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Managing interval-valued multiplicative hesitant fuzzy information in GDM problems

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Abstract. Inspired by the idea of multiplicative preference relation as well as interval-valued hesitant fuzzy set, we introduce a new preference relation called interval-valued multiplicative hesitant fuzzy preference structure. It is a powerful technique to describe the preference information assessed by different decision makers. The predominant feature is that it allows decision makers to use the interval-valued hesitant fuzzy information in describing the preference structure. We focus on the information aggregation methods for interval-valued multiplicative hesitant fuzzy preference information and propose a series of useful aggregation operators. Moreover, we propose a Group Decision-Making (GDM) method based on the proposed aggregation operators under interval-valued multiplicative hesitant fuzzy environment. A real case about college application problem is presented to show usefulness and effectiveness of our proposed method.

1. Introduction

Decision analysis is a regular activity encountered in our daily life and it refers to selection of the most desirable alternatives from a set of possible schemes by using some quantitative methods [1-3]. Multiplicative and additive preference relations are two main techniques to capture the preference structure of the decision makers when they compare alternatives with each other [1,4-6].

Multiplicative preference relations are based on the Saaty's 1-9 scale and have been investigated by many scholars. Chiclana et al. [7] focused on the aggregation methods of the multiplicative fuzzy preference information in Group Decision-Making (GDM) problems. Based on the C-OWA operator [8], Xu [9] extended the fuzzy preference to interval-valued fuzzy environment and introduced an effective method for the GDM problem. Fan et al. [10] focused on the GDM problem, where the decision information is expressed by multiplicative fuzzy preference relations, and proposed a fuzzy programming model. Genç et al. [11] extended the fuzzy preference orderings and studied the consistent problem of the interval fuzzy preference relation. Xia et al. [12] extended the multiplicative fuzzy preference relations to intuitionistic fuzzy environment and proposed the intuitionistic multiplicative preference relations. Based on Xia et al. [12], Jiang et al. [13] studied the consistency problem and proposed a series of useful compatibility measures and consensus models for intuitionistic multiplicative GDM problems. Xia and Xu [14] introduced the generalized operations for intuitionistic multiplicative preference information based on Archimedes rules and then introduced some generalized aggregation operators. For capturing the interrelations between the aggregated arguments, the power average and Choquet Integral were used for the aggregation operators. Some other research results about intuitionistic multiplicative preference information are shown in Jiang and

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Xu [15] and Yu and Fang [16]. Yu and Xu [17] introduced the concept of intuitionistic multiplicative triangular fuzzy set and developed a series of aggregation operators for multi-attribute decision making.

Multiplicative Hesitant Fuzzy Set (MHFS) has been introduced by Xia [18] based on the multiplicative preferences relations. The main difference between the MHFS and Hesitant Fuzzy Set (HFS) [19-21] is the difference in adopted preference information. MHFS is complementary to HFS. Recently, MHFS has attracted attention of many researchers. Xia and Xu [22] proposed a series of multiplicative hesitant fuzzy information aggregation operators using algebraic operational laws, based on which a new GDM method is studied. Unlike the aggregation operators proposed by Xia and Xu [22], Yu [23] proposed a group of Einstein operations based operators for the multiplicative hesitant fuzzy information. Wang et al. [24] focused on the compatibility measures and aggregation operators combined with the power mean. Liao et al. [25] investigated the problem of consistency of the multiplicative hesitant fuzzy decision-making matrix. Some concepts such as multiplicative consistency, acceptable multiplicative consistency, and perfect multiplicative consistency are introduced.

It should be noted that the membership degree of the MHFS is expressed by a determined value between $\frac{1}{9}$ and 9. However, in many practical situations, it is hard to use the crisp numbers to express the membership degree and the interval number is a very good choice. In this regard, it will be more applicable to practical problems if we use the interval numbers that belong to $[\frac{1}{9}, 9]$ instead of determined numbers to express the membership degree of the MHFS. Therefore, in this paper, we first propose an extended MHFS, called Interval MHFS (IMHFS), and then study its applications.

The next section gives some basic concepts and then introduces the IMHFS. Section 3 proposes a series of aggregation operators based on three different scenarios. Section 4 proposes a new GDM method using Interval Multiplicative Hesitant Fuzzy Numbers (IMHFNs) and an actual application case about voluntary choice of college entrance examination in China is presented. Section 5 concludes this paper and the future directions in IMHFS are also outlined.

2. MHFS and IMHFS

MHFS was proposed by Xia [18] and its structure is very similar to that of HFS. However, MHFS is based on 1-9 ratio scale while the HFS takes advantage of the 0.1-0.9 scale. The definition of MHFS is given as follows [18].

Definition 1. Suppose there is an objective set and marked as X; the MHFS is defined as:

$$D = \{ \langle x, \rho_D(x) \rangle | \, x \in X \} . \tag{1}$$

In Eq. (1), the function $\rho_D(x)$ is valued in the interval $[1/9, 9]$.

As the basic component of MHFS, Multiplicative Hesitant Fuzzy Number (MHFN) was marked as $\rho = \rho(x)$, which was also defined by Xia [18].

Based on Definition 1, in the following, we give the definition of IMHFS.

