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Research Note

Notes on mathematical formulation and complexity considerations for blocks relocation problem

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Abstract. In a recent paper, Caserta et al. [M. Caserta, S. Schwarze, and S. Voβ. "A mathematical formulation and complexity considerations for the blocks relocation problem", *European Journal of Operational Research*, 219, pp. 96-104 (2012)] proposed two mathematical models for the blocks relocation problem. Because of the complexity of their first model, called BRP-I, they employed a simplifying assumption and introduced a relatively fast model, called BRP-II, to solve medium-sized instances. In this paper, it is first proven that the BRP-II model is incorrect. Then, the corrected and improved formulation of BRP-II, called BRP2c and BRP2ci, respectively, are presented. By correcting a constraint in BRP-II, the reported optimal solution is either corrected or improved in many instances. Also, it is proven that some results of BRP-II reported by Caserta et al. are incorrect. Incorporating some new cut constraints into BRP2ci, the computational time of solving instances is decreased 25 times, on average.

1. Introduction

Caserta et al. [1] proposed a mathematical formulation for the blocks relocation problem and proved the NP-hardness of the problem. Because of the complexity of their first mathematical model, called BRP-I, it could not be solved in a reasonable time. So, they simplified the BRP-I model with an additional assumption, named A1. Their simplified model, called BRP-II, was able to solve some small and medium-sized problem instances. For larger instances, they proposed a fast heuristic algorithm based upon a set of relocation rules.

Petering and Hussein [2] have also proposed a new mixed integer formulation of BRP, called BRP-III. This new formulation, in comparison to the BRP-I from Caserta et al. [1], has the advantages of fewer decision variables and lower runtime performance.

In this paper, we show that BRP-II actually oversatisfies assumption A1, placing more restrictions on the decision maker than there exist in assumption A1. Then, the corrected BRP-II model, called BRP2c, is presented. In addition, using some new cut constraints, the corrected and improved form of BRP-II, called BRP2ci, is presented. In addition, it is shown that some reported results by Caserta et al. [1] for BRP-II are incorrect.

In Section 2, the BRP-II model is presented. In Section 3, it is shown that the BRP-II model, with respect to assumption A1, is not correct. The corrected form of BRP-II, as well as its improved form, is presented in Sections 4 and 5, respectively. Section 6 provides some numerical results.

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2. BRP-II model

The block relocation problem is defined as follows. Let N homogenous blocks be stored in W stacks with the maximum height of H blocks. Each slot in the stacks is shown with the coordinate (i, j), where $i \in \{1, \cdots, W\}$

and $j \in \{1, \cdots, H\}$ indicate the stack and the tier within the stacking area, respectively. In order to decrease the size of the feasible region and speed up solving BRP-II, Caserta et al. [1] used assumption A1, as follows:

"*When retrieving a target block, we are allowed to relocate only blocks found above the target block ($\cdots$ to maintain feasibility, blocks must be relocated according to the last-in-first-out (LIFO) policy).*"

Four sets of variables defined by Caserta et al. [1] that are used in this paper are:

$$b_{ijnt} = \begin{cases} 1, & \text{if block } n \text{ is in } (i,j) \text{ in time period } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$x_{ijklnt} = \begin{cases} 1, & \text{if block } n \text{ is relocated from } (i,j) \\ & \text{to } (k,l) \text{ in time period } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{ijnt} = \begin{cases} 1, & \text{if block } n \text{ is retrieved from } (i,j) \\ & \text{in time period } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$v_{nt} = \begin{cases} 1, & \text{if block } n \text{ has been retrieved in time} \\ & \text{period } t', \text{ with } t' \in \{1, \cdots, t-1\} \\ 0, & \text{otherwise.} \end{cases}$$

It is necessary to note that since, in BRP-II, the assumption A1 is applied, in the BRP-II model, v_{nt} is no longer a variable and it is a parameter with the following values [1]:

$$v_{nt} = \begin{cases} 0, & n = 1, \cdots, N, \quad t = 1, \cdots, n; \\ 1, & n = 1, \cdots, N-1, \quad t = n+1, \cdots, N. \end{cases}$$

So, in the rest of this paper, we consider v_{nt} as a parameter. Caserta et al. [1] proposed the mathematical formulation of the BRP-II model as follows:

BRP-II:

$$\min \sum_{i=1}^{W} \sum_{j=1}^{H} \sum_{k=1}^{W} \sum_{l=1}^{H} \sum_{n=1}^{N} \sum_{t=1}^{N} x_{ijklnt}$$

