega_i}} = 1$$

$$- \frac{\gamma \prod_{i=1}^n (1-a_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}$$

$$\leq 1 - \prod_{i=1}^n (1-a_i)^{\omega_i} \leq 1.$$

Also:

$$1 + (\gamma-1)a_i \geq (1-a_i)$$

$$\Leftrightarrow \prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i)^{\omega_i} \geq 0$$

$$\Leftrightarrow \frac{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}} \geq 0.$$

Thus, $0 \leq a_\alpha^p \leq 1$. On the other hand:

$$\frac{\gamma\{\prod_{i=1}^n (1-a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i-c_i)^{\omega_i}\}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}$$

$$\leq \frac{\gamma \prod_{i=1}^n (1-a_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}$$

$$\leq \prod_{i=1}^n (1-a_i)^{\omega_i} \leq 1.$$

Also:

$$\prod_{i=1}^n (1-a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i-c_i)^{\omega_i} \geq 0$$

$$\Leftrightarrow \frac{\gamma\{\prod_{i=1}^n (1-a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i-c_i)^{\omega_i}\}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}} \geq 0.$$

Thus $0 \leq c_\alpha^p \leq 1$. Moreover:

$$a_\alpha^p + c_\alpha^p = 1$$

$$- \frac{\gamma \prod_{i=1}^n (1-a_i-c_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}$$

$$\leq 1 - \prod_{i=1}^n (1-a_i-c_i)^{\omega_i} \leq 1.$$

Similarly, $0 \leq b_\alpha^p \leq 1$, $0 \leq d_\alpha^p \leq 1$ and $b_\alpha^p + d_\alpha^p \leq 1$.

Hence, IVIFHIWA $\in [0, 1]$. Therefore, the aggregated IVIFN is again an IVIFN.□

Based on Theorem 3.1, we have some properties of the proposed IVIFHIWA operator for a collection of IVIFNs $\alpha_i = \langle[a_i, b_i], [c_i, d_i]\rangle (i = 1, 2, \ldots, n)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the associated weighted vector satisfying $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$.

Property 3.1 (Idempotency)

If $\alpha_i = \alpha_0 = \langle[a_0, b_0], [c_0, d_0]\rangle$ for all i, then:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_0.$$

Proof

Since $\alpha_i = \alpha_0 = \langle[a_0, b_0], [c_0, d_0]\rangle (i = 1, 2, \ldots, n)$, and $\sum_{i=1}^n \omega_i = 1$, by Theorem 3.1, we have:

$$\text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= \left\langle \left[\frac{\prod_{i=1}^n (1+(\gamma-1)a_0)^{\omega_i} - \prod_{i=1}^n (1-a_0)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_0)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_0)^{\omega_i}}, \right.\right.$$

$$\left. \frac{\prod_{i=1}^n (1+(\gamma-1)b_0)^{\omega_i} - \prod_{i=1}^n (1-b_0)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)b_0)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-b_0)^{\omega_i}} \right],$$

$$\left[\frac{\gamma\left\{\prod_{i=1}^n (1-a_0)^{\omega_i} - \prod_{i=1}^n (1-a_0-c_0)^{\omega_i}\right\}}{\prod_{i=1}^n (1+(\gamma-1)a_0)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_0)^{\omega_i}}, \right.$$

$$\left.\left. \frac{\gamma\left\{\prod_{i=1}^n (1-b_0)^{\omega_i} - \prod_{i=1}^n (1-b_0-d_0)^{\omega_i}\right\}}{\prod_{i=1}^n (1+(\gamma-1)b_0)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-b_0)^{\omega_i}} \right] \right\rangle$$

$$= \left\langle \left[\frac{(1+(\gamma-1)a_0)^{\sum_{i=1}^n \omega_i} - (1-a_0)^{\sum_{i=1}^n \omega_i}}{(1+(\gamma-1)a_0)^{\sum_{i=1}^n \omega_i} + (\gamma-1)(1-a_0)^{\sum_{i=1}^n \omega_i}}, \frac{(1+(\gamma-1)b_0)^{\sum_{i=1}^n \omega_i} - (1-b_0)^{\sum_{i=1}^n \omega_i}}{(1+(\gamma-1)b_0)^{\sum_{i=1}^n \omega_i} + (\gamma-1)(1-b_0)^{\sum_{i=1}^n \omega_i}} \right], \left[\frac{\gamma\left\{(1-a_0)^{\sum_{i=1}^n \omega_i} - (1-a_0-c_0)^{\sum_{i=1}^n \omega_i}\right\}}{(1+(\gamma-1)a_0)^{\sum_{i=1}^n \omega_i} + (\gamma-1)(1-a_0)^{\sum_{i=1}^n \omega_i}}, \frac{\gamma\left\{(1-b_0)^{\sum_{i=1}^n \omega_i} - (1-b_0-d_0)^{\sum_{i=1}^n \omega_i}\right\}}{(1+(\gamma-1)b_0)^{\sum_{i=1}^n \omega_i} + (\gamma-1)(1-b_0)^{\sum_{i=1}^n \omega_i}} \right] \right\rangle$$

$$= \left\langle \left[\frac{(1+(\gamma-1)a_0) - (1-a_0)}{(1+(\gamma-1)a_0) + (\gamma-1)(1-a_0)}, \frac{(1+(\gamma-1)b_0) - (1-b_0)}{(1+(\gamma-1)b_0) + (\gamma-1)(1-b_0)} \right], \left[\frac{\gamma\left\{(1-a_0) - (1-a_0-c_0)\right\}}{(1+(\gamma-1)a_0) + (\gamma-1)(1-a_0)}, \frac{\gamma\left\{(1-b_0) - (1-b_0-d_0)\right\}}{(1+(\gamma-1)b_0) + (\gamma-1)(1-b_0)} \right] \right\rangle$$

$$= \langle [a_0, b_0], [c_0, d_0] \rangle = \alpha_0. \square$$

Property 3.2 (Boundedness)

Let $\alpha^- = \min(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha^+ = \max(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then:

$$\alpha^- \leq \text{IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.$$

Proof

Let $f(x) = \frac{1-x}{1+(\gamma-1)x}, x \in [0,1]$ then $f^{'}(x) = \frac{-\gamma}{(1+(\gamma-1)x)^2} < 0$; thus, $f(x)$ is decreasing function. Since $a_{i,\min} \leq a_i \leq a_{i,\max}$, for all $i = 1, 2, \ldots, n$, $f(a_{i,\max}) \leq f(a_i) \leq f(a_{i,\min})$ for all i, i.e., $\frac{1-a_{i,\max}}{1+(\gamma-1)a_{i,\max}} \leq \frac{1-a_i}{1+(\gamma-1)a_i} \leq \frac{1-a_{i,\min}}{1+(\gamma-1)a_{i,\min}}$ for all i. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the associated weighted vector satisfying $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$; then, for all i, we have:

$$\left(\frac{1-a_{i,\max}}{1+(\gamma-1)a_{i,\max}}\right)^{\omega_i} \leq \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i} \leq \left(\frac{1-a_{i,\min}}{1+(\gamma-1)a_{i,\min}}\right)^{\omega_i}.$$

Thus:

$$\prod_{i=1}^n \left(\frac{1-a_{i,\max}}{1+(\gamma-1)a_{i,\max}}\right)^{\omega_i} \leq \prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i} \leq \prod_{i=1}^n \left(\frac{1-a_{i,\min}}{1+(\gamma-1)a_{i,\min}}\right)^{\omega_i}$$

$$\Leftrightarrow (\gamma-1)\left(\frac{1-a_{i,\max}}{1+(\gamma-1)a_{i,\max}}\right) \leq (\gamma-1)\prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i} \leq (\gamma-1)\left(\frac{1-a_{i,\min}}{1+(\gamma-1)a_{i,\min}}\right)$$

$$\Leftrightarrow \left(\frac{\gamma}{1+(\gamma-1)a_{i,\max}}\right) \leq 1+(\gamma-1)\prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i} \leq \left(\frac{\gamma}{1+(\gamma-1)a_{i,\min}}\right)$$

$$\Leftrightarrow \left(\frac{1+(\gamma-1)a_{i,\min}}{\gamma}\right) \leq \frac{1}{1+(\gamma-1)\prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i}} \leq \left(\frac{1+(\gamma-1)a_{i,\max}}{\gamma}\right)$$

$$\Leftrightarrow 1+(\gamma-1)a_{i,\min} \leq \frac{\gamma}{1+(\gamma-1)\prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i}} \leq 1+(\gamma-1)a_{i,\max}$$

$$\Leftrightarrow (\gamma-1)a_{i,\min} \leq \frac{\gamma}{1+(\gamma-1)\prod_{i=1}^n \left(\frac{1-a_i}{1+(\gamma-1)a_i}\right)^{\omega_i}} - 1$$

$$\leq (\gamma-1)a_{i,\max}$$

$$\Leftrightarrow a_{i,\min}$$

$$\leq \frac{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}-\prod_{i=1}^{n}(1-a_i)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}}$$

$$\leq a_{i,\max}. \tag{2}$$

Similarly:

$$b_{i,\min} \leq \frac{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}-\prod_{i=1}^{n}(1-b_i)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}}$$

$$\leq b_{i,\max}. \tag{3}$$

On the other hand, let $g(y)=\frac{\gamma-(\gamma-1)y}{(\gamma-1)y}, y \in [0,1]$; then, $g'(y)=-\gamma/((\gamma-1))^2y^2<0$; thus $g(y)$ is decreasing function on (0,1]. Since $1-a_{i,\max} \leq 1-a_i \leq 1-a_{i,\min}$ for all i, $g(1-a_{i,\min}) \leq g(1-a_i) \leq g(1-a_{i,\max})$, i.e., $\frac{\gamma-(\gamma-1)(1-a_{i,\min})}{(\gamma-1)(1-a_{i,\min})} \leq \frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)} \leq \frac{\gamma-(\gamma-1)(1-a_{i,\max})}{(\gamma-1)(1-a_{i,\max})}$ for all $i=1,2,\ldots,n$. Let $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the associated weighted vector satisfying $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$; then, for all i, we have:

$$\left(\frac{\gamma-(\gamma-1)(1-a_{i,\min})}{(\gamma-1)(1-a_{i,\min})}\right)^{\omega_i}$$

$$\leq \left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}$$

$$\leq \left(\frac{\gamma-(\gamma-1)(1-a_{i,\max})}{(\gamma-1)(1-a_{i,\max})}\right)^{\omega_i}.$$

