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Some generalized Einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making

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Abstract. In this paper, for the Multiple Attribute Group Decision Making (MAGDM) problems where attribute values are the Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs), the group decision making method based on some generalized Einstein aggregation operators was developed. Firstly, Interval-Valued Intuitionistic Fuzzy Generalized Einstein Weighted Averaging (IVIFGEWA) operator, Interval-Valued Intuitionistic Fuzzy Generalized Einstein Ordered Weighted Averaging (IVIFGEOWA) operator, and Interval-Valued Intuitionistic Fuzzy Generalized Einstein Hybrid Weighted Averaging (IVIFGE-HWA) operator were proposed. Some general properties of these operators such as idempotency, commutativity, monotonicity, and boundedness, were discussed, and some special cases in these operators were analyzed. Furthermore, the method for MAGDM problems based on these operators was developed, and the operational processes were illustrated in detail. Finally, an illustrative example was given to show the decision steps of the proposed method and to demonstrate their effectiveness.

1. Introduction

Fuzzy theory is an important tool to process fuzzy information. Zadeh [1] firstly proposed the fuzzy set theory, then Atanassov [2,3] proposed the Intuitionistic Fuzzy Set (IFS) by adding a non-membership function. Furthermore, Atanassov and Gargov [4], and Atanassov [5] proposed the Interval-Valued Intuitionistic Fuzzy Set (IVIFS) in which the membership and non-membership degrees were extended to interval numbers. Chen et al. [6] proposed a fuzzy ranking method for Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs) based on likelihood-based comparison relations, and then presented a new multi-attribute decision making method based on the proposed interval-valued intuitionistic fuzzy weighted average operator. Gomathi Nayagam et al. [7] proposed a new accuracy function for IVIFNs. Liu et al. [8] proposed the interval-valued intuitionistic fuzzy entropy. Wei et al. [9] proposed an entropy measure for IVIFS, which generalizes three entropy measures of intuitionistic fuzzy sets defined independently by Szmidt, Wang, and Huang. Zhang and Yu [10] constructed an optimization model to determine the attribute weights based on cross-entropy for multi-attribute decision making problems with interval-valued intuitionistic fuzzy information; then, an extended TOPSIS method was proposed to rank all the alternatives. Wang et al. [11] defined a new score function for the IVIFNs based on the prospect value functions. Xu and Yager [12] developed a new similarity measure between IVIFSs

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and utilized it to solve the group decision making problems where attribute values are interval-valued intuitionistic fuzzy numbers. Wan and Dong [13] proposed a possibility degree decision method for multi-attribute group decision making with interval-valued intuitionistic fuzzy information. Wang et al. [14] proposed a decision approach based on interval belief degrees and fuzzy evidential reasoning for multi-criteria decision problems in which the criteria weights are interval numbers and criteria values are triangular intuitionistic fuzzy numbers. Wang and Zhang [15] proposed a new decision-making method based on the evidential reasoning algorithms for multicriteria fuzzy decision-making problems in which the criteria's weight is not completely certain, and the criteria values are Atanassov's Intuitionistic Fuzzy Sets (A-IFSs). The above-mentioned researchers have investigated the basic methods for ranking the intuitionistic fuzzy numbers or some extensions of them based on some extended theory such as entropy, possibility degree, similarity, etc. However, these methods can only give the ranking of the alternatives with intuitionistic fuzzy information, and cannot give the overall evaluation values of each alternative.

The information aggregation operators, based on IVIFS, are the important decision making tools which can give the overall evaluation values of each alternative, and then rank the alternatives based on these overall values. The aggregation operators are receiving more and more attentions [16-31]. Xu [23], and Xu and Chen [24,25] proposed some aggregation operators for IVIFSs such as the Interval Intuitionistic Fuzzy Weighted Aggregation operator (IIFWA), the Interval Intuitionistic Fuzzy Ordered Weighted Aggregation operator (IIFOWA), Interval Intuitionistic Fuzzy Hybrid Aggregation operator (IIFHA), the Interval Intuitionistic Fuzzy Weighted Geometric operator (IIFWG), the Interval Intuitionistic Fuzzy Ordered Weighted Geometric operator (IIFOWG), and the Interval Intuitionistic Fuzzy Hybrid Geometric operator (IIFHG). Furthermore, some properties of these operators were investigated. Wei and Yi [26] developed an Induced Interval Intuitionistic Fuzzy Ordered Weighted Geometric operator (I-IIFOWG) by adding the induced parameters. Yu et al. [27] proposed the Interval-Valued Intuitionistic Fuzzy Prioritized Weighted Average (IVIFPWA) operator and the Interval-Valued Intuitionistic Fuzzy Prioritized Weighted Geometric (IVIFPWG) operator according to the priority of the different attributes, and studied some of their desirable properties. Furthermore, a decision making approach based on these operators is given to solve the MADM problems in which attribute values take the form of interval-valued intuitionistic fuzzy numbers. Zhao [28] proposed generalized interval intuitionistic fuzzy aggregation operators, including arithmetic weighted averaging operator, ordered weighted averaging operator, and hybrid weighted averaging operator, which extended the generalized aggregation operators to the environment in which the attribute values are interval-valued intuitionistic fuzzy sets. The generalized aggregation operators are a generalization of arithmetic aggregation operators and geometric aggregation operators. Wang et al. [29,30] defined intuitionistic interval numbers, and proposed intuitionistic interval weighted arithmetic averaging operator, intuitionistic interval ordered weighted averaging operator, and intuitionistic interval heavy averaging operator, etc. Wang et al. [31] defined the Normal Intuitionistic Fuzzy Numbers (NIFNs), and proposed some normal intuitionistic fuzzy aggregation operators, including Normal Intuitionistic Fuzzy-Induced Generalized Ordered Weighted Averaging operator (NIFIGOWA), etc.