Definition 2. Suppose there is an objective set and marked as X; the IMHFS is defined as:

$$D = \{ \langle x, \tilde{\rho}_D(x) \rangle | \, x \in X \} , \tag{2}$$

where $\tilde{\rho}_D(x) = \left[\tilde{\rho}_D^-(x), \tilde{\rho}_D^+(x)\right]$ is the membership degree interval with the conditions that $\tilde{\rho}_D(x) \in [\frac{1}{9}, 9]$.

Inspired by the idea of MHFN, we define $\tilde{\rho}_D(x)$ as the IMHFN. For any IMHFN $\tilde{\rho}_D(x)$, the score is defined as:

$$s(\tilde{\rho}) = \sqrt[1/\triangle_g]{\prod\nolimits_{\eta \in g} \eta^+ \cdot \eta^-}. \tag{3}$$

$[\eta^-, \eta^+]$ is an interval number among the possible intervals in $\tilde{\rho}_D(x)$. For any given two IMHFNs $\tilde{\rho}_1$ and $\tilde{\rho}_2$, if $s(\tilde{\rho}_1) > s(\tilde{\rho}_2)$, then $\tilde{\rho}_1 \succ \tilde{\rho}_2$.

3. Aggregation operators for IMHFNs

The information aggregation is essential to many fields. In this section, some series of new types of operators, called IMHF aggregation operators, are proposed. Three different operational laws, i.e. algebraic operational laws, Einstein operational laws, and Archimedes operational laws, are adopted.

3.1. Operation laws for IMHFNs

In our previous work [26], some algebraic operational laws and corresponding aggregation operators for IMHF information have been investigated, based on which we introduce some IMHF information operational laws.

Definition 3. Suppose $\tilde{\rho}_1$ and $\tilde{\rho}_2$ to be two IMHFNs and λ be a real number greater than zero; then, the algebraic operational laws of the IMHF information are defined as follows:

(1) $\tilde{\rho}_1^\lambda = \cup_{\tilde{\eta}_1 \in \tilde{\rho}_1}$

$$\left\{ \left[\frac{\eta_1^{-\lambda}}{(1+\eta_1^-)^\lambda - \eta_1^{-\lambda}}, \frac{\eta_1^{+\lambda}}{(1+\eta_1^+)^\lambda - \eta_1^{+\lambda}} \right] \right\},$$

(2) $\lambda\tilde{\rho}_1 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1}\left\{\left[(1+\eta_1^-)^\lambda - 1, (1+\eta_1^+)^\lambda - 1\right]\right\}$,

(3) $\tilde{\rho}_1 \oplus \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left\{\left[\eta_1^- + \eta_2^- + \eta_1^-\eta_2^-, \eta_1^+ + \eta_2^+ + \eta_1^+\eta_2^+\right]\right\},$$

(4) $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left(\left[\frac{\eta_1^-\eta_2^-}{\eta_1^- + \eta_2^- + 1}, \frac{\eta_1^+\eta_2^+}{\eta_1^+ + \eta_2^+ + 1}\right]\right).$$

Based on the Einstein operations, Xia et al. [12] introduced some operational laws for multiplicative intuitionistic fuzzy information, inspired by which we propose some Einstein operational laws for IMHF information.

Definition 4. Suppose $\tilde{\rho}_1$ and $\tilde{\rho}_2$ to be two IMHFNs and λ be a real number greater than zero; then, the Einstein operational laws of the IMHF information are defined as follows:

(5) $\tilde{\rho}_1^\lambda = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1}$

$$\left\{\left[\frac{2\eta_1^{-\lambda}}{(2+\eta_1^-)^\lambda - \eta_1^{-\lambda}} \frac{2\eta_1^{+\lambda}}{(2+\eta_1^+)^\lambda - \eta_1^{+\lambda}}\right]\right\},$$

(6) $\lambda\tilde{\rho}_1 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1}$

$$\left\{\left[\frac{(1+2\eta_1^-)^\lambda - 1}{2}, \frac{(1+2\eta_1^+)^\lambda - 1}{2}\right]\right\},$$

(7) $\tilde{\rho}_1 \oplus \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left\{\left[\frac{(1+2\eta_1^-)(1+2\eta_2^-) - 1}{2}, \frac{(1+2\eta_1^+)(1+2\eta_2^+) - 1}{2}\right]\right\},$$

(8) $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left(\left[\frac{2\eta_1^-\eta_2^-}{(2+\eta_1^-)(2+\eta_2^-) - \eta_1^-\eta_2^-}, \frac{2\eta_1^+\eta_2^+}{(2+\eta_1^+)(2+\eta_2^+) - \eta_1^+\eta_2^+}\right]\right).$$

Definition 3 is based on algebraic operational laws while Definition 4 is based on Einstein operational laws. In the following, we generalize the two kinds of operational laws above through the introduction of the Archimedean operations.