Thus:

$$\prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_{i,\min})}{(\gamma-1)(1-a_{i,\min})}\right)^{\omega_i}$$

$$\leq \prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}$$

$$\leq \prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_{i,\max})}{(\gamma-1)(1-a_{i,\max})}\right)^{\omega_i}$$

$$\Leftrightarrow \frac{\gamma-(\gamma-1)(1-a_{i,\min})}{(\gamma-1)(1-a_{i,\min})}$$

$$\leq \prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}$$

$$\leq \frac{\gamma-(\gamma-1)(1-a_{i,\max})}{(\gamma-1)(1-a_{i,\max})}$$

$$\Leftrightarrow \frac{\gamma}{(\gamma-1)(1-a_{i,\min})}$$

$$\leq \prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}+1$$

$$\leq \frac{\gamma}{(\gamma-1)(1-a_{i,\max})}$$

$$\Leftrightarrow \frac{(\gamma-1)(1-a_{i,\max})}{\gamma}$$

$$\leq \frac{1}{\prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}+1}$$

$$\leq \frac{(\gamma-1)(1-a_{i,\min})}{\gamma}$$

$$\Leftrightarrow 1-a_{i,\max}$$

$$\leq \frac{\gamma}{(\gamma-1)\prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}+(\gamma-1)}$$

$$\leq 1-a_{i,\min}.$$

Also:

$$1-a_{i,\max}-c_{i,\min} \leq 1-a_i-c_i \leq 1$$

$$-a_{i,\min}-c_{i,\max}$$

$$\Leftrightarrow \frac{1-a_{i,\max}-c_{i,\min}}{1-a_{i,\min}} \leq \frac{1-a_i-c_i}{1-a_i}$$

$$\leq \frac{1-a_{i,\min}-c_{i,\max}}{1-a_{i,\max}}$$

$$\Leftrightarrow \frac{1-a_{i,\max}-c_{i,\min}}{1-a_{i,\min}} \leq \prod_{i=1}^{n}\left(\frac{1-a_i-c_i}{1-a_i}\right)^{\omega_i}$$

$$\leq \frac{1-a_{i,\min}-c_{i,\max}}{1-a_{i,\max}}$$

$$\Leftrightarrow \frac{c_{i,\max}+a_{i,\min}-a_{i,\max}}{1-a_{i,\max}} \leq 1-\prod_{i=1}^{n}\left(\frac{1-a_i-c_i}{1-a_i}\right)^{\omega_i}$$

$$\leq \frac{c_{i,\min}+a_{i,\max}-a_{i,\min}}{1-a_{i,\min}}$$

$$\Leftrightarrow c_{i,\max}+a_{i,\min}-a_{i,\max}$$

$$\leq \frac{\gamma\{1-\prod_{i=1}^{n}\left(\frac{1-a_i-c_i}{1-a_i}\right)^{\omega_i}\}}{(\gamma-1)\prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}+(\gamma-1)}$$

$$\leq c_{i,\min}+a_{i,\max}-a_{i,\min}$$

$\Leftrightarrow c_{i,\max}$

$$\leq \frac{\gamma\{1-\prod_{i=1}^{n}\left(\frac{1-a_i-c_i}{1-a_i}\right)^{\omega_i}\}}{(\gamma-1)\prod_{i=1}^{n}\left(\frac{\gamma-(\gamma-1)(1-a_i)}{(\gamma-1)(1-a_i)}\right)^{\omega_i}+(\gamma-1)}$$

$$\leq c_{i,\min},$$

$$c_{i,\max}\leq \frac{\gamma\left\{\prod_{i=1}^{n}(1-a_i)^{\omega_i}-\prod_{i=1}^{n}(1-a_i-c_i)^{\omega_i}\right\}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}}\leq c_{i,\min}. \tag{4}$$

Similarly:

$$d_{i,\max}\leq \frac{\gamma\left\{\prod_{i=1}^{n}(1-b_i)^{\omega_i}-\prod_{i=1}^{n}(1-b_i-d_i)^{\omega_i}\right\}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}}\leq d_{i,\min}. \tag{5}$$

Take:

$$a_{\min}=\min_i(a_{i,\min}),\quad a_{\max}=\max_i(a_{i,\max}),$$

$$b_{\min}=\min_i(b_{i,\min}),\quad b_{\max}=\max_i(b_{i,\max}),$$

$$c_{\min}=\min_i(c_{i,\min}),\quad c_{\max}=\max_i(c_{i,\max}),$$

$$d_{\min}=\min_i(d_{i,\min}),\quad d_{\max}=\max_i(d_{i,\max}).$$

Let:

$$\mathrm{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\alpha=\langle[a_\alpha,b_\alpha],[c_\alpha,d_\alpha]\rangle,$$

then, Eqs. (2) to (5) are transformed into the following forms, respectively:

$$a_{\min}\leq a_\alpha\leq a_{\max},\quad c_{\max}\leq c_\alpha\leq c_{\min},$$

$$b_{\min}\leq b_\alpha\leq b_{\max},\quad d_{\max}\leq d_\alpha\leq d_{\min}.$$

Thus:

$$S(\alpha)=\frac{a_\alpha+b_\alpha-c_\alpha-d_\alpha}{2}$$

$$\leq\frac{a_{\max}+b_{\max}-c_{\max}-d_{\max}}{2}=S(\alpha^+),$$

and:

$$S(\alpha)=\frac{a_\alpha+b_\alpha-c_\alpha-d_\alpha}{2}$$

$$\geq\frac{a_{\min}+b_{\min}-c_{\min}-d_{\min}}{2}=S(\alpha^-).$$

If $S(\alpha)<S(\alpha^+)$ and $S(\alpha)>S(\alpha^-)$, then, by order relation between two IVIFNs, we have:

$$\alpha^-<\mathrm{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\alpha^+.\square$$

Property 3.3 (Monotonicity)

Let α_i and $\beta_i,(i=1,2,\ldots,n)$ be two collections of IVIFNs such that $\alpha_i\leq\beta_i$ for all i; then,

$$\mathrm{IVIFHIWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$\leq\mathrm{IVIFHIWA}(\beta_1,\beta_2,\ldots,\beta_n).$$

Property 3.4 (Shift-invariance)

If $\beta=\langle[a_\beta,b_\beta],[c_\beta,d_\beta]\rangle$ is an IVIFN, then:

$$\mathrm{IVIFHIWA}(\alpha_1\oplus\beta,\alpha_2\oplus\beta,\ldots,\alpha_n\oplus\beta)$$

$$=\mathrm{IVIFHIWA}(\alpha_1,\alpha_2\ldots,\alpha_n)\oplus\beta.$$

Proof

As α_i and β are IVIFN, we have the equation shown in Box I. Therefore the equation shown in Box II is obtained. Hence:

$$\mathrm{IVIFHIWA}(\alpha_1\oplus\beta,\alpha_2\oplus\beta,\ldots,\alpha_n\oplus\beta)$$

$$=\mathrm{IVIFHIWA}(\alpha_1,\alpha_2\ldots,\alpha_n)\oplus\beta.$$

Property 3.5 (Homogeneity)

If $\beta>0$ is a real number, then:

$$\mathrm{IVIFHIWA}(\beta\alpha_1,\beta\alpha_2,\ldots,\beta\alpha_n)$$

$$=\beta\ \mathrm{IVIFHIWA}(\alpha_1,\alpha_2\ldots,\alpha_n).$$

Proof

$\alpha_i=\langle[a_i,b_i],[c_i,d_i]\rangle$ is an IVIFN for $i=1,2,\ldots,n$. Therefore, for $\beta>0$, we have:

$$\beta\alpha_i=\left\langle\left[\frac{(1+(\gamma-1)a_i)^\beta-(1-a_i)^\beta}{(1+(\gamma-1)a_i)^\beta+(\gamma-1)(1-a_i)^\beta},\right.\right.$$

$$\left.\frac{(1+(\gamma-1)b_i)^\beta-(1-b_i)^\beta}{(1+(\gamma-1)b_i)^\beta+(\gamma-1)(1-b_i)^\beta}\right],$$

$$\left[\frac{\gamma\left[(1-a_i)^\beta-(1-a_i-c_i)^\beta\right]}{(1+(\gamma-1)a_i)^\beta+(\gamma-1)(1-a_i)^\beta},\right.$$

$$\left.\left.\frac{\gamma\left[(1-b_i)^\beta-(1-b_i-d_i)^\beta\right]}{(1+(\gamma-1)b_i)^\beta+(\gamma-1)(1-b_i)^\beta}\right]\right\rangle.$$

$$\alpha_i \oplus \beta = \left\langle \left[\frac{(1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta)-(1-a_i)(1-a_\beta)}{(1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta)+(\gamma-1)(1-a_i)(1-a_\beta)}, \frac{(1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta)-(1-b_i)(1-b_\beta)}{(1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta)+(\gamma-1)(1-b_i)(1-b_\beta)} \right] \left[\frac{\gamma\left[(1-a_i)(1-a_\beta)-(1-a_i-c_i)(1-a_\beta-c_\beta)\right]}{(1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta)+(\gamma-1)(1-a_i)(1-a_\beta)}, \frac{\gamma\left[(1-b_i)(1-b_\beta)-(1-b_i-d_i)(1-b_\beta-d_\beta)\right]}{(1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta)+(\gamma-1)(1-b_i)(1-b_\beta)} \right] \right\rangle.$$

Box I

Therefore:

IVIFHIWA$(\beta\alpha_1, \beta\alpha_2, \ldots, \beta\alpha_n)$

$$= \left\langle \left[\frac{\prod_{i=1}^n [(1+(\gamma-1)a_i)^\beta]^{\omega_i} - \prod_{i=1}^n [(1-a_i)^\beta]^{\omega_i}}{\prod_{i=1}^n [(1+(\gamma-1)a_i)^\beta]^{\omega_i} + (\gamma-1)\prod_{i=1}^n [(1-a_i)^\beta]^{\omega_i}}, \frac{\prod_{i=1}^n [(1+(\gamma-1)b_i)^\beta]^{\omega_i} - \prod_{i=1}^n [(1-b_i)^\beta]^{\omega_i}}{\prod_{i=1}^n [(1+(\gamma-1)b_i)^\beta]^{\omega_i} + (\gamma-1)\prod_{i=1}^n [(1-b_i)^\beta]^{\omega_i}} \right], \left[\frac{\gamma\left\{\prod_{i=1}^n [(1-a_i)^\beta]^{\omega_i} - \prod_{i=1}^n [(1-a_i-c_i)^\beta]^{\omega_i}\right\}}{\prod_{i=1}^n [(1+(\gamma-1)a_i)^\beta]^{\omega_i} + (\gamma-1)\prod_{i=1}^n [(1-a_i)^\beta]^{\omega_i}}, \frac{\gamma\left\{\prod_{i=1}^n [(1-b_i)^\beta]^{\omega_i} - \prod_{i=1}^n [(1-b_i-d_i)^\beta]^{\omega_i}\right\}}{\prod_{i=1}^n [(1+(\gamma-1)b_i)^\beta]^{\omega_i} + (\gamma-1)\prod_{i=1}^n [(1-b_i)^\beta]^{\omega_i}} \right] \right\rangle$$

$$= \left\langle \left[\frac{\left(\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i}\right)^\beta - \left(\prod_{i=1}^n (1-a_i)^{\omega_i}\right)^\beta}{\left(\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i}\right)^\beta + (\gamma-1)\left(\prod_{i=1}^n (1-a_i)^{\omega_i}\right)^\beta}, \frac{\left(\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i}\right)^\beta - \left(\prod_{i=1}^n (1-b_i)^{\omega_i}\right)^\beta}{\left(\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i}\right)^\beta + (\gamma-1)\left(\prod_{i=1}^n (1-b_i)^{\omega_i}\right)^\beta} \right], \left[\frac{\gamma\left\{\left(\prod_{i=1}^n (1-a_i)^{\omega_i}\right)^\beta - \left(\prod_{i=1}^n (1-a_i-c_i)^{\omega_i}\right)^\beta\right\}}{\left(\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i}\right)^\beta + (\gamma-1)\left(\prod_{i=1}^n (1-a_i)^{\omega_i}\right)^\beta}, \frac{\gamma\left\{\left(\prod_{i=1}^n (1-b_i)^{\omega_i}\right)^\beta - \left(\prod_{i=1}^n (1-b_i-d_i)^{\omega_i}\right)^\beta\right\}}{\left(\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i}\right)^\beta + (\gamma-1)\left(\prod_{i=1}^n (1-b_i)^{\omega_i}\right)^\beta} \right] \right\rangle$$

$$= \beta \left\langle \left[\frac{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}, \frac{\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i} - \prod_{i=1}^n (1-b_i)^{\omega_i}}{\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-b_i)^{\omega_i}} \right], \left[\frac{\gamma\left\{\prod_{i=1}^n (1-a_i)^{\omega_i} - \prod_{i=1}^n (1-a_i-c_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1+(\gamma-1)a_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-a_i)^{\omega_i}}, \frac{\gamma\left\{\prod_{i=1}^n (1-b_i)^{\omega_i} - \prod_{i=1}^n (1-b_i-d_i)^{\omega_i}\right\}}{\prod_{i=1}^n (1+(\gamma-1)b_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^n (1-b_i)^{\omega_i}} \right] \right\rangle$$

$$= \beta \text{ IVIFHIWA}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Hence:

$$\text{IVIFHIWA}(\beta\alpha_1, \ldots, \beta\alpha_n) = \beta \text{ IVIFHIWA}(\alpha_1, \ldots, \alpha_n).\square$$

Property 3.6

If $\alpha_i = \langle [a_{\alpha_i}, b_{\alpha_i}], [c_{\alpha_i}, d_{\alpha_i}] \rangle$ and $\beta_i = \langle [a_{\beta_i}, b_{\beta_i}], [c_{\beta_i}, d_{\beta_i}] \rangle, (i = 1, 2, \ldots, n)$ are two collections of IVIFNs, then:

$$\mathrm{IVIFHIWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \ldots, \alpha_n \oplus \beta)$$

$$= \left\langle \left[\frac{\prod_{i=1}^{n}((1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta))^{\omega_i} - \prod_{i=1}^{n}((1-a_i)(1-a_\beta))^{\omega_i}}{\prod_{i=1}^{n}((1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta))^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}((1-a_i)(1-a_\beta))^{\omega_i}}, \right.\right.$$

$$\left. \frac{\prod_{i=1}^{n}((1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta))^{\omega_i} - \prod_{i=1}^{n}((1-b_i)(1-b_\beta))^{\omega_i}}{\prod_{i=1}^{n}((1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta))^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}((1-b_i)(1-b_\beta))^{\omega_i}} \right],$$

$$\left[\frac{\gamma\left\{ \prod_{i=1}^{n}((1-a_i)(1-a_\beta))^{\omega_i} - \prod_{i=1}^{n}((1-a_i-c_i)(1-a_\beta-c_\beta))^{\omega_i} \right\}}{\prod_{i=1}^{n}((1+(\gamma-1)a_i)(1+(\gamma-1)a_\beta))^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}((1-a_i)(1-a_\beta))^{\omega_i}}, \right.$$

$$\left.\left. \frac{\gamma\left\{ \prod_{i=1}^{n}((1-b_i)(1-b_\beta))^{\omega_i} - \prod_{i=1}^{n}((1-b_i-d_i)(1-b_\beta-d_\beta))^{\omega_i} \right\}}{\prod \lim its_{i=1}^{n}((1+(\gamma-1)b_i)(1+(\gamma-1)b_\beta))^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}((1-b_i)(1-b_\beta))^{\omega_i}} \right] \right\rangle$$

$$= \left\langle \left[\frac{\prod_{i=1}^{n}((1+(\gamma-1)a_i))^{\omega_i}(1+(\gamma-1)a_\beta)^{\omega_i} - \prod_{i=1}^{n}((1-a_i))^{\omega_i}(1-a_\beta)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}(1+(\gamma-1)a_\beta)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}(1-a_\beta)^{\omega_i}}, \right.\right.$$

$$\left. \frac{\prod_{i=1}^{n}((1+(\gamma-1)b_i))^{\omega_i}(1+(\gamma-1)b_\beta)^{\omega_i} - \prod_{i=1}^{n}((1-b_i))^{\omega_i}(1-b_\beta)^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}(1+(\gamma-1)b_\beta)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}(1-b_\beta)^{\omega_i}} \right],$$

$$\left[\frac{\gamma\left\{ \prod_{i=1}^{n}(1-a_i)^{\omega_i}(1-a_\beta)^{\omega_i} - \prod_{i=1}^{n}(1-a_i-c_i)^{\omega_i}(1-a_\beta-c_\beta)^{\omega_i} \right\}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i}(1+(\gamma-1)a_\beta)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}(1-a_i)^{\omega_i}(1-a_\beta)^{\omega_i}}, \right.$$

$$\left.\left. \frac{\gamma\left\{ \prod_{i=1}^{n}(1-b_i)^{\omega_i}(1-b_\beta)^{\omega_i} - \prod_{i=1}^{n}(1-b_i-d_i)^{\omega_i}(1-b_\beta-d_\beta)^{\omega_i} \right\}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i}(1+(\gamma-1)b_\beta)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}(1-b_i)^{\omega_i}(1-b_\beta)^{\omega_i}} \right] \right\rangle$$

$$= \left\langle \left[\frac{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i} \right\}(1+(\gamma-1)a_\beta) - \left\{ \prod_{i=1}^{n}(1-a_i)^{\omega_i} \right\}(1-a_\beta)}{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i} \right\}(1+(\gamma-1)a_\beta) + (\gamma-1)\left\{ \prod_{i=1}^{n}(1-a_i)^{\omega_i} \right\}(1-a_\beta)}, \right.\right.$$

$$\left. \frac{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i} \right\}(1+(\gamma-1)b_\beta) - \left\{ \prod_{i=1}^{n}(1-b_i)^{\omega_i} \right\}(1-b_\beta)}{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i} \right\}(1+(\gamma-1)b_\beta) + (\gamma-1)\left\{ \prod_{i=1}^{n}(1-b_i)^{\omega_i} \right\}(1-b_\beta)} \right],$$

$$\left[\frac{\gamma\left(\left\{ \prod_{i=1}^{n}(1-a_i)^{\omega_i} \right\}(1-a_\beta) - \left\{ \prod_{i=1}^{n}(1-a_i-c_i)^{\omega_i} \right\}(1-a_\beta-c_\beta) \right)}{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)a_i)^{\omega_i} \right\}(1+(\gamma-1)a_\beta) + (\gamma-1)\left\{ \prod_{i=1}^{n}(1-a_i)^{\omega_i} \right\}(1-a_\beta)}, \right.$$

$$\left.\left. \frac{\gamma\left(\left\{ \prod_{i=1}^{n}(1-b_i)^{\omega_i} \right\}(1-b_\beta) - \left\{ \prod_{i=1}^{n}(1-b_i-d_i)^{\omega_i} \right\}(1-b_\beta-d_\beta) \right)}{\left\{ \prod_{i=1}^{n}(1+(\gamma-1)b_i)^{\omega_i} \right\}(1+(\gamma-1)b_\beta) + (\gamma-1)\left\{ \prod_{i=1}^{n}(1-b_i)^{\omega_i} \right\}(1-b_\beta)} \right] \right\rangle$$

$$= \mathrm{IVIFHIWA}(\alpha_1, \alpha_2 \ldots, \alpha_n) \oplus \beta$$

Box II

$$\text{IVIFHIWA}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \ldots, \alpha_n \oplus \beta_n)$$

$$= \text{IVIFHIWA}(\alpha_1, \alpha_2 \ldots, \alpha_n)$$

$$\oplus \text{IVIFHIWA}(\beta_1, \beta_2 \ldots, \beta_n).$$

Property 3.7

If $\eta > 0$ is any real number and $\beta = \langle [a_\beta, b_\beta], [c_\beta, d_\beta] \rangle$ is an IVIFN, then:

$$\text{IVIFHIWA}(\eta\alpha_1 \oplus \beta, \eta\alpha_2 \oplus \beta, \ldots, \eta\alpha_n \oplus \beta)$$

$$= \eta\,\text{IVIFHIWA}(\alpha_1, \alpha_2 \ldots, \alpha_n) \oplus \beta.$$

Proof

By using the properties 3.1, 3.5, and 3.6, we get the required proof; therefore, it is omitted here.□

3.2. Interval valued intuitionistic fuzzy Hamacher interactive ordered weighted averaging operator

In this section, we intend to take the idea of OWA into IVIFHIWA operator and propose a new operator called an IVIF Hamacher interactive ordered weighted averaging (IVIFHIOWA) operator. In the following, we first introduce the concept of IVIFHIOWA operator and then illustrate it with a numerical example.