All the above aggregation operators are based on the algebraic operational rules of IVIFNs, and the keys of the algebraic operations are Algebraic product and Algebraic sum, which are one type of operations that can be chosen to model the intersection and union of IVIFNs. In general, a general T-norm and T-conorm can be used to model the intersection and union of IVIFNs [32,33]. Wang and Liu [34] proposed the intuitionistic fuzzy Einstein aggregation operators based on Einstein operations which meet the typical T-norm and T-conorm and have the same smooth approximations as the algebraic operators such as the Intuitionistic Fuzzy Einstein Weighted Geometric operator (IFEWG) and the Intuitionistic Fuzzy Einstein Ordered Weighted Geometric operator (IFEOWG), and established some general properties of these operators such as idempotency, commutativity, and monotonicity. Wang and Liu [35] proposed the Intuitionistic Fuzzy Einstein Weighted Averaging operator (IFEWA) and the Intuitionistic Fuzzy Einstein Ordered Weighted Averaging operator (IFEOWA), and studied various properties of these operators and analyzed the relations between the existing intuitionistic fuzzy aggregation operators and them. Maris and Iliadis [36] further explained the advantages of Einstein operations by using some T-norms to unify the risk indices and to produce a unified means of risk measure. The algebraic T-norm estimated the risky areas under average rainfall conditions, and the Einstein T-norm offered a good approach for an overall evaluation. The computer system has proven its ability to work more effectively compared to the older methods.

Because the interval-valued intuitionistic fuzzy numbers are easier to express the fuzzy decision information and Einstein operations have the good smooth approximations, and the generalized aggregation operators are a generalization of most aggregation operators such as arithmetic aggregation operators and geometric aggregation operators, how to extend the

Einstein operations to aggregate the interval-valued intuitionistic fuzzy information based on the generalized aggregation operators is a meaningful works, which is also the focus of this paper. In the following, we will investigate interval-valued intuitionistic generalized Einstein aggregation operators which combine Einstein aggregation operators with the generalized aggregation operators based on interval-valued intuitionistic fuzzy information in order to generalize the existing aggregation operators.

In order to do so, the remainder of this paper is shown as follows. In Section 2, we briefly review some basic concepts of interval-valued intuitionistic fuzzy sets, generalized aggregation operators, and Einstein operations. In Section 3, we establish Einstein operations of IVIFNs and their characteristics, furthermore, develop some generalized Einstein aggregation operators such as Interval-Valued Intuitionistic Fuzzy Generalized Einstein Weighted Averaging (IVIFGEWA) operator, Interval-Valued Intuitionistic Fuzzy Generalized Einstein Ordered Weighted Averaging (IVIFGEOWA) operator, and Interval-Valued Intuitionistic Fuzzy Generalized Einstein Hybrid Averaging (IVIFGEHA) operator. We also study some desirable properties of them such as idempotency, commutativity, monotonicity and boundedness and some special cases of them. In Section 4, based on the operators introduced in Section 3, we propose a decision making method for multiple attribute group decision making problems with interval-valued intuitionistic fuzzy information. In Section 5, we give an example to illustrate the application of proposed method, and compare the developed methods with the existing methods. In Section 6, we conclude the paper.

2. Preliminaries

2.1. Interval-valued intuitionistic fuzzy set

Definition 1 [2]. If $X = \{x_1, x_2, \cdots, x_n\}$ is a universe of discourse, then an Intuitionistic Fuzzy Set (IFS) A in X is given by:

$$A = \{< x, u_A(x), v_A(x) > x \in X\}, \tag{1}$$

where $u_A : X \to [0,1]$ and $v_A : X \to [0,1]$, on condition that $0 \le u_A(x) + v_A(x) \le 1$, $\forall x \in X$. The numbers $u_A(x)$ and $v_A(x)$ represent the membership degree and non-membership degree of the element x to the set A, respectively.