Definition 5. Suppose $\tilde{\rho}_1$ and $\tilde{\rho}_2$ to be two IMHFNs and λ be a real number greater than zero; then, Archimedean operational laws of the IMHF information are defined as follows:

(9) $\tilde{\rho}_1^\lambda = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1}$

$$\left\{\left[g^{-1}\left(g(h(\eta_1^-))^\lambda\right), g^{-1}\left((g(\eta_1^+))^\lambda\right)\right]\right\},$$

(10) $\lambda\tilde{\rho}_1 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1}$

$$\left\{\left[h^{-1}\left((h(\eta_1^-))^\lambda\right), h^{-1}\left(\left(h(\eta_1^+)\right)^\lambda\right)\right]\right\},$$

(11) $\tilde{\rho}_1 \oplus \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left\{\left[h^{-1}(h(\eta_1^-).h(\eta_2^-)), h^{-1}(h(\eta_1^+).h(\eta_2^+))\right]\right\},$$

(12) $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \cup_{\tilde{\eta}_1\in\tilde{\rho}_1,\tilde{\eta}_2\in\tilde{\rho}_2}$

$$\left(\left[\left[g^{-1}(g(\eta_1^-).g(\eta_2^-)), g^{-1}(g(\eta_1^+).g(\eta_2^+))\right]\right]\right).$$

In Definition 5, g represents a strictly decreasing function and has a very close relationship with h, satisfying $h(t) = g(\frac{1}{t})$. The function g is a generalized form and when it takes some common forms, (9)-(12) would be transformed.

3.2. Some aggregation operators for IMHFNs

Definition 6. Suppose $\tilde{\rho}_i (i = 1, 2, \cdots, n)$ to be a collection of IMHFNs and w_i be the weight of $\tilde{\rho}_i$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the IMHF Weighted Average operator (IMHFWA) and IMHF Weighted Heometric operator (IMHFWG) are defined as follows:

IMHFWA:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = w_1\tilde{\rho}_1 \oplus w_2\tilde{\rho}_2 \oplus \cdots \oplus w_n\tilde{\rho}_n, \tag{4}$$

IMHFWG:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \tilde{\rho}_1^{w_1} \otimes \tilde{\rho}_2^{w_2} \otimes \cdots \otimes \tilde{\rho}_n^{w_n}. \tag{5}$$

Combined with the well-known OWA operator (Yager, 1988), in the following, we introduce the IMHFOWA and IMHFOWG operators.

Definition 7. Suppose $\tilde{\rho}_i (i = 1, 2, \cdots, n)$ to be a collection of IMHFNs; the IMHFOWA and IMHFOWG operators are defined as follows:

IMHFOWA:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \omega_1\tilde{\rho}_{\sigma(1)} \oplus \omega_2\tilde{\rho}_{\sigma(2)} \oplus \cdots \oplus \omega_n\tilde{\rho}_{\sigma(n)}, \tag{6}$$

IMHFOWG:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \tilde{\rho}_{\sigma(1)}^{\omega_1} \otimes \tilde{\rho}_{\sigma(2)}^{\omega_2} \otimes \cdots \otimes \tilde{\rho}_{\sigma(n)}^{\omega_n}, \tag{7}$$

where $(\omega_1, \omega_2, \cdots, \omega_n)$ is the associate weight vector of $\tilde{\rho}_i (i = 1, 2, \cdots, n)$.

In order to apply the IMHFOWA and IMHFOWG operators to the decision-making area, we should further research these two operators and transform them to some more simple forms. Based on the algebraic operational laws of IMHF information defined in Definition 3, the following useful results can be obtained.

Theorem 1. Suppose $\tilde{\rho}_i (i = 1, 2, \cdots, n)$ to be a collection of IMHFNs and w_i be the weight of $\tilde{\rho}_i$, $w_i \in [0, 1]$, and $\sum\limits_{i=1}^{n} w_i = 1$. Then:

IMHFWA:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i \in \tilde{\rho}_i} \left\{ \left[\prod_{i=1}^{n} \left(\eta_i^- + 1\right)^{w_i} - 1, \prod_{i=1}^{n} \left(\eta_i^+ + 1\right)^{w_i} - 1 \right] \right\}, \quad (8)$$

IMHFWG:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i} \in \tilde{\rho}_i \left\{ \left[\frac{\prod_{i=1}^{n} \eta_i^{-w_i}}{\prod_{i=1}^{n} (1 + \eta_i^-)^{w_i} - \prod_{i=1}^{n} \eta_i^{-w_i}}, \frac{\prod_{i=1}^{n} \eta_i^{+w_i}}{\prod_{i=1}^{n} (1 + \eta_i^+)^{w_i} - \prod_{i=1}^{n} \eta_i^{+w_i}} \right] \right\}, \quad (9)$$

IMHFOWA:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i \in \tilde{\rho}_i} \left\{ \left[\prod_{i=1}^{n} \left(\eta_{\sigma(i)}^- + 1\right)^{\omega_i} - 1, \prod_{i=1}^{n} \left(\eta_{\sigma(i)}^+ + 1\right)^{\omega_i} - 1 \right] \right\}, \quad (10)$$

IMHFOWG:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i \in \tilde{\rho}_i} \left\{ \left[\frac{\prod_{i=1}^{n} \eta_{\sigma(i)}^{-\omega_i}}{\prod_{i=1}^{n} (1 + \eta_{\sigma(i)}^-)^{\omega_i} - \prod_{i=1}^{n} \eta_{\sigma(i)}^{-\omega_i}}, \frac{\prod_{i=1}^{n} \eta_{\sigma(i)}^{+\omega_i}}{\prod_{i=1}^{n} (1 + \eta_{\sigma(i)}^+)^{\omega_i} - \prod_{i=1}^{n} \eta_{\sigma(i)}^{+\omega_i}} \right] \right\}. \quad (11)$$

In Eqs. (10) and (11), $(\omega_1, \omega_2, \cdots, \omega_n)$ is the associate weight vector of $\tilde{\rho}_i (i = 1, 2, \cdots, n)$ and $\eta_{\sigma(i)}$ is the ith largest of η_i.