Subject to:

$$\sum_{i=1}^{W} \sum_{j=1}^{H} b_{ijnt} + v_{nt} = 1, \qquad n = 1, \cdots, N, \qquad t = 1, \cdots, T, \tag{1}$$

$$\sum_{n=1}^{N} b_{ijnt} \leq 1, \qquad i = 1, \cdots, W, \qquad j = 1, \cdots, H, \qquad t = 1, \cdots, T, \tag{2}$$

$$\sum_{n=1}^{N} b_{ijnt} \geq \sum_{n=1}^{N} b_{ij+1nt}, \qquad i = 1, \cdots, W, \qquad j = 1, \cdots, H-1, \qquad t = 1, \cdots, T, \tag{3}$$

$$b_{ijnt} = b_{ijnt-1} + \sum_{k=1}^{W} \sum_{l=1}^{H} x_{klijnt-1} - \sum_{k=1}^{W} \sum_{l=1}^{H} x_{ijklnt-1} - y_{ijnt-1}, \qquad i = 1, \cdots, W, \quad j = 1, \cdots, H, \quad n = 1, \cdots, N, \quad t = 2, \cdots, T, \tag{4}$$

$$v_{nt} = \sum_{i=1}^{W} \sum_{j=1}^{H} \sum_{t'=1}^{t-1} y_{ijnt'}, \qquad n = 1, \cdots, N, \qquad t = 1, \cdots, T, \tag{5}$$

$$1 - \sum_{n=1}^{N} x_{ijklnt} \geq \sum_{n=1}^{N} \sum_{j'=j+1}^{H} \sum_{l'=l+1}^{H} x_{ij'kl'nt}, \qquad i,k = 1, \cdots, W, \quad j,l = 1, \cdots, H, \quad t = 1, \cdots, N-1, \tag{6}$$

$$M\left(1 - \sum_{j=1}^{H} b_{ijtt}\right) \geq \sum_{k=1}^{W} \sum_{j=1}^{H} \sum_{l=1}^{H} \sum_{n=1}^{N} \left(\sum_{i'=1}^{i-1} x_{i'jklnt} + \sum_{i''=i+1}^{W} x_{i''jklnt} \right), \qquad i = 1, \cdots, W, \quad t = 1, \cdots, N, \tag{7}$$

$$x_{ijilnt} = 0, \qquad i = 1, \cdots, W, \qquad j,l = 1, \cdots, H, \quad n = 1, \cdots, N, \quad t = 1, \cdots, N. \tag{8}$$

The objective of the BRP-II model is to minimize the total number of relocations. The eight following explanations correspond to the abovementioned eight constraints, respectively:

1. In each time period, each block must be either within the stack or in the outside region;

2. In each time period, each slot (i, j) must be occupied by, at most, one block;
3. In each time period, if a slot is empty, slots above it in the same stack must be empty;
4. Eq. (4) determines the configuration of the stack in time period t, according to the configuration and the moves that are done in time period $t-1$;
5. Eq. (5) provides information about removed blocks;
6. The LIFO policy for relocations must be satisfied. In other words, if, in time period t, block m is originally located in the same stack as block n and in a tier below block n, and if these blocks are relocated to the same stack, then, block m must be relocated to a tier above the tier to which block n is relocated;
7. Relocations are only allowed to originate from the target stack;
8. Relocations within the same stack are not allowed.

Please note that at each stage of BRP-II, first, all blocks that have blocked the target block should be relocated into other stacks. Then, the target block should be removed, and all these movements should be done at one stage of the BRP-II model. This fact is used to prove the incorrectness of the BRP-II model in the next section.

3. Proving incorrectness

According to Caserta et al. [1], assumption A1 has been widely used in literature [3,4]. In this section, it is proven that the BRP-II model over-satisfies assumption A1 and this is why it is not correct. According to A1, relocations are only allowed for blocks found above the target stack, but no limitations are placed on the destination stack and on the number of relocations that are allowed from the target stack into a given destination stack. However, according to Constraint (6) of BRP-II, at each stage of the mathematical model, the number of relocations from the target stack into slots $2 \cdots H$ of any destination stack is limited, at most, to one. More details on this claim are provided in Lemma 1.