For each IFS A in X, if $\pi(x) = 1 - u_A(x) - v_A(x)$, $\forall x \in X$, then $\pi(x)$ is called the degree of indeterminacy of x to the set A [2,3]. It is obvious that $0 \le \pi(x) \le 1$, $\forall x \in X$.

To give element x, the pair $(u_A(x), v_A(x))$ is called an Intuitionistic Fuzzy Value (IFV) [37] which for convenience can be denoted as $\tilde{a} = (u_{\tilde{a}}, v_{\tilde{a}})$ such that $u_{\tilde{a}} \in [0,1]$, $v_{\tilde{a}} \in [0,1]$ and $0 \le u_{\tilde{a}} + v_{\tilde{a}} \le 1$.

Atanassov and Gargov [4] further extended the intuitionistic fuzzy set to the Interval-Valued Intuitionistic Fuzzy Set (IVIFS) in which membership and non-membership functions are expressed by interval numbers.

Definition 2 [3,5]. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse, then an IVIFS $\tilde{A}$ in X is given by:

$$\tilde{A} = \{< x, \tilde{u}_{\tilde{A}}(x), \tilde{v}_{\tilde{A}}(x) > x \in X\}, \tag{2}$$

where $\tilde{u}_{\tilde{A}}(x) \subseteq [0,1]$ and $\tilde{v}_{\tilde{A}}(x) \subseteq [0,1]$ are interval numbers, on condition that $0 \le \sup(\tilde{u}_{\tilde{A}}(x)) + \sup(\tilde{v}_{\tilde{A}}(x)) \le 1$, $\forall x \in X$. The numbers $\tilde{u}_{\tilde{A}}(x)$ and $\tilde{v}_{\tilde{A}}(x)$ represent the membership degree and non-membership degree of the element x to the set $\tilde{A}$, respectively. For convenience, if $\tilde{u}_{\tilde{A}}(x_i) = [a,b]$ and $\tilde{v}_{\tilde{A}}(x_i) = [c,d]$, then $\tilde{a} = ([a,b],[c,d])$ is called an Interval-Valued Intuitionistic Fuzzy Number (IVIFN).

Definition 3 [6]. If $\tilde{a} = ([a,b],[c,d])$ be an IVIFN, a score function S of the IVIFN $\tilde{a}$ can be represented as follows:

$$S(\tilde{a}) = \frac{a+b-c-d}{2}. \tag{3}$$

Obviously, $S(\tilde{a}) \in [-1,1]$.

Definition 4 [6]. Let $\tilde{a} = ([a,b],[c,d])$ be an IVIFN, then an accuracy function H of the IVIFN $\tilde{a}$ can be represented as follows:

$$H(\tilde{a}) = \frac{a+b+c+d}{2}. \tag{4}$$

Definition 5 [6]. If $\tilde{a}_1 = ([a_1,b_1],[c_1,d_1])$ and $\tilde{a}_2 = ([a_2,b_2],[c_2,d_2])$ are any two IVIFNs, then:

1. If $S(\tilde{a}_1) > S(\tilde{a}_2)$, then, $\tilde{a}_1 > \tilde{a}_2$;
2. If $S(\tilde{a}_1) = S(\tilde{a}_2)$, then:
   If $L(\tilde{a}_1) > L(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;
   If $L(\tilde{a}_1) = L(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

2.2. GHWA operator

Definition 6 [28]. A GWA operator of dimension n is a mapping GWA: $(R^+)^n \to R^+$. Such that:

$$\mathrm{GWA}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} w_j a_j^{\lambda}\right)^{1/\lambda}, \tag{5}$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is a weight vector of $(a_1, a_2, \cdots, a_n)$, satisfying $w_j \in [0,1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$. λ is a parameter such that $\lambda \in (0, +\infty)$, and R^+ is the set of all nonnegative real numbers.

Definition 7 [28]. A GOWA operator of dimension n is a mapping GOWA: $(R^+)^n \rightarrow R^+$, such that:

$$\text{GOWA}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda}\right)^{1/\lambda}, \tag{6}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is a weight vector which is correlative with GOWA, satisfying $\omega_j \in [0,1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$; b_j is the jth largest among real numbers $a_k (k = 1, 2, \cdots, n)$. λ is a parameter, such that $\lambda \in (0, +\infty)$, and R^+ is the set of all nonnegative real numbers.

Definition 8. A GHWA operator of dimension n is a mapping GHWA: $(R^+)^n \rightarrow R^+$, such that:

$$\text{GHWA}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda}\right)^{1/\lambda}, \tag{7}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is a weight vector which is correlative with GOWA, satisfying $\omega_j \in [0,1] (j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$; b_j is the jth largest among real numbers $(nw_k a_k)(k = 1, 2, \cdots, n)$; $w = (w_1, w_2, \cdots, w_n)^T$ is a weight vector of $(a_1, a_2, \cdots, a_n)$, satisfying $w_k \in [0,1](k = 1, 2, \cdots, n)$ and $\sum_{k=1}^{n} w_k = 1$. λ is a parameter, such that $\lambda \in (0, +\infty)$, and R^+ is the set of all nonnegative real numbers.