Definition 3.3

Suppose there is a family of IVIFNs $\alpha_i = \langle [a_i, b_i], [c_i, d_i] \rangle$ for $i = 1, 2, \ldots, n$ and $IVIFHIOWA : \Omega^n \longrightarrow \Omega$, if:

$$\text{IVIFHIOWA}(\alpha_1, \ldots, \alpha_n) = \omega_1\alpha_{\delta(1)} \oplus \omega_2\alpha_{\delta(2)}$$

$$\oplus \ldots \oplus \omega_n\alpha_{\delta(n)},$$

where $\omega = (\omega_1, \omega_2 \ldots, \omega_n)^T$ is the weight vector associated with IVIFHIOWA, such that $\omega_i > 0$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\alpha_{\delta(i-1)} \geq \alpha_{\delta(i)}$ for any i. Thus, IVIFHIOWA is called an IVIF Hamacher interactive OWA operator.

Theorem 3.3

Let $\alpha_i = \langle [a_i, b_i], [c_i, d_i] \rangle (i = 1, 2, \ldots, n)$ be the collection of IVIFNs; then, based on IVIFHIOWA operator, the aggregated IVIFN can be expressed as:

$$\text{IVIFHIOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= \Bigg\langle \Bigg[\frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i}},$$

$$\frac{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i}} \Bigg]$$

$$\Bigg[\frac{\gamma\{\prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-a_{\delta(i)}-c_{\delta(i)})^{\omega_i}\}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)a_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-a_{\delta(i)})^{\omega_i}},$$

$$\frac{\gamma\{\prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i} - \prod\limits_{i=1}^{n}(1-b_{\delta(i)}-d_{\delta(i)})^{\omega_i}\}}{\prod\limits_{i=1}^{n}(1+(\gamma-1)b_{\delta(i)})^{\omega_i} + (\gamma-1)\prod\limits_{i=1}^{n}(1-b_{\delta(i)})^{\omega_i}} \Bigg] \Bigg\rangle. \tag{6}$$

The proof of this theorem is similar to that of Theorem 3.1 and, hence, it is omitted here.

Corollary 3.2

The IVIFHIOWA and IVIFHOWA operators have the following relation for a collections of IVIFNs $\alpha_i (i = 1, 2, \ldots, n)$:

$$\text{IVIFHIOWA}(\alpha_1, \ldots, \alpha_n)$$

$$\leq \text{IVIFHOWA}(\alpha_1, \ldots, \alpha_n).$$

Property 3.8

Let $\alpha_i = \langle [a_{\alpha_i}, b_{\alpha_i}], [c_{\alpha_i}, d_{\alpha_i}] \rangle)$, $\beta_i = \langle [a_{\beta_i}, b_{\beta_i}], [c_{\beta_i}, d_{\beta_i}] \rangle, (i = 1, 2, \ldots, n)$, and $\beta =< [a_\beta, b_\beta],\ [c_\beta, d_\beta] >$ be collections of IVIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighting vector of the IVIFHIOWA operator, $\omega_i > 0, i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n} \omega_i = 1$; then, we have the following:

(i) Idempotency: If all $\alpha_i, (i = 1, 2, \ldots, n)$ are equal, i.e. $\alpha_i = \alpha$ for all i, then:

$$\text{IVIFHIOWA}(\alpha_1, \ldots, \alpha_n) = \alpha.$$

(ii) Boundedness:

$$\alpha_{\min} \leq \text{IVIFHIOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$\leq \alpha_{\max},$$

where $\alpha_{\min} = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\alpha_{\max} = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

(iii) Monotonicity: If $\alpha_i \leq \beta_i$, then, for every weight vector ω, we have:

$$\text{IVIFHIOWA}(\alpha_1, \ldots, \alpha_n) \leq \text{IVIFHIOWA}(\beta_1, \ldots, \beta_n).$$

(iv) Shift-invariance:

$$\text{IVIFHIOWA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta \oplus \ldots \oplus \alpha_n \oplus \beta) = \text{IVIFHIOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \beta.$$

(v) Homogeneity:

$$\text{IVIFHIOWA}(\beta\alpha_1, \beta\alpha_2, \ldots, \beta\alpha_n) = \beta\ IVIFHIOWA(\alpha_1, \alpha_2 \ldots, \alpha_n).$$

The proof of these properties is similar to that of IVIFHIWA operator properties and hence, it is omitted here.

Example 3.3

Let $\alpha_1 = \langle[0.22, 0.31], [0.23, 0.54]\rangle$, $\alpha_2 = \langle[0.04, 0.21], [0.35, 0.46]\rangle$, and $\alpha_3 = \langle[0.25, 0.27], [0.23, 0.40]\rangle$ be three IVIFNs, i.e. $a_1 = 0.22, a_2 = 0.04, a_3 = 0.25$; $b_1 = 0.31, b_2 = 0.21, b_3 = 0.27$; $c_1 = 0.23, c_2 = 0.35, c_3 = 0.23$; and $d_1 = 0.54, d_2 = 0.46, d_3 = 0.40$, and $\omega = (0.314, 0.355, 0.331)^T$ be the weight vectors of $\alpha_i (i = 1, 2, 3)$. Assume $\gamma = 2$; then, score functions of α_i are $S(\alpha_1) = -0.12$, $S(\alpha_2) = -0.28$, and $S(\alpha_3) = -0.055$. Thus, the ranking of IVIFNs is $\alpha_3 > \alpha_1 > \alpha_2$. Therefore, $\alpha_{\delta(1)} = \langle[0.25, 0.27], [0.23, 0.40]\rangle$, $\alpha_{\delta(2)} = \langle[0.22, 0.31], [0.23, 0.54]\rangle$, and $\alpha_{\delta(3)} = \langle[0.04, 0.21], [0.35, 0.46]\rangle$. Therefore, based on these IVIFNs, we have:

$$\prod_{i=1}^{3}(1 + (\gamma - 1)a_{\delta(i)})^{\omega_i} = (1.25)^{0.314} \times (1.22)^{0.355} \times (1.04)^{0.331} = 1.661,$$

$$\prod_{i=1}^{3}(1 - a_{\delta(i)})^{\omega_i} = (0.75)^{0.314} \times (0.78)^{0.355} \times (0.96)^{0.331} = 0.8253,$$

$$\prod_{i=1}^{3}(1 + (\gamma - 1)b_{\delta(i)})^{\omega_i} = (1.27)^{0.314} \times (1.31)^{0.355} \times (1.21)^{0.331} = 1.2637,$$

$$\prod_{i=1}^{3}(1 - b_{\delta(i)})^{\omega_i} = (0.73)^{0.314} \times (0.69)^{0.355} \times (0.79)^{0.331} = 0.7345,$$

$$\prod_{i=1}^{3}(1 + (\gamma - 1)(1 - c_{\delta(i)}))^{\omega_i} = (1.77)^{0.314} \times (1.77)^{0.355} \times (1.65)^{0.331} = 1.7293,$$

$$\prod_{i=1}^{3}(d_{\delta(i)})^{\omega_i} = (0.40)^{0.311} \times (0.54)^{0.355} \times (0.46)^{0.331} = 0.4660,$$

$$\prod_{i=1}^{3}(1 + (\gamma - 1)(1 - d_{\delta(i)}))^{\omega_i} = (1.6)^{0.314} \times (1.46)^{0.355} \times (1.54)^{0.331} = 1.5294,$$

$$\prod_{i=1}^{3}(1 + (\gamma - 1)(1 - a_{\delta(i)} - c_{\delta(i)}))^{\omega_i} = (1.52)^{0.314} \times (1.55)^{0.355} \times (1.61)^{0.331} = 1.5600,$$

$$\prod_{i=1}^{3}(1 + (\gamma - 1)(1 - b_{\delta(i)} - d_{\delta(i)}))^{\omega_i} = (1.33)^{0.314} \times (1.15)^{0.355} \times (1.33)^{0.331} = 1.2631.$$

Thus:

$$\text{IVIFHIOWA}(\alpha_1, \alpha_2, \alpha_3) = \langle[0.1711, 0.2648], [0.2672, 0.4855]\rangle,$$

$$\text{IVIFHOWA}(\alpha_1, \alpha_2, \alpha_3) = \langle[0.1711, 0.2648], [0.2651, 0.4671]\rangle.$$

Thus, it is clear from these results that:

$$\text{IVIFHIOWA}(\alpha_1, \alpha_2, \alpha_3) < \text{IVIFHOWA}(\alpha_1, \alpha_2, \alpha_3).$$

3.3. Interval valued intuitionistic fuzzy Hamacher interactive hybrid weighted averaging operator

Definition 3.4

Suppose there is a family of IVIFNs, $\alpha_i = \langle[a_i, b_i], [c_i, d_i]\rangle, (i = 1, 2, \ldots, n)$ and IVIFHIHWA : $\Omega^n \longrightarrow \Omega$, if:

$$\text{IVIFHIHWA}(\alpha_1,\ldots,\alpha_n)=\omega_1\dot{\alpha}_{\sigma(1)}\oplus\omega_2\dot{\alpha}_{\sigma(2)}\oplus\ldots\oplus\omega_n\dot{\alpha}_{\sigma(n)},$$

where $(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weighted vector associated with IVIFHIHWA; $\dot{\alpha}_{\sigma(i)}$ is the i^{th} largest weighted IVIFNs $\dot{\alpha}_i$ given by $\dot{\alpha}_i=nw_i\alpha_i, i=1,2,\ldots,n$; and $w=(w_1,w_2,\ldots,w_n)^t$ is the weight vector of α_i such that $w_i>0, \sum_{i=1}^{n}w_i=1$; then, IVIFHIHWA is called IVIF Hamacher interactive hybrid weighted averaging operator.