Example 1. Suppose $\tilde{\rho}_1 = \{[1/6, 1/5], [2/3, 3/4]\}$, $\tilde{\rho}_2 = \{[3/4, 2], [1/3, 3], [1, 2]\}$, and $\tilde{\rho}_2 = \{[4, 5]\}$ to be three IMHFNs; then, based on Theorem 1, the aggregated IMHFNs can be obtained as follows:

1. The results based on IMHFWA are obtained as follows:

$$\begin{aligned}\tilde{\rho} &= \mathrm{IMHFWA}\left(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3\right) \\ &= \{[1.1693, 1.7850], [0.9813, 2.0652], \\ &\quad [1.2680, 1.7850], [1.4432, 2.1582], \\ &\quad [1.2314, 2.4760], [1.5544, 2.1582]\} s\left(\tilde{\rho}\right) \\ &= 2.5945.\end{aligned}$$

2. The results based on IMHFWG are obtained as follows:

$$\begin{aligned}\tilde{\rho} &= \mathrm{IMHFWG}\left(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3\right) \\ &= \{[0.5770, 0.8262], [0.4403, 0.8886], \\ &\quad [0.6265, 0.8262], [1.0648, 1.6302], \\ &\quad [0.7571, 1.8139], [1.1876, 1.6302]\} s\left(\tilde{\rho}\right) \\ &= 0.8739.\end{aligned}$$

Based on the Einstein operational laws of IMHF information defined in Definition 4, we can get the following useful results.

Theorem 2. Suppose $\tilde{\rho}_i (i = 1, 2, \cdots, n)$ to be a collection of IMHFNs, and w_i be the weight of $\tilde{\rho}_i$, $w_i \in [0, 1]$, and $\sum\limits_{i=1}^{n} w_i = 1$. Then:

IMHFWA:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i \in \tilde{\rho}_i} \left\{ \left[\prod_{i=1}^{n} \frac{(1 + 2\eta_i^-)^{w_i} - 1}{2}, \prod_{i=1}^{n} \frac{(1 + 2\eta_i^+)^{w_i} - 1}{2} \right] \right\}, \quad (12)$$

IMHFWG:

$$(\tilde{\rho}_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_n) = \cup_{\tilde{\eta}_i \in \tilde{\rho}_i} \left\{ \left[\frac{2 \prod_{i=1}^{n} \eta_i^{-w_i}}{\prod_{i=1}^{n} (2 + \eta_i^-)^{w_i} - \prod_{i=1}^{n} \eta_i^{-w_i}}, \right.\right.$$

$$\left.\left.\frac{2\prod\limits_{i=1}^{n}\eta_i^{+w_i}}{\prod\limits_{i=1}^{n}(2+\eta_i^{+})^{w_i}-\prod\limits_{i=1}^{n}\eta_i^{+w_i}}\right]\right\},\tag{13}$$

IMHFOWA:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[\prod_{i=1}^{n}\frac{\left(1+2\eta_{\sigma(i)}^{-}\right)^{\omega_i}-1}{2},\right.\right.$$

$$\left.\left.\prod_{i=1}^{n}\frac{\left(1+2\eta_{\sigma(i)}^{+}\right)^{\omega_i}-1}{2}\right]\right\},\tag{14}$$

IMHFOWG:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)$$

$$=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[\frac{2\prod\limits_{i=1}^{n}\eta_{\sigma(i)}^{-\omega_i}}{\prod\limits_{i=1}^{n}(2+\eta_{\sigma(i)}^{-})^{\omega_i}-\prod\limits_{i=1}^{n}\eta_{\sigma(i)}^{-\omega_i}},\right.\right.$$

$$\left.\left.\frac{2\prod\limits_{i=1}^{n}\eta_{\sigma(i)}^{+\omega_i}}{\prod\limits_{i=1}^{n}(2+\eta_{\sigma(i)}^{+})^{\omega_i}-\prod\limits_{i=1}^{n}\eta_{\sigma(i)}^{+\omega_i}}\right]\right\}.\tag{15}$$

Example 2. Take the three IMHFNs in Example 1; for example, based on Theorem 2, the aggregated IMHFNs can be obtained as follows:

1. The results based on IMHFWA:

$$\tilde{\rho}=\text{IMHFWA}\left(\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_3\right)$$
$$=\{[1.0536,1.6272],[0.8572,1.8796],$$
$$[1.1510,1.6272],[1.3722,2.0807],$$
$$[1.1355,2.3870],[1.4895,2.0807]\}s\left(\tilde{\rho}\right)$$
$$=2.2333.$$

2. The results based on IMHFWG:

$$\tilde{\rho}=\text{IMHFWG}\left(\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_3\right)$$
$$=\{[0.6348,0.9369],[0.4821,1.0258],$$
$$[0.6940,0.9369],[1.1099,1.7044],$$
$$[0.8077,1.9134],[1.2340,1.7044]\}s\left(\tilde{\rho}\right)$$
$$=1.0274.$$

Based on Archimedean operational laws of the IMHF information defined in Definition 6, we can get the following useful results.