2.3. Einstein operators

The t-operators are intersection and union operators in fuzzy set theory which are composed of T-norm (T) and T-conorm (T^*), respectively [38]. Although they are originated from the field of statistical metric spaces, they also have important applications in the fuzzy inference and fuzzy decision making.

Based on a T-norm and T-conorm, a generalized union and a generalized intersection of intuitionistic fuzzy sets were introduced by Deschrijver and Kerre [39].

Definition 9 [39]. Let A and B be any two intuitionistic fuzzy sets, then the generalized intersection and union are defined as follows:

$$A \cap_{T,T^*} B = \{ < x, T(u_A(x), u_B(x)), T^*(v_A(x), v_B(x)) > x \in X\}, \tag{8}$$

$$A \cup_{T,T^*} B = \{ < x, T^*(u_A(x), u_B(x)), T(v_A(x), v_B(x)) > x \in X\}, \tag{9}$$

where T denotes a T-norm and T^* a T-conorm.

For instance, the algebraic product $\tilde{a}_1 \otimes \tilde{a}_2$ and the algebraic sum $\tilde{a}_1 \oplus \tilde{a}_2$ on two IVIFNs $\tilde{a}_1$ and $\tilde{a}_2$ can be obtained by defining T-norm and T-conorm. When $T(x,y) = xy$ and $T_\gamma^*(x,y) = x + y - xy$, we can obtain:

$$\tilde{a}_1 \oplus \tilde{a}_2 =([a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2], [c_1c_2, d_1d_2]), \tag{10}$$

$$\tilde{a}_1 \otimes \tilde{a}_2 =([a_1a_2, b_1b_2], [c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2]), \tag{11}$$

$$n\tilde{a}_1 = ([1 - (1 - a_1)^n, 1 - (1 - b_1)^n], [c_1^n, d_1^n]) \quad n > 0, \tag{12}$$

$$\tilde{a}_1^n = ([a_1^n, b_1^n], [1 - (1 - c_1)^n, 1 - (1 - c_1)^n]) \quad n > 0. \tag{13}$$

Obviously, the above operational laws are the same as those given by Atanassov [5].

Furthermore, based on the T-norm and T-conorm, Einstein operations are defined as follows [40]:

$$T(x,y) = \frac{xy}{1 + (1 - x)(1 - y)}, \tag{14}$$

$$T_\gamma^*(x,y) = \frac{x + y}{1 + xy}, \tag{15}$$

where T denotes a T-norm and T^* a T-conorm.

Wang and Liu [34] defined the Einstein operational rules of IFVs as shown in the following.

Let $\tilde{a}_1 = (a_1, b_1)$ and $\tilde{a}_2 = (a_2, b_2)$ be two IFVs, then the operational rules based on Einstein T-norm and T-conorm are defined as follows:

$$\tilde{a}_1 \oplus_E \tilde{a}_2 = \left(\frac{a_1 + a_2}{1 + a_1a_2}, \frac{b_1b_2}{1 + (1 - b_1)(1 - b_2)}\right), \tag{16}$$

$$\tilde{a}_1 \otimes_E \tilde{a}_2 = \left(\frac{a_1a_2}{(1 + (1 - a_1)(1 - a_2)}, \frac{b_1 + b_2}{1 + b_1b_2}\right), \tag{17}$$

$$n\tilde{a}_1 = \left(\frac{(1 + a_1)^n - (1 - a_1)^n}{(1 + a_1)^n + (1 - a_1)^n}, \frac{2b_1^n}{(2 - b_1)^n + b_1^n}\right) \quad n > 0, \tag{18}$$

$$\tilde{a}_1^n = \left(\frac{2a_1^n}{(2 - a_1)^n + a_1^n}, \frac{(1 + b_1)^n - (1 - b_1)^n}{(1 + b_1)^n + (1 - b_1)^n}\right) \quad n > 0. \tag{19}$$

3. Einstein operations of IVIFNs

3.1. The operational rules based on Einstein T-norm and T-conorm

In order to establish Einstein operation rules of IVIFNs, firstly, we can give the following definitions.

Definition 10. Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs, then, the generalized intersection and union are defined as follows:

$$\tilde{a}_1 \otimes_{T,T^*} \tilde{a}_2 = ([T(a_1, a_2), T(b_1, b_2)], [T^*(c_1, c_2), T^*(d_1, d_2)]), \quad (20)$$

$$\tilde{a}_1 \oplus_{T^*,T} \tilde{a}_2 = ([T^*(a_1, a_2), T^*(b_1, b_2], [T(c_1, c_2), T(d_1, d_2)]). \quad (21)$$

Based on Definition 10, Einstein T-norm and T-conorm, we can establish the Einstein operational rules for two IVIFNs, respectively.

Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs, then the operational rules based on Einstein T-norm and T-conorm are defined as follows:

$$\tilde{a}_1 \oplus_E \tilde{a}_2 = \left(\left[\frac{a_1 + a_2}{1 + a_1 a_2}, \frac{b_1 + b_2}{1 + b_1 b_2} \right], \left[\frac{c_1 c_2}{1 + (1 - c_1)(1 - c_2)}, \frac{d_1 d_2}{1 + (1 - d_1)(1 - d_2)} \right] \right), \quad (22)$$

$$\tilde{a}_1 \otimes_E \tilde{a}_2 = \left(\left[\frac{a_1 a_2}{1 + (1 - a_1)(1 - a_2)}, \frac{b_1 b_2}{1 + (1 - b_1)(1 - b_2)} \right], \left[\frac{c_1 + c_2}{1 + c_1 c_2}, \frac{d_1 + d_2}{1 + d_1 d_2} \right] \right), \quad (23)$$

$$n\tilde{a}_1 = \left(\left[\frac{(1 + a_1)^n - (1 - a_1)^n}{(1 + a_1)^n + (1 - a_1)^n}, \frac{(1 + b_1)^n - (1 - b_1)^n}{(1 + b_1)^n + (1 - b_1)^n} \right], \left[\frac{2c_1^n}{(2 - c_1)^n + c_1^n}, \frac{2d_1^n}{(2 - d_1)^n + d_1^n} \right] \right)$$
$$n > 0, \quad (24)$$

$$\tilde{a}_1^n = \left(\left[\frac{2a_1^n}{(2 - a_1)^n + a_1^n}, \frac{2b_1^n}{(2 - b_1)^n + b_1^n} \right], \left[\frac{(1 + c_1)^n - (1 - c_1)^n}{(1 + c_1)^n + (1 - c_1)^n}, \frac{(1 + d_1)^n - (1 - d_1)^n}{(1 + d_1)^n + (1 - d_1)^n} \right] \right)$$
$$n > 0. \quad (25)$$

Theorem 1. Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be any two IVIFNs, and $\gamma > 0$, then:

$$\tilde{a}_1 \oplus_E \tilde{a}_2 = \tilde{a}_2 \oplus_E \tilde{a}_1, \quad (26)$$

$$\tilde{a}_1 \otimes_E \tilde{a}_2 = \tilde{a}_2 \otimes_E \tilde{a}_1, \quad (27)$$

$$\eta(\tilde{a}_1 \oplus_E \tilde{a}_2) = \eta\tilde{a}_1 \oplus_E \eta\tilde{a}_2, \qquad \eta \geq 0, \quad (28)$$

$$\eta_1\tilde{a}_1 \oplus_E \eta_2\tilde{a}_1 = (\eta_1 + \eta_2)\tilde{a}_1, \quad \eta_1, \eta_2 \geq 0, \quad (29)$$

$$\tilde{a}_1^{\eta_1} \otimes_E \tilde{a}_1^{\eta_2} = (\tilde{a}_1)^{\eta_1 + \eta_2}, \quad \eta_1, \eta_2 \geq 0, \quad (30)$$

$$\tilde{a}_1^{\eta} \otimes_E \tilde{a}_2^{\eta} = (\tilde{a}_1 \otimes_E \tilde{a}_2)^{\eta}, \quad \eta \geq 0. \quad (31)$$

It is easy to prove that the formulas in Theorem 1 are all right, the proofs are omitted in here.

3.2. The interval-valued intuitionistic fuzzy generalized Einstein hybrid averaging operators

We can give definition of the interval-valued intuitionistic fuzzy generalized Einstein averaging operators.

Definition 11. Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of the IVIFNs, and IVIFGEWA : $\Omega^n \to \Omega$, if:

$$\text{IVIFGEWA } (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\mathop{\oplus_E}_{j=1}^{n} \left(w_j \tilde{a}_j^{\lambda} \right) \right)^{1/\lambda}, \quad (32)$$

where Ω is the set of all IVIFNs, and $\lambda > 0$. $w = (w_1, w_2, \cdots, w_n)^T$ is weight vector of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then IVIFGEWA is called the interval-valued intuitionistic fuzzy generalized Einstein weighted averaging operator.

Based on the Einstein operational rules of the IVIFNs, we can derive the result shown as Theorem 2.