From Definition 3.4, it can be concluded that:

- It firstly weights the IVIFNs α_i by the associated weights $w_i(i=1,2,\ldots,n)$ and multiplies these values by a balancing coefficient n, hence, getting the weighted IVIFNs $\dot{\alpha}_i=nw_i\alpha_i$ $(i=1,2,\ldots,n)$;
- It reorders the weighted arguments in descending order $(\dot{\alpha}_{\sigma(1)},\dot{\alpha}_{\sigma(2)},\ldots,\dot{\alpha}_{\sigma(n)})$, where $\dot{\alpha}_{\sigma(i)}$ is the ith largest $\dot{\alpha}_i(i=1,2,\ldots,n)$;
- It weights these ordered weighted IVIFNs $\dot{\alpha}_{\sigma(i)}$ by the IVIFHIWA weights $\omega_i(i=1,2,\ldots,n)$ and then aggregates all these values into a collective one.

Theorem 3.4

Suppose that there is a family of IVIFNs $\alpha_i=\langle[a_i,b_i],[c_i,d_i]\rangle,(i=1,2,\ldots,n)$; then, based on the IVIFHIHWA operator, the aggregated IVIFN can be expressed as:

$$\text{IVIFHIHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$=\Bigg\langle\Bigg[\frac{\prod_{i=1}^{n}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}},$$

$$\frac{\prod_{i=1}^{n}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}}\Bigg]$$

$$\Bigg[\frac{\gamma\{\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)}-\dot{c}_{\sigma(i)})^{\omega_i}\}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{a}_{\sigma(i)})^{\omega_i}},$$

$$\frac{\gamma\{\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)}-\dot{d}_{\sigma(i)})^{\omega_i}\}}{\prod_{i=1}^{n}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}(1-\dot{b}_{\sigma(i)})^{\omega_i}}\Bigg]\Bigg\rangle. \quad (7)$$

The proof is similar to that of Theorem 3.1; thus, it is omitted here.

Corollary 3.3

The IVIFHIHWA and IVIFHHWA operators have the following relation for a collection of IVIFNs α_i $(i=1,2,\ldots,n)$:

$$\text{IVIFHIHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\text{IVIFHHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n).$$

Similar to the IVIFHIWA and IVIFHIOWA operators, the IVIFHIHWA operator follows the properties described in Property 3.8.

Example 3.4

Let $\alpha_1=\langle[0.22,0.31],[0.23,0.54]\rangle$, $\alpha_2=\langle[0.04,0.21],[0.35,0.46]\rangle$, and $\alpha_3=\langle[0.25,0.27],[0.23,0.40]\rangle$ be three IVIFNs, i.e. $a_1=0.22, a_2=0.04, a_3=0.25$; $b_1=0.31, b_2=0.21, b_3=0.27$; $c_1=0.23, c_2=0.35, c_3=0.23$; and $d_1=0.54, d_2=0.46, d_3=0.40$. Assume $\gamma=2$ and $w=(0.314,0.355,0.331)^T$ is the weight vector of $\alpha_i(i=1,2,3)$; then, $\dot{\alpha}_i=(3w_i)\alpha_i=\langle[\dot{a}_i,\dot{b}_i],[\dot{c}_i,\dot{d}_i]\rangle$ for $i=1,2,3$ is calculated as shown in Box III.

Similarly, $\dot{\alpha}_2=\langle[0.0426,0.2232],[0.3667,0.4702]\rangle$, and $\dot{\alpha}_3=\langle[0.2483,0.2682],[0.2292,0.3991]\rangle$. Thus, their corresponding score values are $S(\dot{\alpha}_1)=-0.3175$, $S(\dot{\alpha}_2)=-0.4888$, and $S(\dot{\alpha}_3)=-0.1195$; hence, ranking of IVIFNs is $\dot{\alpha}_3>\dot{\alpha}_1>\dot{\alpha}_2$. Therefore, $\dot{\alpha}_{\sigma(1)}=\langle[0.2483,0.2682],[0.2292,0.3991]\rangle$, $\dot{\alpha}_{\sigma(2)}=\langle[0.2076,0.2931],[0.2222,0.5390]\rangle$, and $\dot{\alpha}_{\sigma(3)}=\langle[0.0426,0.2232],[0.3667,0.4702]\rangle$. Let $\omega=(0.25,0.50,0.25)^T$ be the position weighted vector; then, based on these IVIFNs, we have:

$$\prod_{i=1}^{3}(1+(\gamma-1)\dot{a}_{\sigma(i)})^{\omega_i}=1.1737,$$

$$\prod_{i=1}^{3}(1-\dot{a}_{\sigma(i)})^{\omega_i}=0.8199,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)\dot{b}_{\sigma(i)})^{\omega_i}=1.2691,$$

$$\prod_{i=1}^{3}(1-\dot{b}_{\sigma(i)})^{\omega_i}=0.7301,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)\dot{c}_{\sigma(i)})^{\omega_i}=1.2586,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)\dot{d}_{\sigma(i)})^{\omega_i}=1.4857$$

$$\dot{\alpha}_1 = \left\langle \left[\frac{(1+(2-1)\times 0.22)^{3\times 0.314} - (1-0.22)^{3\times 0.314}}{(1+(2-1)\times 0.22)^{3\times 0.314} + (2-1)\times(1-0.22)^{3\times 0.314}}, \right. \right.$$

$$\left. \frac{(1+(2-1)\times 0.31)^{3\times 0.314} - (1-0.31)^{3\times 0.314}}{(1+(2-1)\times 0.31)^{3\times 0.314} + (2-1)\times(1-0.31)^{3\times 0.314}} \right]$$

$$\left[\frac{2\{(1-0.22)^{3\times 0.314} - (1-0.22-0.23)^{3\times 0.314}\}}{(1+(2-1)\times 0.22)^{3\times 0.314} + (2-1)\times(1-0.22)^{3\times 0.314}}, \right.$$

$$\left. \left. \frac{2\{(1-0.31)^{3\times 0.314} - (1-0.31-0.54)^{3\times 0.314}\}}{(1+(2-1)\times 0.31)^{3\times 0.314} + (2-1)\times(1-0.31)^{3\times 0.314}} \right] \right\rangle$$

$$= \langle [0.2076, 0.2931], [0.2222, 0.5390] \rangle$$

Box III

$$\prod_{i=1}^{3}(1-\dot{c}_{\sigma(i)})^{\omega_i} = 0.7372,$$

$$\prod_{i=1}^{3}(1-\dot{d}_{\sigma(i)})^{\omega_i} = 0.5100,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)(1-\dot{c}_{\sigma(i)}))^{\omega_i} = 1.7388,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)(1-\dot{a}_{\sigma(i)}-\dot{c}_{\sigma(i)}))^{\omega_i} = 1.5632,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)(1-\dot{b}_{\sigma(i)}-\dot{d}_{\sigma(i)}))^{\omega_i} = 1.2414,$$

$$\prod_{i=1}^{3}(1+(\gamma-1)(1-\dot{d}_{\sigma(i)}))^{\omega_i} = 1.5121,$$

$$\prod_{i=1}^{3}\dot{d}_{\sigma(i)}^{\omega_i} = 0.4832.$$

Therefore:

$$\text{IVIFHIHWA}(\alpha_1,\alpha_2,\alpha_3) = \langle [0.1775, 0.2696], [0.2579, 0.4987] \rangle,$$

$$\text{IVIFHHWA}(\alpha_1,\alpha_2,\alpha_3) = \langle [0.1775, 0.2696], [0.2646, 0.4867] \rangle.$$

Thus, it is clear from these results that:

$$\text{IVIFHIHWA}(\alpha_1,\alpha_2,\alpha_3) < \text{IVIFHHWA}(\alpha_1,\alpha_2,\alpha_3).$$

4. MCDM method using the proposed operators

MCDM is one of the most trustful approaches for finding the best alternative among the set of some feasible criteria. Assume that a set of different alternatives $A = \{A_1, A_2, \ldots, A_m\}$ exists to be considered by the decision makers whose target is to find the best alternative. These alternatives have to be evaluated by the decision maker(s) according to the different criteria $G = \{G_1, G_2, \ldots, G_n\}$, for which there is a linear ordering $G_1 \succ G_2 \succ \ldots \succ G_n$ (indicating that the attribute G_i has a higher priority than G_j, if $i < j$) prioritization. The evaluation of these alternatives under the different criteria is performed by the decision makers in the form of IVIFSs and the procedure for computing the best alternative is summarized in the following steps:

- **Step 1.** *Construction of IVIF decision-making matrix:* The preferences of the decision maker among the alternatives with different criteria are arranged in the form of interval-valued intuitionistic fuzzy decision matrix as $D_{m\times n}(x_{ij}) = \langle [a_{ij}, b_{ij}], [c_{ij}, d_{ij}] \rangle$, where $[a_{ij}, b_{ij}]$ indicates the degree that the alternative A_i satisfies the attribute and $[c_{ij}, d_{ij}]$ indicates that it does not satisfy the attribute G_j given by the decision maker such that $[a_{ij}, b_{ij}] \subseteq [0,1]$, $[c_{ij}, d_{ij}] \subseteq [0,1]$, $b_{ij} + d_{ij} \leq 1$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Therefore, the decision matrix is expressed as shown in Box IV.
- **Step 2.** *Normalizing the decision matrix*: If all the attributes are of the same type, then the rating values do not need normalization. On the other hand, if there are different types of criteria, namely, benefit (C_1) and cost (C_2), then we transform the rating values of benefit into cost by using the

$$D_{m\times n}(x_{ij}) = \begin{bmatrix} \langle[a_{11},b_{11}],[c_{11},d_{11}]\rangle & \langle[a_{12},b_{12}],[c_{12},d_{12}]\rangle & \ldots & \langle[a_{1n},b_{1n}],[c_{1n},d_{1n}]\rangle \\ \langle[a_{21},b_{21}],[c_{21},d_{21}]\rangle & \langle[a_{22},b_{22}],[c_{22},d_{22}]\rangle & \ldots & \langle[a_{2n},b_{2n}],[c_{2n},d_{2n}]\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle[a_{m1},b_{m1}],[c_{m1},d_{m1}]\rangle & \langle[a_{m2},b_{m2}],[c_{m2},d_{m2}]\rangle & \ldots & \langle[a_{mn},b_{mn}],[c_{mn},d_{mn}]\rangle \end{bmatrix} \quad (8)$$

Box IV

following normalization formula [37]:

$$r_{ij} = \begin{cases} \alpha_{ij}^c & ; \quad j \in C_1 \\ \alpha_{ij} & ; \quad j \in C_2 \end{cases} \quad (9)$$

where $\alpha_{ij}^c = \langle[c_{ij},d_{ij}],[a_{ij},b_{ij}]\rangle$ is the complement of $\alpha_{ij} = \langle[a_{ij},b_{ij}],[c_{ij},d_{ij}]\rangle$. Hence, we obtain the normalized IVIF decision matrix $R = (r_{ij})_{m\times n}$.

