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Research Note

Sample size determination for C_p comparisons

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Abstract. Comparison of quality for products (supplies and goods) is extremely important for manufacturers and consumers. Based on correct comparisons, manufacturers and consumers can find better suppliers to cooperate and better merchandise to purchase, respectively. Quality is often measured and compared by process capability indices, among which C_p is very effective, simple to apply, and particularly useful for the first round of comparison. In practice, C_p is unknown and should be estimated from observations. Let $\widehat{C_{pi}}$ denote the maximum likelihood estimator obtained from normal process, $\mathcal{X}_i$, with index value $C_{pi}, i = 1, 2$. If $\widehat{C_{p1}} > (<)\widehat{C_{p2}}$ is observed, we will conclude that $C_{p1} > (<)C_{p2}$ and decide that $\mathcal{X}_1$ is better (worse) than $\mathcal{X}_2$. Given a small and positive number, ϵ, there is no need to make comparison when $(1-\epsilon)C_{p2} < C_{p1} < (1+\epsilon)C_{p2}$ since C_{p1} is close to C_{p2}. It is desirable to observe $\widehat{C_{p1}} > \widehat{C_{p2}}$ with high probability when $(1+\epsilon)C_{p2} < C_{p1}$ and with low probability when $(1-\epsilon)C_{p2} > C_{p1}$. Given $0 < \epsilon_1, \epsilon_2 < 1$, based on the table constructed from $P(\widehat{C_{p1}} > \widehat{C_{p2}})$, we demonstrate how to find the smallest sample size needed to ensure observing $\widehat{C_{p1}} > \widehat{C_{p2}}$ with probability greater than $1-\epsilon_1$ when $(1+\epsilon)C_{p2} < C_{p1}$ and smaller than ϵ_2 when $(1-\epsilon)C_{p2} > C_{p1}$.

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1. Introduction

In order to build up a good credit, producers have to make sure that their products meet customers' requests with high probability. Therefore, process capability analysis is, a necessity. Once a stable process is confirmed in phase I study, process capability analysis can then be designed to estimate the proportion of parts that meets engineering requirements in a stable production process. There are several ways for the purpose; a well-known summary quantitative measure is the process capability index. It is designed for industrial field for the purpose of process assessment and improvement for suppliers and consumers; however, it is also used extensively in many different fields which include education [1] and medical science [2].

In the past decades, there have been varieties of innovative process capability indices. All of these indices were designed to evaluate the potential ability of processes to meet different kinds of requirements. But, as pointed out in [3], none of the new indices has surpassed C_p and C_{pk} in popularity or ease of use.

Kane [4] introduced index C_p for the very first time defined as the fraction between the allowable range of measurements over which the process measurement can vary and the range of measurements in which the process is actually varying. It can be expressed as $C_p = \frac{\text{USL}-\text{LSL}}{6\sigma}$, where USL and LSL are the upper and the lower specification limits, respectively, and σ is the process standard deviation. The index was named differently by different authors. For example, Finley [5] used the term "capability potential index"

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to which Montgomery [6] referred as process capability ratio. Notice that C_p concerns only the spread of a process not the process mean at all. So, it provides valuable information only when the mean is on target. The index C_p is particularly suitable when the independent data are from a normal distribution and it is meaningless to use it if the process is not under statistical control.

The index C_p has been proved to be useful in automotive and manufacturing industries, among others. For instance, it has been applied to piston-rings for an automotive engine [6] and bolts used in bridge construction [7]. Applications to fan housing weight, roller width, exhaust valves, and acrylic coating can be found in [8]. Recent applications of C_p include engine axle manufacturing process [9], yogurt production [10], and calibration [11,12], etc.

In practice, C_p is rarely known and has to be estimated from sample. Most of the statistical research concerning C_p focus on the standard inferences including point and interval estimations and test. For examples, the maximum likelihood estimator of C_p was proposed by Kane [4], its moments were discussed in Kotz and Johnson [7], and its sampling distribution was investigated by Chan et al. [13], Chou and Owen [14], Chou et al. [15], and Li et al. [16]. It was further shown that the maximum likelihood estimator is biased, but it is asymptotically unbiased when the sample size is large enough (see [3,7,8,17] and references therein for more details). In this paper, we present a different and useful study for C_p comparison to be described explicitly below.

It is not uncommon that different manufactories may have different qualities for the same product. It is important to recognize the better process, since the better process can serve the consumers better and the worse process can be investigated by the manufacturers for improvements.

Let $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ denote the distributions of measurements from two manufacturing processes $\mathcal{X}_1$ and $\mathcal{X}_2$ with potential capability indices $C_{p1} = \frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_1}$ and $C_{p2} = \frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_2}$, respectively. Note that we assume both process have the same upper and lower specification limits. Since it is desirable to have a C_p as large as possible, process $\mathcal{X}_1$ is considered to be better (worse) than process $\mathcal{X}_2$ if $C_{p1} > (<) C_{p2}$.

Unfortunately, C_{p1} and C_{p2} are often unknown, and hence cannot be applied to make comparisons. In practice, C_{p1} and C_{p2} can be inferred from samples. Let $X_{11}, \cdots, X_{1n}$ and $X_{21}, \cdots, X_{2n}$ denote two independent samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Let $S_1^2 = \frac{1}{n}\sum_{i=1}^n (X_{1i} - \bar{X}_1)^2$ and $S_2^2 = \frac{1}{n}\sum_{i=1}^n (X_{2i} - \bar{X}_2)^2$ denote the sample variances, where $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^n X_{1i}$ and $\bar{X}_2 = \frac{1}{n}\sum_{i=1}^n X_{2i}$ are the sample means. Since $\bar{X}_1, S_1^2, \bar{X}_2$, and S_2^2 are maximum likelihood estimators of μ_1, σ_1^2, μ_2, and σ_2^2, respectively, $\widehat{C_{p1}} = \frac{\mathrm{USL}-\mathrm{LSL}}{6S_1}$ and $\widehat{C_{p2}} = \frac{\mathrm{USL}-\mathrm{LSL}}{6S_2}$ are maximum likelihood estimators of $C_{p1} = \frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_1}$ and $C_{p2} = \frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_2}$, respectively. See Hogg and Craig [18] for more details.

The purpose of this paper is to provide an efficient way to find the smallest sample sizes that can help choosing the better process between $\mathcal{X}_1$ and $\mathcal{X}_2$ based on $\widehat{C_{p1}}$ and $\widehat{C_{p2}}$. No such study can be found in the literatures, to the best of our knowledge.

Without loss of generality, assume that $\widehat{C_{p1}} > \widehat{C_{p2}}$ is observed; naturally we decide that $C_{p1} > C_{p2}$. We will judge this decision under three cases: $C_{p1} \in M$, $C_{p1} \in R$, and $C_{p1} \in L$, where $L = (0, (1-\epsilon)C_{p2}]$, $M = ((1-\epsilon)C_{p2}, (1+\epsilon)C_{p2})$, $R = [(1+\epsilon)C_{p2}, \infty)$, and ϵ is a small positive number.

We treat M as an indifferent zone, since C_{p1} is close to C_{p2} when $C_{p1} \in M$, and there is no need to make comparisons between $\mathcal{X}_1$ and $\mathcal{X}_2$. In this case, any decision will be good.