Theorem 2. Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2 \cdots, n)$ be a collection of the IVIFNs, then, the result aggregated from Definition 11 is still an IVIFN, and even:

$$\text{IVIFGEWA}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=$$

$$\left(\left[\frac{2\left(\prod_{j=1}^n x_{a_j}^{w_j}-\prod_{j=1}^n y_{a_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n x_{a_j}^{w_j}+3\prod_{j=1}^n y_{a_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n x_{a_j}^{w_j}-\prod_{j=1}^n y_{a_j}^{w_j}\right)^{1/\lambda}},\right.\right.$$

$$\left.\frac{2\left(\prod_{j=1}^n x_{b_j}^{w_j}-\prod_{j=1}^n y_{b_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n x_{b_j}^{w_j}+3\prod_{j=1}^n y_{b_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n x_{b_j}^{w_j}-\prod_{j=1}^n y_{b_j}^{w_j}\right)^{1/\lambda}}\right],$$

$$\left[\frac{\left(\prod_{j=1}^n z_{c_j}^{w_j}+3\prod_{j=1}^n t_{c_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n z_{c_j}^{w_j}-\prod_{j=1}^n t_{c_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n z_{c_j}^{w_j}+3\prod_{j=1}^n t_{c_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n z_{c_j}^{w_j}-\prod_{j=1}^n t_{c_j}^{w_j}\right)^{1/\lambda}},\right.$$

$$\left.\left.\frac{\left(\prod_{j=1}^n z_{d_j}^{w_j}+3\prod_{j=1}^n t_{d_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n z_{d_j}^{w_j}-\prod_{j=1}^n t_{d_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n z_{d_j}^{w_j}+3\prod_{j=1}^n t_{d_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n z_{d_j}^{w_j}-\prod_{j=1}^n t_{d_j}^{w_j}\right)^{1/\lambda}}\right]\right), \tag{33}$$

where:

$$x_{a_j}=(2-a_j)^\lambda+3a_j^\lambda,\qquad y_{a_j}=(2-a_j)^\lambda-a_j^\lambda,$$

$$x_{b_j}=(2-b_j)^\lambda+3b_j^\lambda,\qquad y_{b_j}=(2-b_j)^\lambda-b_j^\lambda,$$

$$z_{c_j}=(1+c_j)^\lambda+3(1-c_j)^\lambda,$$

$$t_{c_j}=(1+c_j)^\lambda-(1-c_j)^\lambda,$$

$$z_{d_j}=(1+d_j)^\lambda+3(1-d_j)^\lambda,$$

$$t_{d_j}=(1+d_j)^\lambda-(1-d_j)^\lambda.$$

Proof.

1. From Eq. (32), we can calculate $\tilde{a}_j^\lambda$ firstly, and obtain:

$$\tilde{a}_1^\lambda=\left(\left[\frac{2a_1^\lambda}{(2-a_1)^\lambda+a_1^\lambda},\frac{2b_1^\lambda}{(2-b_1)^\lambda+b_1^\lambda}\right],\right.$$

$$\left[\frac{(1+c_1)^\lambda-(1-c_1)^\lambda}{(1+c_1)^\lambda+(1-c_1)^\lambda},\right.$$

$$\left.\left.\frac{(1+d_1)^\lambda-(1-d_1)^\lambda}{(1+d_1)^\lambda+(1-d_1)^\lambda}\right]\right).$$

2. Calculate $w_j\tilde{a}_j^\lambda$, and obtain the equation as shown in Box I.
3. Calculate $\overset{n}{\underset{j=1}{\oplus}}_E(w_j\tilde{a}_j^\lambda)$. For convenience, let:

$$x_{a_j}=(2-a_j)^\lambda+3a_j^\lambda,\qquad y_{a_j}=(2-a_j)^\lambda-a_j^\lambda,$$

$$x_{b_j}=(2-b_j)^\lambda+3b_j^\lambda,\qquad y_{b_j}=(2-b_j)^\lambda-b_j^\lambda,$$

$$z_{c_j}=(1+c_j)^\lambda+3(1-c_j)^\lambda,$$

$$t_{c_j}=(1+c_j)^\lambda-(1-c_j)^\lambda,$$

$$z_{d_j}=(1+d_j)^\lambda+3(1-d_j)^\lambda,$$

$$t_{d_j}=(1+d_j)^\lambda-(1-d_j)^\lambda.$$

Then:

$$w_j\tilde{a}_j^\lambda=\left(\left[\frac{x_{a_j}^{w_j}-y_{a_j}^{w_j}}{x_{a_j}^{w_j}+y_{a_j}^{w_j}},\frac{x_{b_j}^{w_j}-y_{b_j}^{w_j}}{x_{b_j}^{w_j}+y_{b_j}^{w_j}}\right],\left[\frac{2t_{c_j}^{w_j}}{z_{c_j}^{w_j}+t_{c_j}^{w_j}},\frac{2t_{d_j}^{w_j}}{z_{d_j}^{w_j}+t_{d_j}^{w_j}}\right]\right).$$