Theorem 3. Suppose $\tilde{\rho}_i(i=1,2,\cdots,n)$ to be a collection of IMHFNs and w_i be the weight of $\tilde{\rho}_i$, $w_i\in[0,1]$, and $\sum\limits_{i=1}^{n}w_i=1$. Then:

IMHFWA:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[h^{-1}\left(\prod_{i=1}^{n}(h(\eta_i^{-}))^{w_i}\right),\right.\right.$$

$$\left.\left.h^{-1}\left(\prod_{i=1}^{n}(h(\eta_i^{+}))^{w_i}\right)\right]\right\},\tag{16}$$

IMHFWG:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[g^{-1}\left(\prod_{i=1}^{n}(g(\eta_i^{-}))^{w_i}\right),\right.\right.$$

$$\left.\left.g^{-1}\left(\prod_{i=1}^{n}(g(\eta_i^{+}))^{w_i}\right)\right]\right\},\tag{17}$$

IMHFOWA:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[h^{-1}\left(\prod_{i=1}^{n}(h(\eta_{\sigma(i)}^{-}))^{\omega_i}\right),\right.\right.$$

$$\left.\left.h^{-1}\left(\prod_{i=1}^{n}(h(\eta_{\sigma(i)}^{+}))^{\omega_i}\right)\right]\right\},\tag{18}$$

IMHFOWG:

$$(\tilde{\rho}_1,\tilde{\rho}_2,\cdots,\tilde{\rho}_n)=\cup_{\tilde{\eta}_i\in\tilde{\rho}_i}\left\{\left[g^{-1}\left(\prod_{i=1}^{n}(g(\eta_{\sigma(i)}^{-}))^{\omega_i}\right),\right.\right.$$

$$\left.\left.g^{-1}\left(\prod_{i=1}^{n}(g(\eta_{\sigma(i)}^{+}))^{\omega_i}\right)\right]\right\}.\tag{19}$$

4. Multi-criteria GDM under interval-valued multiplicative hesitant fuzzy environment

Multi-criteria GDM is also called multiple-objective decisions with finite scheme and has widely been applied in technology, engineering, mathematics, etc. Because there is a profound theoretical significance and wide practical background in various fields, research on multi-criteria GDM problem has always drawn close attention [27-29]. The essence of multi-criteria decision making is to rank the finite alternatives based on the decision-making information. It is composed of two important parts:

1. Acquisition of decision information;
2. Aggregation of the decision information for every alternative based on the appropriate aggregation methods.

Suppose there is a group of experts $(e_1, e_2, \cdots, e_p)$ (decision makers) to evaluate a set of alternatives $(x_1, x_2, \cdots, x_n)$. During the evaluation process, they compare each alternative with others and the comparison results are expressed by interval-valued multiplicative hesitant fuzzy numbers:

$$\tilde{\rho}_{ij}^k = \cup_{\tilde{\eta}_{ij}^k \in \tilde{\rho}_{ij}^k} \left\{ \left[\eta_{ij}^{-k}, \eta_{ij}^{+k}\right] \right\}$$

$$(i, j = 1, 2, \cdots, n; k = 1, 2, \cdots, p).$$

The meaning of $\tilde{\rho}_{ij}^k$ is explained as: $[\eta_{ij}^{-k}, \eta_{ij}^{+k}] \subseteq [1/9, 9]$; where $[\eta_{ij}^{-k}, \eta_{ij}^{+k}]$ is the degree range provided by the decision maker, e_k, with the meaning of alternative x_i being priority over the alternative x_j. To find the most appropriate alternative, a decision process is defined:

- **Step 1.** Obtain the comprehensive performance value of alternative x_i provided by expert e^k. This process should take advantage of the IMHFWA, IMHFWG, IMHFOWA, or IMHFOWAG operator to aggregate $(\tilde{\rho}_{1j}^k, \tilde{\rho}_{2j}^k, \cdots, \tilde{\rho}_{nj}^k)$;
- **Step 2.** Obtain the comprehensive performance value for alternative x_i. This process should utilize the operators to aggregate all the performance values provided by expert e^k and get the final value for alternatives;
- **Step 3.** Rank the final performance values for each alternative.

Case analysis: Voluntary choice of college entrance examination in China

College entrance examination in China is a very important event in one's life and it has a very close relationship with the examinee's development in the future. In the last three decades, due to the economy, population, policy, etc., higher education has played a significant role. As the lack of the higher education resources and the number of candidates have increased, the college entrance examination is faced with more and stronger competition. The factors should be considered by students and their parents in voluntary choice of college entrance examination, including school factors, job factors, education factors, geographic location, occupation development factors, etc. Therefore, college entrance examination is a fuzzy multi-criteria GDM problem.