- **Step 3.** *Computing the overall aggregated value of the alternatives*: By using the normalized matrix $R = (r_{ij})_{m\times n}$, all the rating values corresponding to each alternative, $A_i (i = 1,2,\ldots,m)$, are aggregated by utilizing IVIFHIWA, IVIFHIOWA, or IVIFHIHWA operator as given in Eqs. (1), (6), and (7), respectively, and the overall value of r_i is obtained.
- **Step 4.** *Comparing each alternative:* Based on the overall assessment of each alternative r_i, prioritized comparison indices as defined in Section 2 are computed.
- **Step 5.** *Ranking the alternatives:* All the alternatives, A_i $(i = 1,2,\ldots,m)$ are ranked according to score values as obtained from Step 4 for different IVIFNs and the most desirable alternatives are selected in descending order of their function.
- **Step 6.** *Performing the sensitivity analysis:* The sensitivity analysis of the parameter γ is performed according to decision makers' preferences.

5. Numerical example

In this section, an example for multi-criteria fuzzy decision making problems of alternatives is used as a demonstration of the applications and the effectiveness of the proposed decision making method. The aim of this problem is to provide a panel who wants to invest money on four possible alternatives, namely, car, food, computer, and arm companies, respectively denoted by A_1, A_2, A_3, and A_4 with a decision. The panel takes the decision according to the three criteria given by C_1 in risk analysis; C_2 is the growth analysis and C_3 is the environmental impact analysis. The weight vector corresponding to each criterion is given by the committee as $\omega = (0.25, 0.45, 0.3)^T$. These four possible alternatives, A_i $(i = 1,2,3,4)$, are to be evaluated using the interval-valued intuitionistic fuzzy information by the decision-maker under the above three criteria, as listed in the decision matrix $D_{4\times 3}(x_{ij})$ as shown in Box V.

Since C_1 is the cost criterion, and C_2 and C_3 are benefit criteria, the interval-valued intuitionistic fuzzy decision matrix $D = (\alpha_{ij})_{4\times 3}$ can be transformed into the following normalized matrix $R = (r_{ij})_{4\times 3}$ using Eq. (9), as shown in Box VI.

The IVIFHIWA operator, corresponding to $\gamma = 2$, is utilized to aggregate all the performance values r_{ij} $(j = 1,2,3)$ of the ith alternative and the overall

$$D_{4\times 3}(x_{ij}) = \begin{bmatrix} \langle[0.4,0.5],[0.3,0.4]\rangle & \langle[0.2,0.4],[0.4,0.6]\rangle & \langle[0.5,0.6],[0.1,0.3]\rangle \\ \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.2,0.3],[0.6,0.7]\rangle & \langle[0.1,0.2],[0.4,0.7]\rangle \\ \langle[0.3,0.6],[0.3,0.4]\rangle & \langle[0.3,0.4],[0.5,0.6]\rangle & \langle[0.1,0.3],[0.5,0.6]\rangle \\ \langle[0.7,0.8],[0.1,0.2]\rangle & \langle[0.1,0.3],[0.6,0.7]\rangle & \langle[0.1,0.2],[0.3,0.4]\rangle \end{bmatrix} \quad (10)$$

Box V

$$R_{4\times 3}(x_{ij}) = \begin{bmatrix} \langle[0.4,0.5],[0.3,0.4]\rangle & \langle[0.4,0.6],[0.2,0.4]\rangle & \langle[0.1,0.3],[0.5,0.6]\rangle \\ \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.4,0.7],[0.1,0.2]\rangle \\ \langle[0.3,0.6],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.3,0.4]\rangle & \langle[0.5,0.6],[0.1,0.3]\rangle \\ \langle[0.7,0.8],[0.1,0.2]\rangle & \langle[0.6,0.7],[0.1,0.3]\rangle & \langle[0.3,0.4],[0.1,0.2]\rangle \end{bmatrix} \quad (11)$$

Box VI

performance value, r_i, corresponding to alternative A_i ($i = 1, 2, 3, 4$) is obtained by the equations shown in Box VII.

By using the score function, we get the score values of these respective alternatives as:

$$S(r_1) = -0.0019; \quad S(r_2) = 0.3781;$$

$$S(r_3) = 0.2008; \quad S(r_4) = 0.3815.$$

Therefore, the ranking of the four alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$, i.e. arms company $\succ$ food company $\succ$ computer company $\succ$ car company; thus, A_4 (i.e., arms company) is the most desirable one and A_1 (i.e., car company) is the least desirable one. However, for different values of γ, say $\gamma = 1, 2, 3$, the score functions and an overall aggregated IVIFN for alternatives are given in Table 1 by the existing and proposed operators. From these results, it can be concluded that the results of the proposed operators coincide with the results of the existing methodologies and the obtained aggregated IVIFN is more optimistic than the aggregated values of the existing methodologies for taking a decision.

On the other hand, if we aggregate these different IVIFNs by IVIFHIHWA operator, then, firstly we find $\dot{\alpha}_{ij} = (3w_j)\alpha_{ij}$ as:

$$\dot{\alpha}_{11} = \left\langle \left[\frac{(1.4)^{1.2} - (0.6)^{1.2}}{(1.4)^{1.2} + (0.6)^{1.2}}, \frac{(1.5)^{1.2} - (0.5)^{1.2}}{(1.5)^{1.2} + (0.5)^{1.2}} \right], \right.$$

$$r_1 = \text{IVIFHIWA}(r_{11}, r_{12}, r_{13})$$

$$= \left\langle \left[\frac{(1.4)^{0.25}(1.4)^{0.45}(1.1)^{0.3} - (0.6)^{0.25}(0.6)^{0.45}(0.9)^{0.3}}{(1.4)^{0.25}(1.4)^{0.45}(1.1)^{0.3} + (0.6)^{0.25}(0.6)^{0.45}(0.9)^{0.3}}, \frac{(1.5)^{0.25}(1.6)^{0.45}(1.3)^{0.3} - (0.5)^{0.25}(0.4)^{0.45}(0.7)^{0.3}}{(1.5)^{0.25}(1.6)^{0.45}(1.3)^{0.3} + (0.5)^{0.25}(0.4)^{0.45}(0.7)^{0.3}} \right], \right.$$

$$\left. \left[\frac{2\left\{(0.6)^{0.25}(0.6)^{0.45}(0.9)^{0.3} - (0.3)^{0.25}(0.4)^{0.45}(0.4)^{0.3}\right\}}{(1.4)^{0.25}(1.4)^{0.45}(1.1)^{0.3} + (0.6)^{0.25}(0.6)^{0.45}(0.9)^{0.3}}, \frac{2\left\{(0.5)^{0.25}(0.4)^{0.45}(0.7)^{0.3} - (0.1)^{0.25}(0.0)^{0.45}(0.1)^{0.3}\right\}}{(1.5)^{0.25}(1.6)^{0.45}(1.3)^{0.3} + (0.5)^{0.25}(0.4)^{0.45}(0.7)^{0.3}} \right] \right\rangle$$

$$= \langle [0.3155, 0.4946], [0.3085, 0.5054] \rangle$$

$$r_2 = \text{IVIFHIWA}(r_{21}, r_{22}, r_{23})$$

$$= \left\langle \left[\frac{(1.6)^{0.25}(1.6)^{0.45}(1.4)^{0.3} - (0.4)^{0.25}(0.4)^{0.45}(0.6)^{0.3}}{(1.6)^{0.25}(1.6)^{0.45}(1.4)^{0.3} + (0.4)^{0.25}(0.4)^{0.45}(0.6)^{0.3}}, \frac{(1.7)^{0.25}(1.7)^{0.45}(1.7)^{0.3} - (0.3)^{0.25}(0.3)^{0.45}(0.3)^{0.3}}{(1.7)^{0.25}(1.7)^{0.45}(1.7)^{0.3} + (0.3)^{0.25}(0.3)^{0.45}(0.3)^{0.3}} \right], \right.$$

$$\left. \left[\frac{2\left\{(0.4)^{0.25}(0.4)^{0.45}(0.6)^{0.3} - (0.2)^{0.25}(0.2)^{0.45}(0.5)^{0.3}\right\}}{(1.6)^{0.25}(1.6)^{0.45}(1.4)^{0.3} + (0.4)^{0.25}(0.4)^{0.45}(0.6)^{0.3}}, \frac{2\left\{(0.3)^{0.25}(0.3)^{0.45}(0.3)^{0.3} - (0.0)^{0.25}(0.0)^{0.45}(0.1)^{0.3}\right\}}{(1.6)^{0.25}(1.6)^{0.45}(1.4)^{0.3} + (0.4)^{0.25}(0.4)^{0.45}(0.6)^{0.3}} \right] \right\rangle$$

$$= \langle [0.5457, 0.7000], [0.1895, 0.3000] \rangle$$

$$r_3 = \text{IVIFHIWA}(r_{31}, r_{32}, r_{33})$$

$$= \left\langle \left[\frac{(1.3)^{0.25}(1.5)^{0.45}(1.5)^{0.3} - (0.7)^{0.25}(0.5)^{0.45}(0.5)^{0.3}}{(1.3)^{0.25}(1.5)^{0.45}(1.5)^{0.3} + (0.7)^{0.25}(0.5)^{0.45}(0.5)^{0.3}}, \frac{(1.6)^{0.25}(1.6)^{0.45}(1.6)^{0.3} - (0.4)^{0.25}(0.4)^{0.45}(0.4)^{0.3}}{(1.6)^{0.25}(1.6)^{0.45}(1.6)^{0.3} + (0.4)^{0.25}(0.4)^{0.45}(0.4)^{0.3}} \right], \right.$$

$$\left. \left[\frac{2\left\{(0.7)^{0.25}(0.5)^{0.45}(0.5)^{0.3} - (0.4)^{0.25}(0.2)^{0.45}(0.4)^{0.3}\right\}}{(1.3)^{0.25}(1.5)^{0.45}(1.5)^{0.3} + (0.7)^{0.25}(0.5)^{0.45}(0.5)^{0.3}}, \frac{2\left\{(0.4)^{0.25}(0.4)^{0.45}(0.4)^{0.3} - (0.0)^{0.25}(0.0)^{0.45}(0.1)^{0.3}\right\}}{(1.6)^{0.25}(1.6)^{0.45}(1.6)^{0.3} + (0.4)^{0.25}(0.4)^{0.45}(0.4)^{0.3}} \right] \right\rangle$$