If $C_{p1} \in R$, then $C_{p1} > C_{p2}$ and our decision will be good if we observe $\widehat{C_{p1}} > \widehat{C_{p2}}$ with high probability. In other words, it is desirable to have large value of:

$$\min_{C_{p1} \in R} P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right). \tag{1}$$

If $C_{p1} \in L$, then $C_{p1} < C_{p2}$ and our decision will be good if we observe $\widehat{C_{p1}} > \widehat{C_{p2}}$ with low probability. Thus, it is desirable to have small value of:

$$\max_{C_{p1} \in L} P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right). \tag{2}$$

We explain the maximum in Relation (1) and the minimum in Relation (2) as follows. Note that for fixed n, $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ is a function of C_{p1} and C_{p2} (see Appendix), denoted as $F(C_{p1}, C_{p2}) = P(\widehat{C_{p1}} > \widehat{C_{p2}})$. The minimum value of F over $\{(C_{p1}, C_{p2}) | C_{p1} > (1+\epsilon)C_{p2}\}$ is denoted by:

$$\min_{C_{p1} \in R} P(\widehat{C_{p1}} > \widehat{C_{p2}}).$$

The maximum value of F over $\{(C_{p1}, C_{p2}) | C_{p1} < (1-\epsilon)C_{p2}\}$ is denoted by:

$$\max_{C_{p1} \in L} P(\widehat{C_{p1}} > \widehat{C_{p2}}).$$

Given two small positive numbers ϵ_1 and ϵ_2, we aim to find the smallest sample size to ensure:

$$\min_{C_{p1} \in R} P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) > 1 - \epsilon_1, \tag{3}$$

and the smallest sample size to achieve:

$$\max_{C_{p1} \in L} P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) < \epsilon_2. \tag{4}$$

The rest of this paper is organized as follows: in

Section 2, we will give the form of $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ and the proof can be found in the Appendix. In Section 3, illustrations of sample size determinations will be given via some examples. Some remarks concerning the statistical inferences are presented in Section 4. Future study is proposed in Section 5. Conclusions are provided in Section 6.

2. Derivation of probabilities

In order to find the smallest n that achieves Relations (3) and (4), we need to evaluate $P(\widehat{C_{p1}} > \widehat{C_{p2}})$. The proofs of the following results are given in Appendix:

$$P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) = \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{n-2}dt, \qquad (5)$$

where $\alpha = \tan^{-1}(\sigma_1/\sigma_2)$. When $n = 2k+3$, where k denotes any nonnegative integer, Eq. (5) implies that:

$$P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) = \frac{\Gamma(2k+2)}{2^{2k+1}[\Gamma(k+1)]^2} \cdot \sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{[\cos(2\alpha)]^{2i+1}+1}{2i+1}, \qquad (6)$$

where $\cos 2\alpha = \frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2+\sigma_2^2}$. Moreover, when $n = 2k+2$:

$$P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) = \frac{\Gamma(2k+1)}{2^{2k}\left[\Gamma\left(\frac{2k+1}{2}\right)\right]^2}\left\{\frac{(\sin 2\alpha)^{2k-1}(\cos 2\alpha)}{2k} + \frac{2k-1}{(2k)(2k-2)}(\sin 2\alpha)^{2k-3}(\cos 2\alpha) + \frac{(2k-1)(2k-3)}{(2k)(2k-2)(2k-4)}(\sin 2\alpha)^{2k-5}(\cos 2\alpha) + \cdots + \frac{(2k-1)(2k-3)\cdots 3}{(2k)(2k-2)(2k-4)\cdots 2}(\sin 2\alpha)(\cos 2\alpha) + \frac{(2k-1)(2k-3)\cdots 3.1}{(2k)(2k-2)(2k-4)\cdots 2}(\pi - 2\alpha)\right\}, \qquad (7)$$

where:

$$\alpha = \tan^{-1}(\sigma_1/\sigma_2), \sin 2\alpha = \frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2},$$

$$\cos 2\alpha = \frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2+\sigma_2^2}.$$

Given n and $\frac{\sigma_2}{\sigma_1}$ (or $\frac{C_{p1}}{C_{p2}}$), Eqs. (6) and (7) can be calculated. To save the places, the results for $n \in \{3, \cdots, 100\}$ and $\frac{C_{p1}}{C_{p2}} \in \{0.1, 0.2, \cdots, 0.9, 0.95, 1, 1.05, 1.1, 1.2, \cdots, 2\}$, are given in Tables 1 to 3.

It is easy to see from Tables 1 to 3 that $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ increases with $\frac{C_{p1}}{C_{p2}}$. Moreover, $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ increases (decreases) with n if $\frac{C_{p1}}{C_{p2}} > (<)1$. It should be noted that the rounding error produces many 0 in Table 1 and many 1 in Table 3. Each 0 represents a positive number less than 0.00001, and each 1 represents a positive number between 1 and 0.99999.

3. Illustrations

In this section, we explain how to use the above results to find the smallest sample sizes needed to make comparisons so as to achieve the predetermined bounds for probabilities (1) and (2).

Example 1: If $\epsilon = 0.05$, then the indifferent zone defined in Section 1 is $M = [(1-\epsilon)C_{p2}, (1+\epsilon)C_{p2}] = [0.95C_{p2}, 1.05C_{p2}]$, and we treat two processes with C_{p1} and C_{p2} equally well when $C_{p1} \in M$. Note that 0.95 and 1.05 are chosen for convenience; they can be replaced by other suitable positive numbers u and v, such that $u < 1$ and $v > 1$.

If $C_{p1} \in R = [1.05C_{p2}, \infty)$, then $C_{p1} > C_{p2}$ and we want $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ to be large. If $C_{p1} \in L = (0, 0.95C_{p2}]$, then $C_{p1} < C_{p2}$ and we want $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ to be small. For example, if 0.67 and 0.35 are considered large and small enough, respectively, then the problem becomes what sample size n will ensure that:

$$\min_{C_{p1}\in R} P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right) > 0.67, \qquad (8)$$

and:

$$\max_{C_{p1}\in L} P(\widehat{C_{p1}} > \widehat{C_{p2}}) < 0.35. \qquad (9)$$

Note that 0.67 and 0.35 are chosen for convenience; they can be replaced by any number γ and δ, such that $0 < \gamma < 1$, $0 < \delta < 1$, and γ and δ are close to 1 and 0, respectively.

In order to achieve Relation (8), we need $n \geq 83$ by Table 2 (check the rows and columns corresponding to $n \geq 83$ and $\frac{C_{p1}}{C_{p2}} \geq 1.05$, respectively).

Moreover, Relation (9) is true if $n \geq 58$ according to Table 2 (check the rows and columns corresponding to $n \geq 58$ and $\frac{C_{p1}}{C_{p2}} \leq 0.95$, respectively).

Consequently, if $n \geq 83$, then Relations (8) and (9) will be held simultaneously.

Example 2: If $\epsilon = 0.1$, then $R = [(1+\epsilon)C_{p2}, \infty) = [1.1C_{p2}, \infty)$ and $L = (0, (1-\epsilon)C_{p2}] = (0, 0.9C_{p2}]$. What sample size n will ensure for Relations (8) and (9)?

In order to achieve Relation (8), we need $n \geq 23$ by Table 2 (check the rows and columns corresponding to $n \geq 23$ and $\frac{C_{p1}}{C_{p2}} \geq 1.1$, respectively).

Table 1. $P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$ for given n when $\frac{C_{p1}}{C_{p2}} \in \{0.1, \cdots, 0.8\}$.