Generally speaking, voluntary choice of college entrance examination is decided by the examinee (e_1) and his/her father (e_2), and mother (e_3). Meanwhile, we suppose the weight vector of the three decision makers (examinee, his/her father, his/her mother) as $(1/3, 1/3, 1/3)^T$. When facing three universities (x_1, x_2, x_3), the decision makers compare each university with others and construct the following interval-valued multiplicative hesitant fuzzy matrices:

$$\tilde{D}^1 = \begin{pmatrix} \{[1,1]\} & \{[\frac{1}{4}, \frac{2}{3}]\} & \{[\frac{1}{6}, \frac{1}{3}], [\frac{2}{5}, \frac{2}{3}]\} \\ \{[1,2],[2,4]\} & \{[1,1]\} & \{[\frac{1}{5}, \frac{3}{4}]\} \\ \{[\frac{1}{7}, \frac{1}{4}], [\frac{2}{5}, \frac{3}{4}]\} & \{[2,4]\} & \{[1,1]\} \end{pmatrix},$$

$$\tilde{D}^2 = \begin{pmatrix} \{[1,1]\} & \{[1,2],[3,5]\} & \{[\frac{1}{4}, \frac{2}{3}], [\frac{1}{6}, \frac{1}{2}]\} \\ \{[\frac{3}{4}, \frac{5}{3}]\} & \{[1,1]\} & \{[3,4],[3,5]\} \\ \{[\frac{2}{7}, \frac{3}{4}], [\frac{1}{2}, \frac{2}{3}]\} & \{[1,3]\} & \{[1,1]\} \end{pmatrix},$$

$$\tilde{D}^3 = \begin{pmatrix} \{[1,1]\} & \{[\frac{1}{4}, \frac{2}{3}]\} & \{[2,4]\} \\ \{[1,2],[2,4]\} & \{[1,1]\} & \{[\frac{1}{3}, \frac{3}{5}]\} \\ \{[\frac{1}{3}, \frac{2}{5}]\} & \{[\frac{1}{6}, \frac{1}{3}], [\frac{1}{5}, \frac{1}{2}]\} & \{[1,1]\} \end{pmatrix}.$$

In the following, we use the IMHFWA and IMHFWG operators to find the best university for the examinee.

1. Using IMHFWA operator based on algebraic operations:
   - **Step 1.** First of all, adopt the IMHFWA operator to obtain the comprehensive value of university, x_i, expressed by three different decision makers:

$$\tilde{\rho}_1^1 = \{[0.4288, 0.6441], [0.5183, 0.7711]\},$$

$$\tilde{\rho}_2^1 = \{[0.6869, 1.1898], [0.9310, 1.5962]\},$$

$$\tilde{\rho}_3^1 = \{[0.8998, 1.3208], [1.0328, 1.5962]\},$$

$$\tilde{\rho}_1^2 = \{[0.7100, 1.1544], [0.6711, 1.0801],$$

$$[1.1544, 1.7144], [1.1055, 1.6207]\},$$

$$\tilde{\rho}_2^2 = \{[1.4101, 1.9876], [1.4101, 2.1748]\},$$

$$\tilde{\rho}_3^2 = \{[0.7261, 1.4101], [0.8171, 1.3713]\},$$

$$\tilde{\rho}_1^3 = \{[0.6894, 1.5544]\},$$

$$\tilde{\rho}_2^3 = \{[0.7472, 1.1253], [1.0000, 1.5198]\},$$

$$\tilde{\rho}_3^3 = \{[0.4598, 0.5513], [0.4736, 0.6134]\}.$$

   - **Step 2.** Obtain the comprehensive performance value, $\tilde{\rho}_i$, for alternative x_i based on IMHFWA operator (algebraic operations):

$$\tilde{\rho}_1 = \{[0.6041, 1.0838], [0.5918, 1.0595], [0.7325, 1.2506], [0.7193, 1.2244], [0.6369, 1.1361], [0.6244, 1.1112], [0.7680, 1.3071], [0.7545, 1.2803]\},$$

$$\tilde{\rho}_2 = \{[0.9223, 1.4046], [1.0109, 1.5451], [0.9223, 1.4538], [1.0109, 1.5971], [1.0109, 1.5451], [1.1035, 1.6937], [1.0109, 1.5971], [1.1035, 1.7488]\},$$

$$\tilde{\rho}_3 = \{[0.6854, 1.0549], [0.6907, 1.0820], [0.7145, 1.0438], [0.7199, 1.0707], [0.7238, 1.1332], [0.7292, 1.1613], [0.7536, 1.1217], [0.7591, 1.1496]\}.$$

- **Step 3.** Using the score function, $s(\tilde{\rho}) = \sqrt[1/\triangle g]{\prod_{\eta \in g} \eta^+ \cdot \eta^-}$, defined in Section 2, calculate the score of IMHFNs $\tilde{\rho}_i (i = 1, 2, 3)$:

$$s(\tilde{\rho}_1) = 0.7960, \quad s(\tilde{\rho}_2) = 1.5850, \quad s(\tilde{\rho}_3) = 0.7947.$$

Since $s(\tilde{\rho}_2) > s(\tilde{\rho}_3) > s(\tilde{\rho}_1)$, the most suitable university for the examinee is x_2.