Lemma 1. *In each time period t of the BRP-II model, due to Constraint (6), for any two stacks, i and k, at most, one block can be relocated from slots $(i, 2:H)$ into slots $(k, 2:H)$, where $(i, 2:H)$ denotes all slots of the ith stack, except slot $(i, 1)$.*

Proof. In each time period, t, with respect to assumption A1, no block could be relocated from slot $(i, 1)$ into slot $(k, 1)$, because this block is located on the floor and is not a blocking block for the target block. Therefore, it could not be relocated, and, therefore, we have:

$$\forall t, \forall i, \forall k; \qquad \sum_{n=1}^{N} x_{i1k1nt} = 0$$

$$\rightarrow \quad \forall t, \forall i, \forall k; \qquad 1 - \sum_{n=1}^{N} x_{i1k1nt} = 1.$$

Now, with respect to Eq. (8), we have:

$$\rightarrow \quad \forall t, \forall i, \forall k; \qquad \sum_{n=1}^{N} \sum_{j'=2}^{H} \sum_{l'=2}^{H} x_{ij'kl'nt} \leq 1. \qquad \square$$

Thus, according to Lemma 1, the BRP-II model is not created correctly, as, with respect to assumption A1, the number of these relocations is not limited to one. To further explain the issue, please consider Example 2 from Caserta et al. [1], as in Figure 1. For this example: $W = 3$, $H = 4$ and $N = 8$. Assuming A1, it is obvious that this example is feasible. Consequently, we can conclude that BRP-II in this instance must be feasible. However, it is infeasible with respect to Lemma 1 and consequently for BRP-II, because three blocks have blocked block 1 and there are only two other stacks that could each receive, at most, one block.

Now, we show that some results reported by Caserta et al. [1], which are presented under the BRP-II columns in Table 1, are incorrect. Recall that these results are from the BRP-II model and, therefore, Lemma 1 should be satisfied in them. For a better understanding of the problem, the solution procedures for two instances, $3 \times 3 - 1$ and $4 \times 5 - 2$, are illustrated.

Instance $3 \times 3 - 1$: In Figure 2, the middle configuration refers to the initial layout of the instance, $3 \times 3 - 1$. According to assumption A1, after block 1 is retrieved, in order to retrieve block 2, blocks 5 and 6 should be moved to the other stacks. As block 5 can be relocated into either stack 1 or stack 3, the left and right hand side layouts present two first possible children. So far, two relocations are done.

		5
6		4
7		3
8	2	1

Figure 1. Example 2 from Caserta et al. [1].

		6								5
5		4	←	1	5	4	→	6		4
7		9		7	6	9		7		9
3		8		3	2	8		3		8

Figure 2. Instance $3 \times 3 - 1$ and its two first children.

Table 1. Numerical evaluation of BRP2ci versus BRP2c.

Stack size			BRP-II		BRP2c		BRP2ci		$R.$
$H-2$	W	V	n_r	PT	n_r	PT	n_r	PT	
3	3	1	**6**	3	6	1.45	6	0.11	13.2
		2	5	3	5	1.09	5	0.11	9.9
		3	2	3	2	0.31	2	0.08	3.9
		4	4	3	4	0.42	4	0.09	4.7
		5	1	3	1	0.81	1	0.06	13.5
3	4	1	5	3	5	4.48	5	0.28	16
		2	3	5	3	2.12	3	0.22	9.6
		3	7	11	7	6.18	7	0.34	18.2
		4	5	9	5	3.9	5	0.26	15
		5	6	11	6	5.19	6	0.27	19.2
3	5	1	6	37	6	19.03	6	0.72	26.4
		2	7	27	7	45.18	7	1.2	37.7
		3	8	435	8	28.28	8	0.91	31.1
		4	6	20	6	7.89	6	0.8	9.9
		5	9	128	9	26.61	9	1.59	16.7
3	6	1	11	123	11	159.6	11	5.35	29.8
		2	8	113	7	26.85	7	1.59	16.9
		3	11	156	11	81.4	11	3.09	26.3
		4	7	84	7	117.86	7	1.78	66.2
		5	4	37	4	16.29	4	1.23	13.2
3	7	1	7	291	7	112.85	7	2.89	39
		2	10	257	*	*	10	3.29	-
		3	9	3964	*	*	9	3.34	-
		4	8	248	8	163.41	8	3.35	48.8
		5	12	187	12	110.46	12	7.97	13.9
3	8	1	8	2433	-	-	8	7.25	-
		2	**9**	344	-	-	10	9.31	-
		3	11	1781	-	-	9	6.57	-
		4	10	298	-	-	10	11.03	-
		5	**11**	375	-	-	13	11.06	-
4	4	1	11	202	10	100.42	10	1.25	80.3
		2	12	134	13	*	10	0.95	-
		3	11	156	10	*	10	1.56	-
		4	8	46	7	17.19	7	0.78	22
		5	10	45	9	64.88	9	1.15	56.4
4	5	1	**13**	1476	*	*	16	61.62	-
		2	**8**	3611	*	*	10	5.27	-
		3	14	592	22	*	13	8.28	-
		4	10	3981	*	*	8	2.96	-
		5	**12**	2754	24	*	16	81.26	-