	$\frac{C_{p1}}{C_{p2}}$							
n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
3	0.00990	0.03846	0.08257	0.13793	0.20000	0.26471	0.32886	0.39024
4	0.00167	0.01266	0.03927	0.08327	0.14238	0.21187	0.28643	0.36138
5	0.00029	0.00432	0.01933	0.05183	0.10400	0.17311	0.25331	0.33801
6	0.00005	0.00151	0.00972	0.03287	0.07719	0.14327	0.22618	0.31813
7	0.00001	0.00054	0.00496	0.02111	0.05792	0.11963	0.20329	0.30072
8	0.00000	0.00019	0.00255	0.01369	0.04381	0.10054	0.18362	0.28519
9	0.00000	0.00007	0.00133	0.00894	0.03334	0.08493	0.16649	0.27115
10	0.00000	0.00003	0.00069	0.00587	0.02550	0.07203	0.15142	0.25832
11	0.00000	0.00001	0.00036	0.00387	0.01958	0.06129	0.13806	0.24652
12	0.00000	0.00000	0.00019	0.00256	0.01509	0.05230	0.12615	0.23558
13	0.00000	0.00000	0.00010	0.00170	0.01165	0.04473	0.11548	0.22541
14	0.00000	0.00000	0.00005	0.00113	0.00903	0.03833	0.10587	0.21591
15	0.00000	0.00000	0.00003	0.00076	0.00700	0.03291	0.09720	0.20700
16	0.00000	0.00000	0.00002	0.00051	0.00544	0.02830	0.08934	0.19863
17	0.00000	0.00000	0.00001	0.00034	0.00424	0.02436	0.08221	0.19073
18	0.00000	0.00000	0.00000	0.00023	0.00331	0.02100	0.07572	0.18327
19	0.00000	0.00000	0.00000	0.00015	0.00258	0.01813	0.06981	0.17621
20	0.00000	0.00000	0.00000	0.00010	0.00202	0.01566	0.06441	0.16952
21	0.00000	0.00000	0.00000	0.00007	0.00158	0.01354	0.05947	0.16316
22	0.00000	0.00000	0.00000	0.00005	0.00124	0.01172	0.05494	0.15712
23	0.00000	0.00000	0.00000	0.00003	0.00097	0.01015	0.05079	0.15136
24	0.00000	0.00000	0.00000	0.00002	0.00076	0.00880	0.04699	0.14587
25	0.00000	0.00000	0.00000	0.00001	0.00060	0.00763	0.04348	0.14064
26	0.00000	0.00000	0.00000	0.00001	0.00047	0.00662	0.04027	0.13564
27	0.00000	0.00000	0.00000	0.00001	0.00037	0.00575	0.03730	0.13086
28	0.00000	0.00000	0.00000	0.00000	0.00029	0.00499	0.03457	0.12628
29	0.00000	0.00000	0.00000	0.00000	0.00023	0.00434	0.03205	0.12191
30	0.00000	0.00000	0.00000	0.00000	0.00018	0.00377	0.02973	0.11771
31	0.00000	0.00000	0.00000	0.00000	0.00014	0.00328	0.02758	0.11369
32	0.00000	0.00000	0.00000	0.00000	0.00011	0.00286	0.02560	0.10983
33	0.00000	0.00000	0.00000	0.00000	0.00009	0.00249	0.02377	0.10613
34	0.00000	0.00000	0.00000	0.00000	0.00007	0.00217	0.02207	0.10258
35	0.00000	0.00000	0.00000	0.00000	0.00005	0.00189	0.02051	0.09916
36	0.00000	0.00000	0.00000	0.00000	0.00004	0.00164	0.01906	0.09588
37	0.00000	0.00000	0.00000	0.00000	0.00003	0.00143	0.01771	0.09272
38	0.00000	0.00000	0.00000	0.00000	0.00003	0.00125	0.01647	0.08969
39	0.00000	0.00000	0.00000	0.00000	0.00002	0.00109	0.01531	0.08676
40	0.00000	0.00000	0.00000	0.00000	0.00002	0.00095	0.01424	0.08395
41	0.00000	0.00000	0.00000	0.00000	0.00001	0.00083	0.01325	0.08124
42	0.00000	0.00000	0.00000	0.00000	0.00001	0.00072	0.01233	0.07863
43	0.00000	0.00000	0.00000	0.00000	0.00001	0.00063	0.01148	0.07612
44	0.00000	0.00000	0.00000	0.00000	0.00001	0.00055	0.01068	0.07369
45	0.00000	0.00000	0.00000	0.00000	0.00001	0.00048	0.00995	0.07135
46	0.00000	0.00000	0.00000	0.00000	0.00000	0.00042	0.00926	0.06910
47	0.00000	0.00000	0.00000	0.00000	0.00000	0.00037	0.00863	0.06693
48	0.00000	0.00000	0.00000	0.00000	0.00000	0.00032	0.00803	0.06483
49	0.00000	0.00000	0.00000	0.00000	0.00000	0.00028	0.00749	0.06280
50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00025	0.00698	0.06085
51	0.00000	0.00000	0.00000	0.00000	0.00000	0.00022	0.00650	0.05896

Table 1. $P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$ for given n when $\frac{C_{p1}}{C_{p2}} \in \{0.1, \cdots, 0.8\}$ (continued).

	$\frac{C_{p1}}{C_{p2}}$							
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
52	0.00000	0.00000	0.00000	0.00000	0.00000	0.00019	0.00606	0.05713
53	0.00000	0.00000	0.00000	0.00000	0.00000	0.00016	0.00565	0.05537
54	0.00000	0.00000	0.00000	0.00000	0.00000	0.00014	0.00527	0.05367
55	0.00000	0.00000	0.00000	0.00000	0.00000	0.00013	0.00491	0.05203
56	0.00000	0.00000	0.00000	0.00000	0.00000	0.00011	0.00458	0.05044
57	0.00000	0.00000	0.00000	0.00000	0.00000	0.00010	0.00427	0.04890
58	0.00000	0.00000	0.00000	0.00000	0.00000	0.00008	0.00398	0.04741
59	0.00000	0.00000	0.00000	0.00000	0.00000	0.00007	0.00372	0.04598
60	0.00000	0.00000	0.00000	0.00000	0.00000	0.00006	0.00347	0.04459
61	0.00000	0.00000	0.00000	0.00000	0.00000	0.00006	0.00323	0.04324
62	0.00000	0.00000	0.00000	0.00000	0.00000	0.00005	0.00302	0.04194
63	0.00000	0.00000	0.00000	0.00000	0.00000	0.00004	0.00282	0.04068
64	0.00000	0.00000	0.00000	0.00000	0.00000	0.00004	0.00263	0.03947
65	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003	0.00245	0.03829
66	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003	0.00229	0.03715
67	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003	0.00214	0.03604
68	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00200	0.03497
69	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00186	0.03394
70	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00174	0.03293
71	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00163	0.03196
72	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00152	0.03102
73	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00142	0.03011
74	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00132	0.02923
75	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00124	0.02837
76	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00116	0.02754
77	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00108	0.02674
78	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00101	0.02596
79	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00094	0.02521
80	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00088	0.02448
81	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00082	0.02377
82	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00077	0.02308
83	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00072	0.02241
84	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00067	0.02177
85	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00063	0.02114
86	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00059	0.02053
87	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00055	0.01994
88	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00051	0.01937
89	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00048	0.01882
90	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00045	0.01828
91	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00042	0.01776
92	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00039	0.01725
93	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00037	0.01676
94	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00034	0.01628
95	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00032	0.01582
96	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00030	0.01537
97	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00028	0.01494
98	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00026	0.01451
99	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00025	0.01410
100	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00023	0.01371

Table 2. $P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$ for given n when $\frac{C_{p1}}{C_{p2}} \in \{0.9, \cdots, 1.4\}$.