In the following, by mathematical induction, we can prove:

$$\overset{n}{\underset{j=1}{\oplus}}_E(w_j\tilde{a}_j^\lambda)=$$

$$\left(\left[\frac{\prod_{j=1}^n x_{a_j}^{w_j}-\prod_{j=1}^n y_{a_j}^{w_j}}{\prod_{j=1}^n x_{a_j}^{w_j}+\prod_{j=1}^n y_{a_j}^{w_j}},\frac{\prod_{j=1}^n x_{b_j}^{w_j}-\prod_{j=1}^n y_{b_j}^{w_j}}{\prod_{j=1}^n x_{b_j}^{w_j}+\prod_{j=1}^n y_{b_j}^{w_j}}\right],\right.$$

$$\left.\left[\frac{2\prod_{j=1}^n t_{c_j}^{w_j}}{\prod_{j=1}^n z_{c_j}^{w_j}+\prod_{j=1}^n t_{c_j}^{w_j}},\frac{2\prod_{j=1}^n t_{d_j}^{w_j}}{\prod_{j=1}^n z_{d_j}^{w_j}+\prod_{j=1}^n t_{d_j}^{w_j}}\right]\right). \tag{34}$$

(a) When $n=1$,

$\because w_1=1.$

For the left-hand side of Eq. (34):

$$\overset{n}{\underset{j=1}{\oplus}}_E(w_j\tilde{a}_j^\lambda)=\tilde{a}_1^\lambda$$

$$=\left(\left[\frac{2a_1^\lambda}{(2-a_1)^\lambda+a_1^\lambda},\frac{2b_1^\lambda}{(2-b_1)^\lambda+b_1^\lambda}\right],\right.$$

$$\left[\frac{(1+c_1)^\lambda-(1-c_1)^\lambda}{(1+c_1)^\lambda+(1-c_1)^\lambda},\right.$$

$$\left.\left.\frac{(1+d_1)^\lambda-(1-d_1)^\lambda}{(1+d_1)^\lambda+(1-d_1)^\lambda}\right]\right),$$

$$w_j\tilde{a}_j^\lambda=\left(\left[\frac{((2-a_j)^\lambda+3a_j^\lambda)^{w_j}-((2-a_j)^\lambda-a_j^\lambda)^{w_j}}{((2-a_j)^\lambda+3a_j^\lambda)^{w_j}+((2-a_j)^\lambda-a_j^\lambda)^{w_j}},\frac{((2-b_j)^\lambda+3b_j^\lambda)^{w_j}-((2-a_j)^\lambda-a_j^\lambda)^{w_j}}{((2-b_j)^\lambda+3b_j^\lambda)^{w_j}+((2-a_j)^\lambda-a_j^\lambda)^{w_j}}\right],\right.$$
$$\left[\frac{2((1+c_j)^\lambda-(1-c_j)^\lambda)^{w_j}}{\left((1+c_j)^\lambda+3(1-c_j)^\lambda\right)^{w_j}+\left((1+c_j)^\lambda-(1-c_j)^\lambda\right)^{w_j}},\right.$$
$$\left.\left.\frac{2((1+d_j)^\lambda-(1-d_j)^\lambda)^{w_j}}{\left((1+d_j)^\lambda+3(1-d_j)^\lambda\right)^{w_j}+((1+d_j)^\lambda-(1-d_j)^\lambda)^{w_j}}\right]\right).$$

Box I

and for the right-hand side of Eq. (34), we have:

$$\left(\left[\frac{\prod_{j=1}^{1}x_{a_j}^{w_j}-\prod_{j=1}^{1}y_{a_j}^{w_j}}{\prod_{j=1}^{1}x_{a_j}^{w_j}+\prod_{j=1}^{1}y_{a_j}^{w_j}},\frac{\prod_{j=1}^{1}x_{b_j}^{w_j}-\prod_{j=1}^{1}y_{b_j}^{w_j}}{\prod_{j=1}^{1}x_{b_j}^{w_j}+\prod_{j=1}^{1}y_{b_j}^{w_j}}\right],\right.$$
$$\left.\left[\frac{2\prod_{j=1}^{1}t_{c_j}^{w_j}}{\prod_{j=1}^{1}z_{c_j}^{w_j}+\prod_{j=1}^{1}t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{1}t_{d_j}^{w_j}}{\prod_{j=1}^{1}z_{d_j}^{w_j}+\prod_{j=1}^{1}t_{d_j}^{w_j}}\right]\right)$$
$$=\left(\left[\frac{x_{a_j}-y_{a_j}}{x_{a_j}+y_{a_j}},\frac{x_{b_j}-y_{b_j}}{x_{b_j}+y_{b_j}}\right],\left[\frac{2t_{c_j}}{z_{c_j}+t_{c_j}},\frac{2t_{d_j}}{z_{d_j}+t_{d_j}}\right]\right)$$
$$=\left(\left[\frac{2a_1^\lambda}{(2-a_1)^\lambda+a_1^\lambda},\frac{2b_1^\lambda}{(2-b_1)^\lambda+b_1^\lambda}\right],\left[\frac{(1+c_1)^\lambda-(1-c_1)^\lambda}{(1+c_1)^\lambda+(1-c_1)^\lambda},\frac{(1+d_1)^\lambda-(1-d_1)^\lambda}{(1+d_1)^\lambda+(1-d_1)^\lambda}\right]\right).$$

Therefore, Eq. (34) holds for $n=1$.