Table 1. Numerical evaluation of BRP2ci versus BRP2c (continued).

Stack size			BRP-II		BRP2c		BRP2ci		*R.*
$H-2$	W	V	n_r	PT	n_r	PT	n_r	PT	
4	6	1	19	*	-	-	20	*	-
		2	8	2413	-	-	8	5.23	-
		3	14	*	-	-	13	12.86	-
		4	14	4800	-	-	14	23.4	-
		5	22	*	-	-	15	45.22	-
4	7	1	-	-	-	-	17	44.1	-
		2	-	-	-	-	18	20.47	-
		3	-	-	-	-	13	18.95	-
		4	-	-	-	-	16	49.78	-
		5	-	-	-	-	16	85.75	-
5	4	1	-	-	-	-	15	118.2	-
		2	-	-	-	-	19	185.36	-
		3	-	-	-	-	15	5.9	-
		4	-	-	-	-	12	5.01	-
		5	-	-	-	-	17	118.16	-
5	5	1	-	-	-	-	*	*	-
		2	-	-	-	-	16	22.06	-
		3	-	-	-	-	*	*	-
		4	-	-	-	-	24	*	-
		5	-	-	-	-	16	46.3	-
Average									25.3

In both new layouts, blocks 5, 6, 7 and 9 have blocked some blocks with higher priority than themselves, implying that four other relocations are unavoidable. Hence, so far, the number of relocations for both layouts is at least six. In addition, in order to free block 3, at least one other relocation is necessary, because, in relocating blocks 5 and 7 or blocks 6 and 7, at least one of them will locate above a block with higher priority than itself. Therefore, with respect to Lemma 1, the number of relocations obtained from BRP-II is at least seven. However, Caserta et al. [1] have reported that the optimal number of relocations for this instance is six, which is not correct. Please note that through replacing Constraint (6) with Constraint (9), the reported n_r of BRP2c and BRP2ci models improves from seven to six, because Constraint (9) provides more alternative relocations to be followed.

Instance 4 × 5 − 2: As seen in Figure 3, eight blocks shaded in a yellow color have blocked blocks with higher priority than themselves. In order to retrieve block 1, blocks 18, 2, and 16 should be moved to other stacks. However, after moving to other stacks, blocks 18 and 16 will again block some blocks with higher priority than themselves. This means that the optimal number of relocations for this instance is at least ten. However, Caserta et al. [1] reported that the optimal number of relocations for this instance is eight. Again, this could not be correct.

4	3	18	7	20
19	6	2	8	10
17	13	16	15	11
5	12	1	9	14

Figure 3. Instance $4 \times 5 - 2$.

4. The corrected BRP-II model, BRP2c

As discussed in Section 3, due to Constraint (6) of the BRP-II model, it over-satisfies assumption A1. To correct this fault, we suggest replacing Constraint (6) of the BRP-II model with Constraint (9), as follows:

$$H\left(1-\sum_{n=1}^{N}x_{ijklnt}\right)\geq\sum_{n=1}^{N}\sum_{j'=j+1}^{H}\sum_{l'=l+1}^{H}x_{ij'kl'nt},$$

$$i,k=1,\cdots,W,\qquad j,l=1,\cdots,H,$$

$$t=1,\cdots,N-1. \tag{9}$$

In Constraint (9), we have corrected Constraint (6), so that the drawback referred to in Lemma 1 is resolved. In fact, by multiplying the left hand side of the Constraint (6) by H, the number of relocations from one stack to another is unlimited. Using this new constraint, instead of Constraint (6), the BRP-II model is corrected and is called "BRP2c" in this paper. Please note that the only difference between BRP-II and BRP2c is the replacement of Constraint (6) of the BRP-II model with Constraint (9). The other constraints, as well as the objective functions, are the same for both models.