	$\frac{C_{p1}}{C_{p2}}$							
n	**0.9**	**0.95**	**1.0**	**1.05**	**1.1**	**1.2**	**1.3**	**1.4**
3	0.44751	0.47438	0.50000	0.52438	0.54751	0.59016	0.62825	0.66216
4	0.43330	0.46739	0.50000	0.53102	0.56040	0.61418	0.66149	0.70279
5	0.42156	0.46160	0.50000	0.53653	0.57105	0.63378	0.68816	0.73471
6	0.41139	0.45656	0.50000	0.54133	0.58029	0.65059	0.71064	0.76105
7	0.40231	0.45204	0.50000	0.54563	0.58855	0.66543	0.73013	0.78341
8	0.39405	0.44791	0.50000	0.54956	0.59606	0.67878	0.74737	0.80275
9	0.38644	0.44409	0.50000	0.55320	0.60300	0.69094	0.76281	0.81971
10	0.37936	0.44052	0.50000	0.55659	0.60946	0.70215	0.77678	0.83474
11	0.37271	0.43716	0.50000	0.55980	0.61553	0.71254	0.78952	0.84814
12	0.36644	0.43397	0.50000	0.56283	0.62126	0.72225	0.80121	0.86018
13	0.36049	0.43094	0.50000	0.56573	0.62670	0.73135	0.81198	0.87103
14	0.35483	0.42803	0.50000	0.56849	0.63189	0.73993	0.82195	0.88087
15	0.34941	0.42525	0.50000	0.57115	0.63685	0.74803	0.83121	0.88981
16	0.34423	0.42257	0.50000	0.57370	0.64161	0.75572	0.83984	0.89796
17	0.33924	0.41998	0.50000	0.57617	0.64619	0.76302	0.84790	0.90542
18	0.33444	0.41748	0.50000	0.57855	0.65060	0.76997	0.85544	0.91224
19	0.32981	0.41505	0.50000	0.58086	0.65487	0.77661	0.86251	0.91850
20	0.32533	0.41270	0.50000	0.58311	0.65899	0.78295	0.86915	0.92426
21	0.32100	0.41041	0.50000	0.58529	0.66299	0.78903	0.87540	0.92957
22	0.31680	0.40818	0.50000	0.58741	0.66687	0.79485	0.88129	0.93446
23	0.31272	0.40601	0.50000	0.58948	0.67063	0.80044	0.88684	0.93897
24	0.30876	0.40389	0.50000	0.59150	0.67430	0.80582	0.89209	0.94314
25	0.30490	0.40182	0.50000	0.59348	0.67787	0.81098	0.89705	0.94700
26	0.30115	0.39980	0.50000	0.59541	0.68134	0.81596	0.90174	0.95058
27	0.29750	0.39782	0.50000	0.59730	0.68473	0.82076	0.90618	0.95389
28	0.29393	0.39588	0.50000	0.59915	0.68804	0.82539	0.91039	0.95696
29	0.29046	0.39398	0.50000	0.60096	0.69128	0.82985	0.91438	0.95981
30	0.28706	0.39212	0.50000	0.60274	0.69444	0.83416	0.91818	0.96246
31	0.28374	0.39029	0.50000	0.60448	0.69753	0.83833	0.92178	0.96493
32	0.28049	0.38850	0.50000	0.60620	0.70056	0.84236	0.92520	0.96722
33	0.27732	0.38674	0.50000	0.60788	0.70352	0.84627	0.92845	0.96935
34	0.27421	0.38500	0.50000	0.60954	0.70643	0.85004	0.93155	0.97133
35	0.27116	0.38330	0.50000	0.61116	0.70927	0.85370	0.93449	0.97318
36	0.26818	0.38162	0.50000	0.61277	0.71207	0.85725	0.93730	0.97491
37	0.26526	0.37997	0.50000	0.61434	0.71481	0.86068	0.93997	0.97651
38	0.26239	0.37834	0.50000	0.61590	0.71749	0.86402	0.94252	0.97801
39	0.25958	0.37674	0.50000	0.61743	0.72013	0.86725	0.94495	0.97941
40	0.25682	0.37516	0.50000	0.61894	0.72273	0.87039	0.94726	0.98072
41	0.25411	0.37361	0.50000	0.62042	0.72528	0.87344	0.94947	0.98193
42	0.25145	0.37207	0.50000	0.62189	0.72778	0.87640	0.95158	0.98307
43	0.24884	0.37056	0.50000	0.62334	0.73024	0.87927	0.95359	0.98414
44	0.24627	0.36907	0.50000	0.62477	0.73267	0.88207	0.95552	0.98513
45	0.24375	0.36759	0.50000	0.62618	0.73505	0.88478	0.95735	0.98606
46	0.24127	0.36614	0.50000	0.62757	0.73740	0.88743	0.95911	0.98693
47	0.23883	0.36470	0.50000	0.62894	0.73971	0.88999	0.96078	0.98774
48	0.23643	0.36328	0.50000	0.63030	0.74198	0.89249	0.96239	0.98850
49	0.23407	0.36188	0.50000	0.63164	0.74422	0.89493	0.96392	0.98921
50	0.23175	0.36049	0.50000	0.63297	0.74642	0.89729	0.96539	0.98988
51	0.22946	0.35912	0.50000	0.63428	0.74860	0.89960	0.96679	0.99050

Table 2. $P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$ for given n when $\frac{C_{p1}}{C_{p2}} \in \{0.9, \cdots, 1.4\}$ (continued).

				$\frac{C_{p1}}{C_{p2}}$				
n	**0.9**	**0.95**	**1.0**	**1.05**	**1.1**	**1.2**	**1.3**	**1.4**
52	0.22721	0.35777	0.50000	0.63558	0.75074	0.90184	0.96813	0.99109
53	0.22500	0.35643	0.50000	0.63686	0.75285	0.90403	0.96942	0.99163
54	0.22281	0.35510	0.50000	0.63813	0.75493	0.90616	0.97065	0.99215
55	0.22066	0.35379	0.50000	0.63938	0.75698	0.90823	0.97182	0.99263
56	0.21854	0.35249	0.50000	0.64062	0.75900	0.91025	0.97295	0.99308
57	0.21646	0.35121	0.50000	0.64185	0.76100	0.91222	0.97403	0.99350
58	0.21440	0.34994	0.50000	0.64307	0.76297	0.91414	0.97506	0.99390
59	0.21237	0.34868	0.50000	0.64427	0.76491	0.91602	0.97605	0.99427
60	0.21037	0.34744	0.50000	0.64547	0.76683	0.91784	0.97700	0.99461
61	0.20840	0.34621	0.50000	0.64665	0.76872	0.91962	0.97791	0.99494
62	0.20646	0.34499	0.50000	0.64782	0.77059	0.92136	0.97878	0.99525
63	0.20454	0.34378	0.50000	0.64897	0.77244	0.92305	0.97962	0.99553
64	0.20265	0.34258	0.50000	0.65012	0.77426	0.92471	0.98042	0.99580
65	0.20078	0.34140	0.50000	0.65126	0.77605	0.92632	0.98119	0.99605
66	0.19894	0.34022	0.50000	0.65239	0.77783	0.92789	0.98193	0.99629
67	0.19713	0.33906	0.50000	0.65350	0.77959	0.92943	0.98263	0.99651
68	0.19533	0.33790	0.50000	0.65461	0.78132	0.93093	0.98331	0.99672
69	0.19356	0.33676	0.50000	0.65571	0.78303	0.93239	0.98396	0.99692
70	0.19182	0.33562	0.50000	0.65680	0.78472	0.93382	0.98458	0.99710
71	0.19009	0.33450	0.50000	0.65788	0.78640	0.93522	0.98518	0.99728
72	0.18839	0.33338	0.50000	0.65895	0.78805	0.93658	0.98576	0.99744
73	0.18671	0.33228	0.50000	0.66001	0.78968	0.93791	0.98631	0.99759
74	0.18505	0.33118	0.50000	0.66106	0.79130	0.93921	0.98684	0.99774
75	0.18341	0.33009	0.50000	0.66210	0.79290	0.94048	0.98735	0.99787
76	0.18179	0.32901	0.50000	0.66314	0.79447	0.94172	0.98783	0.99800
77	0.18019	0.32794	0.50000	0.66417	0.79604	0.94293	0.98830	0.99812
78	0.17861	0.32688	0.50000	0.66519	0.79758	0.94412	0.98875	0.99823
79	0.17704	0.32583	0.50000	0.66620	0.79911	0.94527	0.98918	0.99833
80	0.17550	0.32478	0.50000	0.66720	0.80062	0.94641	0.98960	0.99843
81	0.17398	0.32375	0.50000	0.66820	0.80211	0.94751	0.99000	0.99852
82	0.17247	0.32272	0.50000	0.66919	0.80359	0.94859	0.99038	0.99861
83	0.17098	0.32169	0.50000	0.67017	0.80505	0.94965	0.99074	0.99869
84	0.16951	0.32068	0.50000	0.67114	0.80650	0.95068	0.99110	0.99877
85	0.16805	0.31967	0.50000	0.67211	0.80793	0.95169	0.99144	0.99884
86	0.16661	0.31867	0.50000	0.67307	0.80934	0.95268	0.99176	0.99891
87	0.16519	0.31768	0.50000	0.67403	0.81074	0.95364	0.99207	0.99897
88	0.16379	0.31670	0.50000	0.67497	0.81213	0.95459	0.99238	0.99903
89	0.16240	0.31572	0.50000	0.67591	0.81350	0.95551	0.99266	0.99909
90	0.16102	0.31475	0.50000	0.67685	0.81486	0.95641	0.99294	0.99914
91	0.15966	0.31378	0.50000	0.67778	0.81621	0.95730	0.99321	0.99919
92	0.15832	0.31283	0.50000	0.67870	0.81754	0.95816	0.99347	0.99924
93	0.15699	0.31187	0.50000	0.67961	0.81886	0.95900	0.99371	0.99929
94	0.15567	0.31093	0.50000	0.68052	0.82016	0.95983	0.99395	0.99933
95	0.15437	0.30999	0.50000	0.68143	0.82145	0.96064	0.99418	0.99937
96	0.15309	0.30906	0.50000	0.68232	0.82273	0.96143	0.99440	0.99940
97	0.15181	0.30813	0.50000	0.68321	0.82400	0.96220	0.99461	0.99944
98	0.15055	0.30721	0.50000	0.68410	0.82525	0.96296	0.99481	0.99947
99	0.14931	0.30630	0.50000	0.68498	0.82650	0.96370	0.99500	0.99950
100	0.14808	0.30539	0.50000	0.68586	0.82773	0.96443	0.99519	0.99953