$$= \langle [0.4537, 0.6000], [0.2522, 0.4000] \rangle$$

$$r_4 = \text{IVIFHIWA}(r_{41}, r_{42}, r_{43})$$

$$= \left\langle \left[\frac{(1.7)^{0.25}(1.6)^{0.45}(1.3)^{0.3} - (0.3)^{0.25}(0.4)^{0.45}(0.7)^{0.3}}{(1.7)^{0.25}(1.6)^{0.45}(1.3)^{0.3} + (0.3)^{0.25}(0.4)^{0.45}(0.7)^{0.3}}, \frac{(1.8)^{0.25}(1.7)^{0.45}(1.4)^{0.3} - (0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3}}{(1.8)^{0.25}(1.7)^{0.45}(1.4)^{0.3} + (0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3}} \right], \right.$$

$$\left. \left[\frac{2\left\{(0.3)^{0.25}(0.4)^{0.45}(0.3)^{0.3} - (0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3}\right\}}{(1.7)^{0.25}(1.6)^{0.45}(1.3)^{0.3} + (0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3}}, \frac{2\left\{(0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3} - (0.0)^{0.25}(0.0)^{0.45}(0.4)^{0.3}\right\}}{(1.8)^{0.25}(1.7)^{0.45}(1.4)^{0.3} + (0.2)^{0.25}(0.3)^{0.45}(0.6)^{0.3}} \right] \right\rangle$$

$$= \langle [0.5522, 0.6596], [0.1084, 0.3404] \rangle.$$

Box VII

Table 1. Comparison with IVIFHIWA and existing operators.

	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
	Xu [3]	Proposed	Wang and Liu [14]	Proposed	Liu [25]	Proposed
	Score value		Score value		Score value	
r_1	0.0395	0.0082	0.0301	-0.0019	0.0247	-0.0082
r_2	0.4101	0.3799	0.4082	0.3781	0.4073	0.3772
r_3	0.2367	0.2025	0.2337	0.2008	0.2323	0.1998
r_4	0.4430	0.3928	0.4355	0.3815	0.4318	0.3758
Ranking	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$

$$\dot{R}_{\sigma(ij)}(x_{ij}0) =$$

$$\begin{bmatrix} \langle[0.4687, 0.5778], [0.3000, 0.3610]\rangle & \langle[0.3075, 0.4776], [0.1816, 0.5224]\rangle & \langle[0.1050, 0.3140], [0.5130, 0.5971]\rangle \\ \langle[0.6814, 0.7782], [0.1799, 0.2218]\rangle & \langle[0.4176, 0.7215], [0.1015, 0.1907]\rangle & \langle[0.4776, 0.5720], [0.2118, 0.4280]\rangle \\ \langle[0.5203, 0.6217], [0.1002, 0.2900]\rangle & \langle[0.3552, 0.6814], [0.3153, 0.3186]\rangle & \langle[0.3902, 0.4776], [0.3031, 0.5224]\rangle \\ \langle[0.7782, 0.8664], [0.0855, 0.1336]\rangle & \langle[0.4776, 0.5720], [0.1014, 0.4280]\rangle & \langle[0.3140, 0.4176], [0.1025, 0.2019]\rangle \end{bmatrix}$$

Box VIII

$$\left[\frac{2\{(0.6)^{1.2} - (0.3)^{1.2}\}}{(1.4)^{1.2} + (0.6)^{1.2}}, \frac{2\{(0.5)^{1.2} - (0.1)^{1.2}\}}{(1.5)^{1.2} + (0.5)^{1.2}}\right]\Bigg\rangle$$

$$= \langle[0.4687, 0.5778], [0.3000, 0.3610]\rangle$$

$$\dot{\alpha}_{12} = \Bigg\langle \left[\frac{(1.4)^{0.75} - (0.6)^{0.75}}{(1.4)^{0.75} + (0.6)^{0.75}}, \frac{(1.6)^{0.75} - (0.4)^{0.75}}{(1.6)^{0.75} + (0.4)^{0.75}}\right],$$

$$\left[\frac{2\{(0.6)^{0.75} - (0.4)^{0.75}\}}{(1.4)^{0.75} + (0.6)^{0.75}}, \frac{2\{(0.4)^{0.75} - (0.0)^{0.75}\}}{(1.6)^{0.75} + (0.4)^{0.75}}\right]\Bigg\rangle$$

$$= \langle[0.3075, 0.4776], [0.1816, 0.5224]\rangle$$

$$\dot{\alpha}_{13} = \Bigg\langle \left[\frac{(1.1)^{1.05} - (0.9)^{1.05}}{(1.1)^{1.05} + (0.9)^{1.05}}, \frac{(1.3)^{1.05} - (0.7)^{1.05}}{(1.3)^{1.05} + (0.7)^{1.05}}\right],$$

$$\left[\frac{2\{(0.7)^{1.05} - (0.4)^{1.05}\}}{(1.1)^{1.05} + (0.9)^{1.05}}, \frac{2\{(0.7)^{1.05} - (0.1)^{1.05}\}}{(1.3)^{1.05} + (0.7)^{1.05}}\right]\Bigg\rangle$$

$$= \langle[0.0990, 0.2972], [0.4973, 0.6004]\rangle$$

$$= \langle[0.1050, 0.3140], [0.5130, 0.5971]\rangle.$$

Similarly:

$$\dot{\alpha}_{21} = \langle[0.6814, 0.7782], [0.1799, 0.2218]\rangle,$$

$$\dot{\alpha}_{22} = \langle[0.4776, 0.5720], [0.2118, 0.4280]\rangle,$$

$$\dot{\alpha}_{23} = \langle[0.4176, 0.7215], [0.1015, 0.1907]\rangle,$$

$$\dot{\alpha}_{31} = \langle[0.3552, 0.6814], [0.3153, 0.3186]\rangle,$$

$$\dot{\alpha}_{32} = \langle[0.3902, 0.4776], [0.3031, 0.5224]\rangle,$$

$$\dot{\alpha}_{33} = \langle[0.5203, 0.6217], [0.1002, 0.2900]\rangle$$

$$\dot{\alpha}_{41} = \langle[0.7782, 0.8664], [0.0855, 0.1336]\rangle,$$

$$\dot{\alpha}_{42} = \langle[0.4776, 0.5720], [0.1014, 0.4280]\rangle,$$

$$\dot{\alpha}_{43} = \langle[0.3140, 0.4176], [0.1025, 0.2019]\rangle.$$

Then by score functions, we can get $\dot{R}_{\sigma(ij)}$ as shown in Box VIII.

Hence, the overall performance value r_i corresponding to each alternative, A_i, is obtained by the aggregated IVIFHIHWA operator corresponding to the weight $\omega_i = (0.25, 0.45, 0.30)^T$ as:

$$r_1 = \langle[0.2929, 0.4591], [0.3191, 0.5409]\rangle,$$

$$r_2 = \langle[0.5110, 0.6989], [0.1772, 0.3011]\rangle,$$

$$r_3 = \langle[0.4094, 0.6121], [0.2556, 0.3879]\rangle,$$

$$r_4 = \langle[0.5311, 0.6386], [0.1100, 0.3614]\rangle.$$

By using the score function, we get the score values of these respective alternatives as:

$$S(r_1) = -0.0540, \quad S(r_2) = 0.3658,$$

$$S(r_3) = 0.1890, \quad S(r_4) = 0.3491.$$

Therefore, the ranking order of the four alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$, i.e., food company $\succ$ arms company $\succ$ computer company $\succ$ car company; thus, A_2 (i.e., food company) is the most desirable one and A_1 (i.e., car company) is the least desirable one.

In order to compare the ranking of these alternatives with the rankings in other aggregating operators, namely, IVIFWA [3], IVIFEWA [14], and IVIFHWA [25], by properly assigning the value of γ

to a desired number, their corresponding score values and an overall aggregated IVIFN for alternatives are given in Table 2 by the existing and proposed operators. From these results, it can be concluded that the results of the proposed operator coincide with results of the existing methodologies and the obtained aggregated IVIFN is more optimistic than the aggregated values of the existing methodologies for taking a decision.

To analyze the effect of γ on the most desirable alternatives in the given attributes, we use different values of γ in the proposed approach to rank the alternatives. The corresponding score values and their ranking order are summarized in Table 3. From this table, it can be seen that the aggregation results by using different values of γ are different, but the corresponding rankings of the alternatives are the same.

Table 2. Comparison with IVIFHIHWA and existing operators.

	$\gamma = 1$		**$\gamma = 2$**		**$\gamma = 3$**	
	Xu [3]	**Proposed**	**Wang and Liu [14]**	**Proposed**	**Liu 25**	**Proposed**
	Score value		**Score value**		**Score value**	
r_1	-0.0157	-0.0397	-0.0312	-0.0540	-0.0411	-0.0635
r_2	0.4102	0.3683	0.4075	0.3658	0.4061	0.3644
r_3	0.2212	0.1885	0.2216	0.1890	0.2227	0.1899
r_4	0.4169	0.3679	0.4042	0.3491	0.3986	0.3384
Ranking	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 3. Ordering of the attributes for different values of γ.