2. Using IMHFWA operator based on Einstein operations:
   - **Step 1.** Obtain the comprehensive value of university, x_i, expressed by three different decision makers:

$$\tilde{\rho}_1^1 = \{[0.4086, 0.6340], [0.5041, 0.7686]\},$$

$$\tilde{\rho}_2^1 = \{[0.6635, 1.1736], [0.8795, 1.5358]\},$$

$$\tilde{\rho}_3^1 = \{[0.8409, 1.2171], [1.0000, 1.5358]\},$$

$$\tilde{\rho}_1^2 = \{[0.6906, 1.1355], [0.6447, 1.0536], [1.0791, 1.6272], [1.0183, 1.5206]\},$$

$$\tilde{\rho}_2^2 = \{[1.3722, 1.9455], [1.3722, 2.1147]\},$$

$$\tilde{\rho}_3^2 = \{[0.7092, 1.3722], [0.8104, 1.3297]\},$$

$$\tilde{\rho}_1^3 = \{[0.6379, 1.4895]\},$$

$$\tilde{\rho}_2^3 = \{[0.7331, 1.1038], [0.9620, 1.4509]\},$$

$$\tilde{\rho}_3^3 = \{[0.4410, 0.5400], [0.4565, 0.6052]\}.$$

   - **Step 2.** Obtain the comprehensive performance value, $\tilde{\rho}_i$, for alternative x_i based on IMHFWA operator (Einstein operations):

$$\tilde{\rho}_1 = \{[0.5717, 1.0453], [0.5577, 1.0190], [0.6775, 1.1868], [0.6622, 1.1581], [0.6080, 1.1042], [0.5936, 1.0769], [0.7174, 1.2510], [0.7016, 1.2213]\},$$

$$\tilde{\rho}_2 = \{[0.8901, 1.3723], [0.9713, 1.4987], [0.8901, 1.4146], [0.9713, 1.5438], [0.9710, 1.4987], [1.0572, 1.6336], [0.9713, 1.5438], [1.0572, 1.6817]\},$$

$$\tilde{\rho}_3 = \{[0.6512, 0.9953], [0.6575, 1.0259], [0.6825, 0.9839], [0.6889, 1.0143], [0.6951, 1.0826], [0.7016, 1.1150], [0.7275, 1.0706], [0.7342, 1.1027]\}.$$

   - **Step 3.** Using the score function, $s(\tilde{\rho}) = \sqrt[1/\triangle g]{\prod_{\eta \in g} \eta^+ \cdot \eta^-}$, defined in Section 2, calculate the score of IMHFNs $\tilde{\rho}_i (i = 1, 2, 3)$:

$$s(\tilde{\rho}_1) = 0.7160, \quad s(\tilde{\rho}_2) = 1.4757, \quad s(\tilde{\rho}_3) = 0.7248.$$

Since $s(\tilde{\rho}_2) > s(\tilde{\rho}_3) > s(\tilde{\rho}_1)$, the most suitable university for the examinee is x_2.

3. Using IMHFWG operator based on algebraic operations:
   - **Step 1.** Obtain the comprehensive value of university ,x_i, expressed by three different decision makers:

$$\tilde{\rho}_1^1 = \{[0.3204, 0.5833], [0.4403, 0.7571]\},$$

$$\tilde{\rho}_2^1 = [0.5306, 1.0954], [0.6170, 1.2498]\},$$

$$\tilde{\rho}_3^1 = [0.5306, 0.7571], [0.8405, 1.2498]\},$$

$$\tilde{\rho}_1^2 = \{[0.5833, 1.0445], [0.4910, 0.9259], [0.7293, 1.2238], [0.6051, 1.0741]\},$$

$$\tilde{\rho}_2^2 = \{[1.1915, 1.7024], [1.1915, 1.7669]\},$$

$$\tilde{\rho}_3^2 = \{[0.6170, 1.1915], [0.7755, 1.1337]\},$$

$$\tilde{\rho}_1^3 = \{[0.4481, 1.1876]\},$$

$$\tilde{\rho}_2^3 = \{[0.6580, 1.0000], [0.7755, 1.1337]\},$$

$$\tilde{\rho}_3^3 = \{[0.3539, 0.4910], [0.3796, 0.5685]\}.$$

- **Step 2.** Obtain the comprehensive performance value, $\tilde{\rho}_i$, for alternative x_i based on IMHFWA operator (Einstein operations):

$$\tilde{\rho}_1 = \{[0.4335, 0.8779], [0.4111, 0.8454], [0.4628, 0.9203], [0.4383, 0.8855], [0.4851, 0.9707], [0.4591, 0.9331], [0.5190, 1.0199], [0.4907, 0.9794]\},$$

$$\tilde{\rho}_2 = \{[0.7281, 1.2129], [0.7700, 1.2692], [0.7281, 1.2252], [0.7700, 1.2824], [0.7700, 1.2692], [0.8154, 1.3297], [0.7700, 1.2824], [0.8154, 1.3439]\},$$

$$\tilde{\rho}_3 = \{[0.4832, 0.7413], [0.4957, 0.7842], [0.5170, 0.7315], [0.5306, 0.7736], [0.5555, 0.8633], [0.5706, 0.9171], [0.5964, 0.8511], [0.6130, 0.9038]\}.$$

- **Step 3.** Using the score function, $s(\tilde{\rho}) = \sqrt[1/\Delta g]{\prod_{\eta\in g} \eta^+ \cdot \eta^-}$ defined in Section 2, calculate the score of IMHFNs $\tilde{\rho}_i (i = 1, 2, 3)$:

$$s(\tilde{\rho}_1)=0.4278, \quad s(\tilde{\rho}_2)=0.9830, \quad s(\tilde{\rho}_3)=0.4445.$$

Since $s(\tilde{\rho}_2) > s(\tilde{\rho}_3) > s(\tilde{\rho}_1)$, the most suitable university for the examinee is x_2.