Table 3. $P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$ for given n when $\frac{C_{p1}}{C_{p2}} \in \{1.5, \cdots, 2\}$.

	$\frac{C_{p1}}{C_{p2}}$											
n	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**	n	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**
3	0.69231	0.71910	0.74293	0.76415	0.78308	0.80000	52	0.99777	0.99948	0.99989	0.99998	1.00000	1.00000
4	0.73868	0.76976	0.79667	0.81995	0.84012	0.85762	53	0.99796	0.99954	0.99990	0.99998	1.00000	1.00000
5	0.77424	0.80762	0.83572	0.85936	0.87925	0.89600	54	0.99813	0.99959	0.99992	0.99998	1.00000	1.00000
6	0.80287	0.83730	0.86551	0.88857	0.90741	0.92281	55	0.99829	0.99963	0.99993	0.99999	1.00000	1.00000
7	0.82660	0.86125	0.88889	0.91084	0.92826	0.94208	56	0.99843	0.99967	0.99994	0.99999	1.00000	1.00000
8	0.84663	0.88095	0.90758	0.92815	0.94399	0.95619	57	0.99856	0.99971	0.99994	0.99999	1.00000	1.00000
9	0.86377	0.89737	0.92273	0.94177	0.95601	0.96666	58	0.99868	0.99974	0.99995	0.99999	1.00000	1.00000
10	0.87859	0.91119	0.93513	0.95260	0.96529	0.97450	59	0.99879	0.99977	0.99996	0.99999	1.00000	1.00000
11	0.89149	0.92290	0.94536	0.96128	0.97251	0.98042	60	0.99890	0.99979	0.99996	0.99999	1.00000	1.00000
12	0.90279	0.93290	0.95384	0.96827	0.97816	0.98491	61	0.99899	0.99982	0.99997	0.99999	1.00000	1.00000
13	0.91275	0.94147	0.96091	0.97393	0.98260	0.98835	62	0.99907	0.99984	0.99997	1.00000	1.00000	1.00000
14	0.92154	0.94885	0.96683	0.97854	0.98610	0.99097	63	0.99915	0.99985	0.99998	1.00000	1.00000	1.00000
15	0.92935	0.95522	0.97181	0.98230	0.98888	0.99300	64	0.99922	0.99987	0.99998	1.00000	1.00000	1.00000
16	0.93629	0.96074	0.97600	0.98537	0.99109	0.99456	65	0.99929	0.99988	0.99998	1.00000	1.00000	1.00000
17	0.94248	0.96554	0.97953	0.98790	0.99284	0.99576	66	0.99934	0.99990	0.99998	1.00000	1.00000	1.00000
18	0.94802	0.96971	0.98253	0.98997	0.99425	0.99669	67	0.99940	0.99991	0.99999	1.00000	1.00000	1.00000
19	0.95298	0.97335	0.98507	0.99168	0.99537	0.99742	68	0.99945	0.99992	0.99999	1.00000	1.00000	1.00000
20	0.95742	0.97653	0.98723	0.99309	0.99627	0.99798	69	0.99949	0.99993	0.99999	1.00000	1.00000	1.00000
21	0.96142	0.97932	0.98906	0.99426	0.99699	0.99842	70	0.99954	0.99993	0.99999	1.00000	1.00000	1.00000
22	0.96502	0.98176	0.99062	0.99522	0.99757	0.99876	71	0.99957	0.99994	0.99999	1.00000	1.00000	1.00000
23	0.96826	0.98390	0.99196	0.99602	0.99804	0.99903	72	0.99961	0.99995	0.99999	1.00000	1.00000	1.00000
24	0.97118	0.98577	0.99310	0.99668	0.99841	0.99924	73	0.99964	0.99995	0.99999	1.00000	1.00000	1.00000
25	0.97382	0.98743	0.99407	0.99723	0.99871	0.99940	74	0.99967	0.99996	1.00000	1.00000	1.00000	1.00000
26	0.97620	0.98888	0.99491	0.99769	0.99896	0.99953	75	0.99970	0.99996	1.00000	1.00000	1.00000	1.00000
27	0.97835	0.99016	0.99562	0.99807	0.99916	0.99963	76	0.99972	0.99997	1.00000	1.00000	1.00000	1.00000
28	0.98031	0.99129	0.99623	0.99839	0.99932	0.99971	77	0.99975	0.99997	1.00000	1.00000	1.00000	1.00000
29	0.98207	0.99228	0.99676	0.99865	0.99945	0.99977	78	0.99977	0.99997	1.00000	1.00000	1.00000	1.00000
30	0.98368	0.99316	0.99721	0.99887	0.99955	0.99982	79	0.99979	0.99998	1.00000	1.00000	1.00000	1.00000
31	0.98513	0.99394	0.99759	0.99906	0.99963	0.99986	80	0.99980	0.99998	1.00000	1.00000	1.00000	1.00000
32	0.98645	0.99462	0.99792	0.99921	0.99970	0.99989	81	0.99982	0.99998	1.00000	1.00000	1.00000	1.00000
33	0.98765	0.99523	0.99821	0.99934	0.99976	0.99991	82	0.99983	0.99998	1.00000	1.00000	1.00000	1.00000
34	0.98874	0.99577	0.99846	0.99945	0.99980	0.99993	83	0.99985	0.99998	1.00000	1.00000	1.00000	1.00000
35	0.98972	0.99624	0.99867	0.99954	0.99984	0.99995	84	0.99986	0.99999	1.00000	1.00000	1.00000	1.00000
36	0.99062	0.99666	0.99885	0.99961	0.99987	0.99996	85	0.99987	0.99999	1.00000	1.00000	1.00000	1.00000
37	0.99144	0.99704	0.99901	0.99967	0.99989	0.99997	86	0.99988	0.99999	1.00000	1.00000	1.00000	1.00000
38	0.99219	0.99737	0.99914	0.99973	0.99991	0.99997	87	0.99989	0.99999	1.00000	1.00000	1.00000	1.00000
39	0.99287	0.99766	0.99926	0.99977	0.99993	0.99998	88	0.99990	0.99999	1.00000	1.00000	1.00000	1.00000
40	0.99348	0.99792	0.99936	0.99981	0.99994	0.99998	89	0.99991	0.99999	1.00000	1.00000	1.00000	1.00000
41	0.99405	0.99815	0.99945	0.99984	0.99995	0.99999	90	0.99992	0.99999	1.00000	1.00000	1.00000	1.00000
42	0.99456	0.99835	0.99952	0.99986	0.99996	0.99999	91	0.99992	0.99999	1.00000	1.00000	1.00000	1.00000
43	0.99503	0.99854	0.99959	0.99989	0.99997	0.99999	92	0.99993	0.99999	1.00000	1.00000	1.00000	1.00000
44	0.99546	0.99870	0.99964	0.99990	0.99997	0.99999	93	0.99994	1.00000	1.00000	1.00000	1.00000	1.00000
45	0.99584	0.99884	0.99969	0.99992	0.99998	0.99999	94	0.99994	1.00000	1.00000	1.00000	1.00000	1.00000
46	0.99620	0.99897	0.99973	0.99993	0.99998	1.00000	95	0.99995	1.00000	1.00000	1.00000	1.00000	1.00000
47	0.99652	0.99908	0.99977	0.99994	0.99999	1.00000	96	0.99995	1.00000	1.00000	1.00000	1.00000	1.00000
48	0.99682	0.99918	0.99980	0.99995	0.99999	1.00000	97	0.99995	1.00000	1.00000	1.00000	1.00000	1.00000
49	0.99709	0.99927	0.99983	0.99996	0.99999	1.00000	98	0.99996	1.00000	1.00000	1.00000	1.00000	1.00000
50	0.99734	0.99935	0.99985	0.99997	0.99999	1.00000	99	0.99996	1.00000	1.00000	1.00000	1.00000	1.00000
51	0.99756	0.99942	0.99987	0.99997	0.99999	1.00000	100	0.99996	1.00000	1.00000	1.00000	1.00000	1.00000