5. The corrected and improved BRP-II model, BRP2ci

In this section, by adding some new cut constraints to the BRP2c model, an improved form of it, called "BRP2ci", is presented. To do so, we remove Constraints (6) and (7) from BRP2c and add the new Constraints (9)-(13) to it. The mathematical formulation of BRP2ci is presented as follows:

$$\text{BRP2ci:}\qquad \min\sum_{i=1}^{W}\sum_{j=1}^{H}\sum_{k=1}^{W}\sum_{l=1}^{H}\sum_{n=1}^{N}\sum_{t=1}^{N}x_{ijklnt}.$$

Subject to Constraints (1)-(5), (8) and (9) from BRP2c and the following:

$$\sum_{i=1}^{W}\sum_{j=1}^{H}y_{ijtt}=1,\qquad t=1,\cdots,T, \tag{10}$$

$$\sum_{i=1}^{W}\sum_{j=1}^{H}y_{ijnt}=0,\qquad t=1,\cdots,T,$$

$$n=1,\cdots,t-1,t+1,\cdots,N, \tag{11}$$

$$\sum_{k=1}^{W}\sum_{l=1}^{H}\sum_{n=1}^{N}x_{ijklnt}\leq\sum_{l=1}^{j-1}y_{iltt},\qquad t=1,\cdots,T,$$

$$i=1,\cdots,W,\qquad j=1,\cdots,H, \tag{12}$$

$$\sum_{k=1}^{W}\sum_{l=1}^{H}x_{ijklnt}+y_{ijnt}\leq b_{ijnt},\qquad t=1,\cdots,T,$$

$$i=1,\cdots,W,\qquad j=1,\cdots,H,\qquad n=1,\cdots,N. \tag{13}$$

In order to speed up solving the BRP2ci model, Constraints (10) to (13) are developed. The aim of these constraints is to cut some areas which do not include optimal solutions. A brief explanation of Constraints (10) to (13) for the BRP2ci model is as follows:

- Constraint (12): In time period t, block t must be retrieved;
- Constraint (13): In time period t, blocks $1,\cdots,t-1$, $t+1,\cdots,T$ could not be retrieved;
- Constraint (14): In each time period, only blocks which are located above the target block could be relocated;
- Constraint (15): In each time period, if a slot is empty, then, no movement from it could happen.

6. Numerical evaluation

In order to evaluate the performance of BRP2ci in comparison with that of BRP2c, we have solved both models with the same settings using a machine with Intel Core i3 CPU and 4 GB of RAM. Both models are implemented and solved using IBM ILOG CPLEX Interactive Optimizer 12.1.0.

Results on the same instances of the BRP, as given by Caserta et al. [1], are reported in Table 1. In the table, the first two columns define the instance size, in terms of number of tiers and number of stacks. Column 3 indicates the instance number ranging from 1 to 5. Next, columns 4 and 5 provide the optimal solution (n_r) and the processing time in seconds (PT) that are required to reach the optimum using commercial MIP software, reported by Caserta et al. [1] for BRP-II. As discussed in Section 3, some results reported by Caserta et al. [1] are incorrectly reported. Some of these instances in Table 1, which have incorrect values for n_r, are identified by crossing them out. In cases where computational time is more than the allotted time of 86,400 seconds (one day), Caserta et al. [1] has placed an asterisk in the column, PT, indicating that the reported solutions in column n_r are not necessarily optimal. The next two sections of the table correspond to the analogous information for the BRP2c and BRP2ci models, respectively. Reported computational time, PT, includes computational time after the problem is read by the software. The last column shows the ratio of improvement in the computational time obtained by BRP2ci over BRP2c. For all instances that are solved using models BRP2c and BRP2ci, a maximum computation time of 900 seconds is allotted.

Note that the BRP2ci model is able to solve larger instances. So, we have solved some new classes of instances (4×7, 5×4, and 5×5) that are not solved by BRP-II, as reported by Caserta et al. [1]. Being almost non-solvable within the allotted time of 900s, some instances for BRP2c are not solved. In Table 1, corresponding values to these instances are shown by a dash. For both BRP2c and BRP2ci, if PT is replaced by an asterisk, this means that the optimal solution is not found within the allotted time. If n_r is also replaced by an asterisk, this means that no feasible solution is found; otherwise, the best feasible solution found is reported. Using new cut constraints, it is seen that in comparison to BRP2c, BRP2ci improves the computational time, PT, by cutting some useless areas from the search space so that the ratio of computational time is decreased, on average, 25 times.

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Biographies

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