γ		**By IVIFHIWA**		**By IVIFHIOWA**		**By IVIFHIHWA**	
		Aggregated IVIFN	**Score values**	**Aggregated IVIFN**	**Score values**	**Aggregated IVIFN**	**Score values**
0.1	A_1	$\langle[0.3319, 0.5108], [0.3011, 0.4892]\rangle$	0.0262	$\langle[0.3319, 0.4862][0.3216, 0.5138]\rangle$	-0.0086	$\langle[0.3134, 0.4820], [0.3099, 0.5180]\rangle$	-0.0162
	A_2	$\langle[0.5542, 0.7000], [0.1860, 0.3000]\rangle$	0.3841	$\langle[0.5542, 0.7000][0.1860, 0.3000]\rangle$	0.3841	$\langle[0.5226, 0.6994], [0.1730, 0.3006]\rangle$	0.3742
	A_3	$\langle[0.4607, 0.6000], [0.2489, 0.4000]\rangle$	0.2059	$\langle[0.4522, 0.6000][0.2578, 0.4000]\rangle$	0.1972	$\langle[0.4171, 0.6070], [0.2522, 0.3930]\rangle$	0.1894
	A_4	$\langle[0.5771, 0.6863], [0.1023, 0.3137]\rangle$	0.4237	$\langle[0.5771, 0.6863], [0.1023, 0.3137]\rangle$	0.4237	$\langle[0.5675, 0.6792], [0.1014, 0.3208]\rangle$	0.4123
Ranking		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$	
0.5	A_1	$\langle[0.3271, 0.5045], [0.3032, 0.4955]\rangle$	0.0164	$\langle[0.3271, 0.4809], [0.3239, 0.5191]\rangle$	-0.0175	$\langle[0.3074, 0.4738], [0.3126, 0.5262]\rangle$	-0.0288
	A_2	$\langle[0.5507, 0.7000], [0.1874, 0.3000]\rangle$	0.3816	$\langle[0.5507, 0.7000], [0.1874, 0.3000]\rangle$	0.3816	$\langle[0.5178, 0.6992], [0.1748, 0.3008]\rangle$	0.3707
	A_3	$\langle[0.4582, 0.6000], [0.2501, 0.4000]\rangle$	0.2040	$\langle[0.4493, 0.6000], [0.2592, 0.4000]\rangle$	0.1950	$\langle[0.4140, 0.6085], [0.2536, 0.3915]\rangle$	0.1887
	A_4	$\langle[0.5668, 0.6736], [0.1048, 0.3264]\rangle$	0.4046	$\langle[0.5668, 0.6736], [0.1048, 0.3264]\rangle$	0.4046	$\langle[0.5533, 0.6613], [0.1048, 0.3387]\rangle$	0.3855
Ranking		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$	
1	A_1	$\langle[0.3224, 0.4997], [0.3054, 0.5003]\rangle$	0.0082	$\langle[0.3224, 0.4769], [0.3262, 0.5231]\rangle$	-0.0250	$\langle[0.3015, 0.4672], [0.3152, 0.5328]\rangle$	-0.0397
	A_2	$\langle[0.5483, 0.7000], [0.1885, 0.3000]\rangle$	0.3799	$\langle[0.5483, 0.7000], [0.1885, 0.3000]\rangle$	0.3799	$\langle[0.5144, 0.6991], [0.1760, 0.3009]\rangle$	0.3683
	A_3	$\langle[0.4561, 0.6000], [0.2511, 0.4000]\rangle$	0.2025	$\langle[0.4469, 0.6000], [0.2603, 0.4000]\rangle$	0.1933	$\langle[0.4117, 0.6100], [0.2546, 0.3900]\rangle$	0.1885
	A_4	$\langle[0.5597, 0.6663], [0.1065, 0.3337]\rangle$	0.3928	$\langle[0.5597, 0.6663], [0.1065, 0.3337]\rangle$	0.3928	$\langle[0.5429, 0.6501], [0.1072, 0.3499]\rangle$	0.3679
Ranking		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$	
2	A_1	$\langle[0.3155, 0.4946], [0.3085, 0.5054]\rangle$	-0.0019	$\langle[0.3155, 0.4725], [0.3295, 0.5274]\rangle$	-0.0344	$\langle[0.2929, 0.4591], [0.3191, 0.5409]\rangle$	-0.0540
	A_2	$\langle[0.5457, 0.7000], [0.1895, 0.3000]\rangle$	0.3781	$\langle[0.5457, 0.7000], [0.1895, 0.3000]\rangle$	0.3781	$\langle[0.5110, 0.6989], [0.1772, 0.3011]\rangle$	0.3658
	A_3	$\langle[0.4537, 0.6000], [0.2522, 0.4000]\rangle$	0.2008	$\langle[0.4441, 0.6000], [0.2616, 0.4000]\rangle$	0.1913	$\langle[0.4094, 0.6121], [0.2556, 0.3879]\rangle$	0.1890
	A_4	$\langle[0.5522, 0.6596], [0.1084, 0.3404]\rangle$	0.3815	$\langle[0.5522, 0.6596], [0.1084, 0.3404]\rangle$	0.3815	$\langle[0.5311, 0.6386], [0.1100, 0.3614]\rangle$	0.3491
Ranking		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_4 \succ A_2 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$	
5	A_1	$\langle[0.3043, 0.4888], [0.3135, 0.5112]\rangle$	-0.0158	$\langle[0.3043, 0.4676], [0.3349, 0.5324]\rangle$	-0.0477	$\langle[0.2782, 0.4479], [0.3258, 0.5521]\rangle$	-0.0759
	A_2	$\langle[0.5431, 0.7000], [0.1906, 0.3000]\rangle$	0.3763	$\langle[0.5431, 0.7000], [0.1906, 0.3000]\rangle$	0.3763	$\langle[0.5074, 0.6986], [0.1785, 0.3014]\rangle$	0.3630
	A_3	$\langle[0.4508, 0.6000], [0.2535, 0.4000]\rangle$	0.1987	$\langle[0.4408, 0.6000], [0.2631, 0.4000]\rangle$	0.1889	$\langle[0.4080, 0.6158], [0.2562, 0.3842]\rangle$	0.1917
	A_4	$\langle[0.5441, 0.6532], [0.1103, 0.3468]\rangle$	0.3701	$\langle[0.5441, 0.6532], [0.1103, 0.3468]\rangle$	0.3701	$\langle[0.4978, 0.6101], [0.1147, 0.3899]\rangle$	0.3016
Ranking		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$	
10	A_1	$\langle[0.2961, 0.4859], [0.3172, 0.5141]\rangle$	-0..0247	$\langle[0.2961, 0.4651], [0.3388, 0.5349]\rangle$	-0.0562	$\langle[0.2667, 0.4395], [0.3310, 0.5605]\rangle$	-0.0926
	A_2	$\langle[0.5419, 0.7000], [0.1911, 0.3000]\rangle$	0.3754	$\langle[0.5419, 0.7000], [0.1911, 0.3000]\rangle$	0.3754	$\langle[0.5055, 0.6982], [0.1792, 0.3018]\rangle$	0.3614
	A_3	$\langle[0.4493, 0.6000], [0.2542, 0.4000]\rangle$	0.1975	$\langle[0.4391, 0.6000], [0.2640, 0.4000]\rangle$	0.1876	$\langle[0.4086, 0.6193], [0.2559, 0.3807]\rangle$	0.1957
	A_4	$\langle[0.5401, 0.6504], [0.1113, 0.3496]\rangle$	0.3648	$\langle[0.5401, 0.6504], [0.1113, 0.3496]\rangle$	0.3648	$\langle[0.4918, 0.6056], [0.1160, 0.3944]\rangle$	0.2935
Ranking		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$	
25	A_1	$\langle[0.2882, 0.4837], [0.3208, 0.5163]\rangle$	-0.0326	$\langle[0.2882, 0.4633], [0.3426, 0.5367]\rangle$	-0.0639	$\langle[0.2533, 0.4290], [0.3370, 0.5710]\rangle$	-0.1129
	A_2	$\langle[0.5411, 0.7000], [0.1915, 0.3000]\rangle$	0.3748	$\langle[0.5411, 0.7000], [0.1915, 0.3000]\rangle$	0.3748	$\langle[0.5040, 0.6978], [0.1798, 0.3022]\rangle$	0.3599
	A_3	$\langle[0.4481, 0.6000], [0.2547, 0.4000]\rangle$	0.1967	$\langle[0.4378, 0.6000], [0.2646, 0.4000]\rangle$	0.1866	$\langle[0.4116, 0.6244], [0.2546, 0.3756]\rangle$	0.2029
	A_4	$\langle[0.5372, 0.6485], [0.1120, 0.3515]\rangle$	0.3611	$\langle[0.5372, 0.6485], [0.1120, 0.3515]\rangle$	0.3611	$\langle[0.4872, 0.6023], [0.1171, 0.3977]\rangle$	0.2874
Ranking		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$	
50	A_1	$\langle[0.2846, 0.4829], [0.3224, 0.5171]\rangle$	-0.0360	$\langle[0.2846, 0.4626], [0.3444, 0.5374]\rangle$	-0.0673	$\langle[0.2450, 0.4211], [0.3407, 0.5789]\rangle$	-0.1267
	A_2	$\langle[0.5408, 0.7000], [0.1916, 0.3000]\rangle$	0.3746	$\langle[0.5408, 0.7000], [0.1916, 0.3000]\rangle$	0.3746	$\langle[0.5032, 0.6975], [0.1800, 0.3025]\rangle$	0.3591
	A_3	$\langle[0.4477, 0.6000], [0.2549, 0.4000]\rangle$	0.1964	$\langle[0.4373, 0.6000], [0.2648, 0.4000]\rangle$	0.1863	$\langle[0.4150, 0.6286], [0.2532, 0.3714]\rangle$	0.2095
	A_4	$\langle[0.5361, 0.6478], [0.1123, 0.3522]\rangle$	0.3597	$\langle[0.5361, 0.6478], [0.1123, 0.3522]\rangle$	0.3597	$\langle[0.4854, 0.6009], [0.1175, 0.3991]\rangle$	0.2849
Ranking		$A_2 \succ A_4 \succ A_3 \succ A_1$		$A_2 \succ A_4 \succ A_3 \succ A_1$		$\boldsymbol{A_2 \succ A_4 \succ A_3 \succ A_1}$	

6. Conclusion

In this manuscript, a series of averaging aggregation operators and their corresponding MCDM approaches have been proposed under the IVIFSs environment. For this, firstly, the short-coming of the existing operational laws and their operators have been highlighted. Then, some new operational laws based on the hesitation degree between the grades of membership and non-membership functions have been given to overcome the drawbacks of existing laws. Based on these new operational laws, a series of aggregation operators, namely, IVIFHIWA, IVIFHIOWA, and IVIFHIHWA, have been developed. Various properties such as idempotency, boundedness, monotonicity, and homogeneity of the operators are investigated. Furthermore, it has been observed from the study that the existing operators, i.e., IVIFWA, IVIFEWA, etc., are the special cases of the proposed operators. By comparison with the existing approaches, it has been concluded that the proposed operators show a more stable, practical, and optimistic nature for the decision makers during the aggregation process. The approach has been illustrated through an example of the MCDM problem and it has been concluded from the aforementioned results that the proposed decision making method can be suitably utilized to solve this type of problems.

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