Moreover, Relation (9) is true if $n \geq 15$ according to Table 2 (check the rows and columns corresponding to $n \geq 15$ and $\frac{C_{p1}}{C_{p2}} \leq 0.9$, respectively).

Consequently, if $n \geq 23$, then Relations (8) and (9) will be held simultaneously.

Example 3: If $\epsilon = 0.1$, what sample size n will ensure that:

$$\min_{C_{p1} \in R} P(\widehat{C_{p1}} > \widehat{C_{p2}}) > 0.8, \tag{10}$$

and:

$$\max_{C_{p1} \in L} P(\widehat{C_{p1}} > \widehat{C_{p2}}) < 0.25? \tag{11}$$

In order to achieve Relation (10), we need $n \geq 80$ by Table 2 (check the rows and columns corresponding to $n \geq 80$ and $\frac{C_{p1}}{C_{p2}} \geq 1.1$, respectively).

Moreover, Relation (11) is true if $n \geq 43$ according to Table 2 (check the rows and columns corresponding to $n \geq 43$ and $\frac{C_{p1}}{C_{p2}} \leq 0.9$, respectively).

Consequently, if $n \geq 80$, then Relations (10) and (11) will be held simultaneously.

Tables can be constructed by Eqs. (6) and (7) for other choices of ϵ, n, and $\frac{C_{p1}}{C_{p2}}$; the smallest sample sizes can be found accordingly to satisfy Relations (3) and (4) for given ϵ_1 and ϵ_2.

In order to find the relation between n and ϵ, we provide Figures 1 and 2 constructed from Tables 1 to 3. It is seen that n is a decreasing function of ϵ. Without any surprises, the reasons are as follows. For $C_{p1} \in L \cup R = (0, (1-\epsilon)C_{p2}] \cup [(1+\epsilon)C_{p2}, \infty)$, increasing ϵ will increase $|C_{p1} - C_{p2}|$, thereby increase $|\widehat{C_{p1}} - \widehat{C_{p2}}|$, since $\widehat{C_{p1}}$ and $\widehat{C_{p2}}$ are close to C_{p1} and C_{p2}, respectively. Large value of $|\widehat{C_{p1}} - \widehat{C_{p2}}|$ will make $\{\widehat{C_{p1}} > \widehat{C_{p2}}\}$ easy to observe when $C_{p1} \in R$ and make $\{\widehat{C_{p1}} > \widehat{C_{p2}}\}$ hard to observe when $C_{p1} \in L$. In other words, large ϵ correspond to small sample size n needed to make $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ large when $C_{p1} \in R$ and make $P(\widehat{C_{p1}} > \widehat{C_{p2}})$ small when $C_{p1} \in L$.

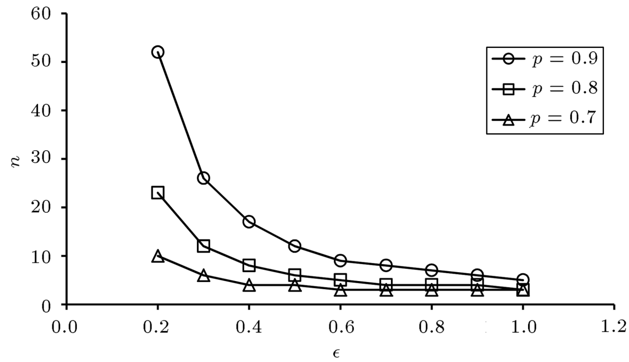

Figure 1. Smallest n to ensure $\min_{C_{p1} \in R} P(\widehat{C_{p1}} > \widehat{C_{p2}}) > p$ with $R = [(1+\epsilon)C_{p2}, \infty)$.

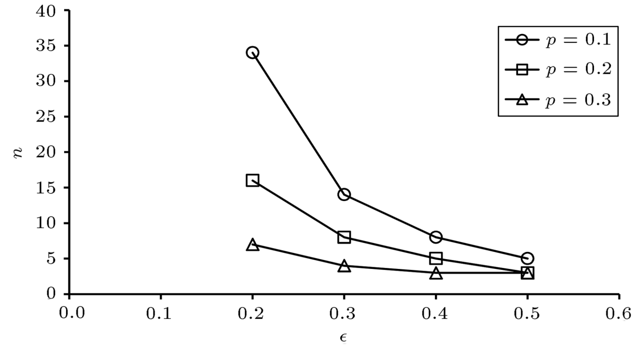

Figure 2. Smallest n to ensure $\max_{C_{p1} \in L} P(\widehat{C_{p1}} > \widehat{C_{p2}}) < p$ with $L = (0, (1-\epsilon)C_{p2}]$.

4. Remarks

In this section, we will discuss some connections between our method and confidence interval and unbiased estimators.

In addition to our comparison method, confidence intervals for $C_{p1} - C_{p2}$ can also be applied to make comparisons between $\widehat{C}_{p1}$ and $\widehat{C}_{p2}$. We will show that these two methods are equivalent when sample size is large. But, our method is better when sample size is small.

Our approach depends heavily on the maximum likelihood estimators of the standard deviations. Since the maximum likelihood estimators are biased, it is interesting to see the effects when the maximum likelihood estimators are replaced by some other unbiased estimators. We will show that estimators might make no difference for the comparisons.

First, we present the connection to confidence interval. When sample size n is large, it is known by central limit theorem that $\widehat{C}_{p1}$ and $\widehat{C}_{p2}$ are asymptotically normally distributed as $N(C_{p1}, \frac{1}{2n}C_{p1}^2)$ and $N(C_{p2}, \frac{1}{2n}C_{p2}^2)$, respectively (see [19]). Therefore, $\widehat{C}_{p1} - \widehat{C}_{p2}$ is asymptotically normally distributed as $N(C_{p1} - C_{p2}, \frac{1}{2n}[C_{p1}^2 + C_{p2}^2])$, since $\widehat{C}_{p1}$ and $\widehat{C}_{p2}$ are independent. Consequently, an approximately level $(1-\alpha)\%$ of confidence interval for $C_{p1} - C_{p2}$ is given as $[A, B]$, where:

$$A = \widehat{C}_{p1} - \widehat{C}_{p2} - z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2 + \widehat{C}_{p2}^2}{2n}},$$

$$B = \widehat{C}_{p1} - \widehat{C}_{p2} + z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2 + \widehat{C}_{p2}^2}{2n}},$$

and $z_{\alpha/2}$ satisfies $P(N(0,1) \geq z_{\alpha/2}) = \alpha/2$.