4. Using IMHFWG operator based on Einstein operations:
- **Step 1.** Obtain the comprehensive value of university, x_i, expressed by three different decision makers:

$$\tilde{\rho}_1^1 = \{[0.3304, 0.5918], [0.4493, 0.7592]\},$$

$$\tilde{\rho}_2^1 = \{[0.5516, 1.1099], [0.6576, 1.2938]\},$$

$$\tilde{\rho}_3^1 = \{[0.5745, 0.8216], [0.8688, 1.2938]\},$$

$$\tilde{\rho}_1^2 = \{[0.6012, 1.0613], [0.5140, 0.9491], [0.7822, 1.2811], [0.6621, 1.1371]\},$$

$$\tilde{\rho}_2^2 = \{[1.2219, 1.7433], [1.2219, 1.8208]\},$$

$$\tilde{\rho}_3^2 = \{[0.6329, 1.2219], [0.7822, 1.1666]\},$$

$$\tilde{\rho}_1^3 = \{[0.4715, 1.2340]\},$$

$$\tilde{\rho}_2^3 = \{[0.6714, 1.0191], [0.8077, 1.1823]\},$$

$$\tilde{\rho}_3^3 = \{[0.3645, 0.4983], [0.3894, 0.5745]\}.$$

- **Step 2.** Obtain the comprehensive performance value, $\tilde{\rho}_i$, for alternative x_i based on IMHFWA operator (Einstein operations):

$$\tilde{\rho}_1 =\{[0.4516, 0.9046], [0.4297, 0.8727], [0.4896, 0.9587], [0.4654, 0.9243], [0.5023, 0.9914], [0.4774, 0.9555], [0.5454, 1.0525], [0.5180, 1.0137]\},$$

$$\tilde{\rho}_2 =\{[0.7554, 1.2406], [0.8042, 1.3072], [0.7554, 1.2561], [0.8042, 1.3239], [0.8042, 1.3072], [0.8571, 1.3789], [0.8042, 1.3239], [0.8571, 1.3969]\},$$

$$\tilde{\rho}_3 =\{[0.5069, 0.7790], [0.5189, 0.8209], [0.5415, 0.7686], [0.5544, 0.8098], [0.5769, 0.8974], [0.5909, 0.9478], [0.6174, 0.8850], [0.6326, 0.9344]\}.$$

- **Step 3.** Using the score function, $s(\tilde{\rho}) = \sqrt[1/\Delta g]{\prod_{\eta\in g} \eta^+ \cdot \eta^-}$ defined in Section 2, calculate the score of IMHFNs $\tilde{\rho}_i (i = 1, 2, 3)$:

$$s(\tilde{\rho}_1)=0.4632, \quad s(\tilde{\rho}_2)=1.0585, \quad s(\tilde{\rho}_3)=0.4826.$$

Since $s(\tilde{\rho}_2) > s(\tilde{\rho}_3) > s(\tilde{\rho}_1)$, the most suitable university for the examinee is x_2.

During the above analysis, we used IMHFWA operator (Algebraic operations), IMHFWA operator (Einstein operations), IMHFWG operator (Algebraic operations), and IMHFWG operator (Einstein operations) to deal with the voluntary choice of college entrance examination problem in China. The results show that the final choice for the examinee is always the second university. However, the scores of the aggregated IMHFNs based on four different aggregation operators are different. For different operations, the scores based on IMHFWA operator are always bigger than those based on IMHFWG operator. On the other hand, for IMHFWA operator, the aggregated results based on algebraic operations are bigger than those based on Einstein operations. For IMHFWG operator, the aggregated results based on algebraic operations are smaller than those based on Einstein operations.

5. Conclusions and future works

In this paper, we introduced the IMHFS, which is properly complementary to the existing HFS theory. The main contributions of this paper include the following three aspects. First of all, we have extended the multiplicative intuitionistic fuzzy preference relations [12] to a more generalized form and proposed the IMHFS, which is a powerful technique to describe the preference information assessed by different appraise subjective and is very useful in decision-making problems. Then, we have proposed a series of aggregation operators for aggregating interval-valued multiplicative hesitant fuzzy information, including IMHFWA, IMHFWG, IMHFOWA, and IMHFOWG operators, based on which a new multi-criteria GDM method is proposed. Finally, we have investigated the problem of voluntary choice of college entrance examination in China based on the theory proposed in this paper.

The future study mainly focuses on the following two aspects:

1. Strengthening the research of aggregation operators for IMHFNs, e.g. considering the relationship between the data to be aggregated;
2. Applying the multi-criteria GDM method proposed in this paper to many other areas such as supply chain management, personnel selection, pattern recognition, and data mining.

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