We conclude that $C_{p1} > C_{p2}$ when:

$$\widehat{C}_{p1} - \widehat{C}_{p2} - z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2 + \widehat{C}_{p2}^2}{2n}} > 0.$$

Thus, we want:

$$P\left(\widehat{C}_{p1}-\widehat{C}_{p2}-z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2+\widehat{C}_{p2}^2}{2n}}>0\right),$$

to be large (small) when $C_{p1} > (<) C_{p2}$.

Similarly, we conclude that $C_{p1} < C_{p2}$ when:

$$\widehat{C}_{p1}-\widehat{C}_{p2}+z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2+\widehat{C}_{p2}^2}{2n}}<0.$$

Thus, we want:

$$P\left(\widehat{C}_{p1}-\widehat{C}_{p2}+z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2+\widehat{C}_{p2}^2}{2n}}<0\right),$$

to be large (small) when $Cp_1 < (>) Cp_2$.

When n is large, $z_{\alpha/2}\sqrt{\frac{\widehat{C}_{p1}^2+\widehat{C}_{p2}^2}{2n}}$ is negligible, and confidence interval based comparison method needs $P(\widehat{C}_{p1}-\widehat{C}_{p2}>0)$ to be large (small) when $C_{p1} > (<) C_{p2}$, equivalent to our method.

When sample size n is small, it is not easy to find a confidence interval for $C_{p1}-C_{p2}$, since the distribution of $\widehat{C}_{p1}-\widehat{C}_{p2}$ is extremely complicated. Clearly, our method works well for small n, and hence better than confidence interval based comparison.

We proceed to present the connection to unbiased estimators. First, consider the unbiased estimator constructed from sample standard deviation. Let:

$$\bar{S}_j=\frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^n(X_{ji}-\bar{X}_j)^2}{(n-1)}}, \qquad j=1,2,$$

where:

$$c_4=\sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}.$$

Then, $\bar{S}_j$ is an unbiased estimator of $\sigma_j, j=1,2$ (see [20]). Define:

$$\bar{C}_{pj}=\frac{\mathrm{USL}-\mathrm{LSL}}{6\bar{S}_j}, \qquad j=1,2.$$

Then:

$$P\left(\bar{C}_{p1}>\bar{C}_{p2}\right)=P\left(\bar{S}_2>\bar{S}_1\right)$$
$$=P(S_2>S_1)=P\left(\widehat{C}_{p1}>\widehat{C}_{p2}\right).$$

Consequently, the final comparisons results made by maximum likelihood estimators S_1 and S_2 will be the same as using unbiased estimators $\bar{S}_1$ and $\bar{S}_2$.

Sample range can also be modified to be unbiased estimator of the standard deviation. Define $X_{j(n)}=\max\{X_{j1},X_{j2},\cdots,X_{jn}\}$, and $X_{j(1)}=\min\{X_{j1},X_{j2},\cdots,X_{jn}\}$, then $R_j=X_{j(n)}-X_{j(1)}$ denotes the sample range of $X_{j1},X_{j2},\cdots,X_{jn}, j=1,2$. Define:

$$\tilde{S}_j=\frac{R_j}{d_2},$$

where $d_2=\frac{\mathrm{ER}_j}{\sigma_j}$, then $\tilde{S}_j$ is an unbiased estimator of σ_j, $j=1,2$. Note that d_2 depends only on the sample size (see [20]). Define:

$$\tilde{C}_{pj}=\frac{\mathrm{USL}-\mathrm{LSL}}{6\tilde{S}_j}, \qquad j=1,2.$$

Then:

$$P\left(\tilde{C}_{p1}>\tilde{C}_{p2}\right)=P\left(\tilde{S}_2>\tilde{S}_1\right)=P(R_2>R_1).$$

When sample size n is large enough, R_j will be close to $4S_j$, denoted by $R_j\sim 4S_j, j=1,2$. In this case:

$$P\left(\tilde{C}_{p1}>\tilde{C}_{p2}\right)=P(R_2>R_1)\sim P(4S_2>4S_1)$$
$$=P(S_2>S_1)=P\left(\widehat{C}_{p1}>\widehat{C}_{p2}\right).$$

Therefore, the final comparisons results produced by S_1 and S_2 will be close to those results made by $\tilde{S}_1$ and $\tilde{S}_2$.

However, when R_j is not close to $4S_j$, $j=1,2$, it is extremely difficult to calculate $P(R_2>R_1)$ and $P(\tilde{C}_{p1}>\tilde{C}_{p2})$ exactly. The evaluation of $P(\tilde{C}_{p1}>\tilde{C}_{p2})$ and the subsequent work for comparison deserves a future research. More topics for future study are presented in the following section.

5. Future study

In this paper, we deal with one-dimensional process capability index C_p comparisons based on maximum likelihood estimators constructed from two normal distributions. There are many ways to extend our findings. We point out below some directions for future research.

- Extension from C_p comparison to C_{pk} comparison: Let μ and σ denote the process mean and standard deviation, respectively. The process capability index C_{pk} is defined as $\frac{d-|\mu-M|}{3\sigma}$ where $d=\frac{\mathrm{USL}-\mathrm{LSL}}{2}$ and $M=\frac{\mathrm{USL}+\mathrm{LSL}}{2}$. Index C_{pk} was created to offset some of the weakness of C_p, primarily the fact that C_p measured capability in terms of processvariation only and did not take process location into consideration [3]. Therefore, it is valuable to make C_{pk} comparison when the mean values of the processes are off-target. Let $\widehat{C}_{pk1}$ and $\widehat{C}_{pk2}$ denote the maximum likelihood estimators of C_{pk}

obtained from two normal processes. The complex distribution of $\widehat{C}_{pk1}-\widehat{C}_{pk2}$ will make the evaluation of $P(\widehat{C}_{pk1}-\widehat{C}_{pk2}>0)$ extremely difficult. Numerical or simulation techniques may be helpful to solve this problem.

- Extension from C_p comparison to multivariate C_p comparison: Let $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ denote the process mean vector and variance-covariance matrix, respectively; and multivariate C_p can be defined based on $|\boldsymbol{\Sigma}|$ or $\mathrm{tr}\boldsymbol{\Sigma}$, the determinant, and trace of $\boldsymbol{\Sigma}$, respectively. Based on observations sampled from multivariate normal distribution $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, comparison can be based on $|\hat{\boldsymbol{\Sigma}}|$ or $\mathrm{tr}\hat{\boldsymbol{\Sigma}}$, where $\hat{\boldsymbol{\Sigma}}$ is the maximum likelihood estimator of $\boldsymbol{\Sigma}$. Multivariate C_p comparison can be made based on many mathematical and statistical properties of $\hat{\boldsymbol{\Sigma}}$ provided in [21].

  Multivariate C_p can also be defined by:

$$C_{pM}=\frac{\sum_{i=1}^{v}(\mathrm{USL}_i-\mathrm{LSL}_i)}{\sum_{i=1}^{v}(\mathrm{UPL}_i-\mathrm{LPL}_i)},$$

  where USL_i and LSL_i are the upper and lower specification limits for the ith quality characteristic; UPL_i and LPL_i are the upper and lower specification limits of a modified process region for the ith quality characteristic, $i=1,\cdots,v$ (see [22] for more details). In this case, we may need help from computer software since the mathematical framework is hard to deal with.

- Extension from normal process comparison to non-normal process comparison: Non-normal observations are frequently found from many processes in industry, for instances, leukocyte filtering process [23] and manufacturing process [24], among others. If non-normal process can be transformed to normal via the Box-Cox transformation [25], then the C_p comparison method for normal processes can be applied. If the Box-Cox transformation is not successful, we seek help from the definition of C_p suitable for non-normal process. For example, quantile based C_P definitions can be found in [26-28], among others. However, the estimation of C_p is then formed from quantile estimators. It is not easy to make inferences based on quantile estimators since the distributions involved are very complicated.

- Extension from two-process comparison to three-process comparison: Given maximum likelihood estimators $\widehat{C}_{p1},\widehat{C}_{p2}$, and $\widehat{C}_{p3}$ obtained from three normal processes, we can make three comparisons based on $\{\widehat{C}_{p1},\widehat{C}_{p2}\}$, $\{\widehat{C}_{p1},\widehat{C}_{p3}\}$, and $\{\widehat{C}_{p2},\widehat{C}_{p3}\}$, separately, from which conclusion can be made. Alternatively, we can also make one comparison based on $\widehat{C}_{p1},\widehat{C}_{p2}$, and $\widehat{C}_{p3}$ simultaneously. In this case, the calculation of $P(\widehat{C}_{p1}>\widehat{C}_{p2}>\widehat{C}_{p3})$ and the subsequent settings for comparison are straightforward and tedious.

6. Conclusions

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two manufacturing processes with process capability indices $C_{p1}=\frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_1}$ and $C_{p2}=\frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma_2}$, respectively. Let $\widehat{C_{p1}}$ and $\widehat{C_{p2}}$ denote the maximum likelihood estimators of C_{p1} and C_{p2} under the normality assumption. We calculate $P(\widehat{C_{p1}}>\widehat{C_{p2}})$ and present a table from which smallest sample sizes can be determined to make $\min_{C_{p1}\in R}P(\widehat{C_{p1}}>\widehat{C_{p2}})$ large and make $\max_{C_{p1}\in L}P(\widehat{C_{p1}}>\widehat{C_{p2}})$ small, where $L=(0,(1-\epsilon)C_{p2}],R=[(1+\epsilon)C_{p2},\infty)$, and $\epsilon>0$. Consequently, comparison of C_{p1} and C_{p2} based on $\widehat{C_{p1}}$ and $\widehat{C_{p2}}$ provide manufacturers and consumers a way to correctly (wrongly) recognize better suppliers to cooperate and better merchandise to purchase, respectively, with high (low) probability. We discuss statistical properties concerning our method. We also point out some directions for future study.

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Appendix

Here, we calculate Eqs. (5), (6), and (7) stated in Section 2. First note that $\frac{S_2^2\sigma_1^2}{S_1^2\sigma_2^2}$ has an F distribution, $F(n-1,n-1)$, with probability density function:

$$\frac{\Gamma(n-1)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\frac{w^{(n-3)/2}}{(1+w)^{n-1}}, \qquad 0<w<\infty, \tag{A.1}$$

(see [18]).

In view of Relation (A.1) with the transformations $w=(\tan\theta)^2$ and $t=2\theta$, we have:

$$\begin{aligned}
&P\left(\widehat{C_{p1}}>\widehat{C_{p2}}\right)\\
&=P(S_2^2>S_1^2)\\
&=P\left(\frac{S_2^2\sigma_1^2}{S_1^2\sigma_2^2}>\frac{\sigma_1^2}{\sigma_2^2}\right)\\
&=P\left(F_{n-1,n-1}>\frac{\sigma_1^2}{\sigma_2^2}\right)\\
&=\int_{\sigma_1^2/\sigma_2^2}^{\infty}\frac{\Gamma(n-1)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\frac{w^{(n-3)/2}}{(1+w)^{n-1}}dw\\
&=\int_{\alpha}^{\pi/2}\frac{\Gamma(n-1)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\frac{[(\tan\theta)^2]^{(n-3)/2}}{[1+(\tan\theta)^2]^{n-1}}.[2\tan\theta][\sec\theta]^2d\theta\\
&=\frac{2\Gamma(n-1)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{\alpha}^{\pi/2}\frac{(\tan\theta)^{n-2}}{(\sec\theta)^{2n-4}}d\theta\\
&=\frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{n-2}dt,
\end{aligned}$$

where $\alpha=\tan^{-1}(\sigma_1/\sigma_2)$, and Eq. (5) follows.

We will calculate Eq. (5) further for $n = 2k+3$ and $n = 2k+2$ separately, where $k \in \{0, 1, 2, \cdots\}$. Consider the case when $n = 2k+3$. To show Eq. (6), combine Eq. (5), the transformation $u = \cos t$, and binomial theorem together to imply that:

$$P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{n-2}dt$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{2k+1}dt$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{-1}^{\cos(2\alpha)}(1-u^2)^k du$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\cdot\int_{-1}^{\cos(2\alpha)}\sum_{i=0}^{k}\binom{k}{i}(-u^2)^i du$$

$$= \frac{\Gamma(2k+2)}{2^{2k+1}[\Gamma(k+1)]^2}$$

$$\cdot\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{[\cos(2\alpha)]^{2i+1}+1}{2i+1},$$

where $\cos(2\alpha) = \frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2+\sigma_2^2}$, as was shown.

We proceed to calculate Eq. (5) with $n = 2k+2$ and verify Eq. (7). First note that integration by parts implies:

$$\int(\sin t)^m dt = -\frac{(\sin t)^{m-1}(\cos t)}{m} + \frac{m-1}{m}\int(\sin t)^{m-2}dt, \quad \text{(A.2)}$$

where $m \geq 2$ is a positive integer. Applying Eq. (A.2) several times, we have:

$$P\left(\widehat{C_{p1}} > \widehat{C_{p2}}\right)$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{n-2}dt$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\int_{2\alpha}^{\pi}(\sin t)^{2k}dt$$

$$= \frac{\Gamma(n-1)}{2^{n-2}\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}\left[\frac{(\sin 2\alpha)^{2k-1}(\cos 2\alpha)}{2k} + \frac{2k-1}{2k}\int_{2\alpha}^{\pi}(\sin t)^{2k-2}dt\right]$$

$$= \frac{\Gamma(n-1)}{2^{n-2}[\Gamma(\frac{n-1}{2})]^2}\left\{\frac{(\sin 2\alpha)^{2k-1}(\cos 2\alpha)}{2k} + \frac{2k-1}{(2k)(2k-2)}(\sin 2\alpha)^{2k-3}(\cos 2\alpha) + \frac{(2k-1)(2k-3)}{(2k)(2k-2)}\int_{2\alpha}^{\pi}(\sin t)^{2k-4}dt\right\}$$

$$= \frac{\Gamma(n-1)}{2^{n-2}[\Gamma(\frac{n-1}{2})]^2}\left\{\frac{(\sin 2\alpha)^{2k-1}(\cos 2\alpha)}{2k} + \frac{2k-1}{(2k)(2k-2)}(\sin 2\alpha)^{2k-3}(\cos 2\alpha) + \frac{(2k-1)(2k-3)}{(2k)(2k-2)(2k-4)}(\sin 2\alpha)^{2k-5}(\cos 2\alpha) + \cdots + \frac{(2k-1)(2k-3)\cdots 3}{(2k)(2k-2)(2k-4)\cdots 2}(\sin 2\alpha)(\cos 2\alpha) + \frac{(2k-1)(2k-3)\cdots 3.1}{(2k)(2k-2)(2k-4)\cdots 2}\left(\int_{2\alpha}^{\pi}1dt\right)\right\}$$

$$= \frac{\Gamma(2k+1)}{2^{2k}\left[\Gamma\left(\frac{2k+1}{2}\right)\right]^2}\left\{\frac{(\sin 2\alpha)^{2k-1}(\cos 2\alpha)}{2k} + \frac{2k-1}{(2k)(2k-2)}(\sin 2\alpha)^{2k-3}(\cos 2\alpha) + \frac{(2k-1)(2k-3)}{(2k)(2k-2)(2k-4)}(\sin 2\alpha)^{2k-5}(\cos 2\alpha) + \cdots + \frac{(2k-1)(2k-3)\cdots 3}{(2k)(2k-2)(2k-4)\cdots 2}(\sin 2\alpha)(\cos 2\alpha) + \frac{(2k-1)(2k-3)\cdots 3.1}{(2k)(2k-2)(2k-4)\cdots 2}(\pi - 2\alpha)\right\},$$

where $\alpha = \tan^{-1}(\sigma_1/\sigma_2), \sin 2\alpha = \frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}$, and $\cos 2\alpha = \frac{\sigma_2^2-\sigma_1^2}{\sigma_1^2+\sigma_2^2}$. The verification is completed.

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