(b) Assume that Eq. (34) holds for $n=k$, then we have:

$$\overset{k}{\underset{j=1}{\oplus_E}}(w_j\tilde{a}_j^\lambda)$$
$$=\left(\left[\frac{\prod_{j=1}^{k}x_{a_j}^{w_j}-\prod_{j=1}^{k}y_{a_j}^{w_j}}{\prod_{j=1}^{k}x_{a_j}^{w_j}+\prod_{j=1}^{k}y_{a_j}^{w_j}},\frac{\prod_{j=1}^{k}x_{b_j}^{w_j}-\prod_{j=1}^{k}y_{b_j}^{w_j}}{\prod_{j=1}^{k}x_{b_j}^{w_j}+\prod_{j=1}^{k}y_{b_j}^{w_j}}\right],\right.$$
$$\left.\left[\frac{2\prod_{j=1}^{k}t_{c_j}^{w_j}}{\prod_{j=1}^{k}z_{c_j}^{w_j}+\prod_{j=1}^{k}t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{k}t_{d_j}^{w_j}}{\prod_{j=1}^{k}z_{d_j}^{w_j}+\prod_{j=1}^{k}t_{d_j}^{w_j}}\right]\right),$$

when $n=k+1$,

$$\overset{k+1}{\underset{j=1}{\oplus_E}}\left(w_j\tilde{a}_j^\lambda\right)=\overset{k+1}{\underset{j=1}{\oplus_h}}\left(w_j\tilde{a}_j^\lambda\right)\oplus_E\left(w_{k+1}\tilde{a}_{k+1}^\lambda\right)$$
$$=\left(\left[\frac{\prod_{j=1}^{k}x_{a_j}^{w_j}-\prod_{j=1}^{k}y_{a_j}^{w_j}}{\prod_{j=1}^{k}x_{a_j}^{w_j}+\prod_{j=1}^{k}y_{a_j}^{w_j}},\frac{\prod_{j=1}^{k}x_{b_j}^{w_j}-\prod_{j=1}^{k}y_{b_j}^{w_j}}{\prod_{j=1}^{k}x_{b_j}^{w_j}+\prod_{j=1}^{k}y_{b_j}^{w_j}}\right],\right.$$
$$\left.\left[\frac{2\prod_{j=1}^{k}t_{c_j}^{w_j}}{\prod_{j=1}^{k}z_{c_j}^{w_j}+\prod_{j=1}^{k}t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{k}t_{d_j}^{w_j}}{\prod_{j=1}^{k}z_{d_j}^{w_j}+\prod_{j=1}^{k}t_{d_j}^{w_j}}\right]\right)$$
$$\oplus_E\left(\left[\frac{x_{a_{j+1}}^{w_{j+1}}-y_{a_{j+1}}^{w_{j+1}}}{x_{a_{j+1}}^{w_{j+1}}+y_{a_{j+1}}^{w_{j+1}}},\frac{x_{b_{j+1}}^{w_{j+1}}-y_{b_{j+1}}^{w_{j+1}}}{x_{b_{j+1}}^{w_{j+1}}+y_{b_{j+1}}^{w_{j+1}}}\right],\left[\frac{2t_{c_{j+1}}^{w_{j+1}}}{z_{c_{j+1}}^{w_{j+1}}+t_{c_{j+1}}^{w_{j+1}}},\frac{2t_{d_{j+1}}^{w_{j+1}}}{z_{d_{j+1}}^{w_{j+1}}+t_{d_{j+1}}^{w_{j+1}}}\right]\right)$$
$$=\left(\left[\frac{\frac{\prod_{j=1}^{k}x_{a_j}^{w_j}-\prod_{j=1}^{k}y_{a_j}^{w_j}}{\prod_{j=1}^{k}x_{a_j}^{w_j}+\prod_{j=1}^{k}y_{a_j}^{w_j}}+\frac{x_{a_{j+1}}^{w_{j+1}}-y_{a_{j+1}}^{w_{j+1}}}{x_{a_{j+1}}^{w_{j+1}}+y_{a_{j+1}}^{w_{j+1}}}}{1+\frac{\prod_{j=1}^{k}x_{a_j}^{w_j}-\prod_{j=1}^{k}y_{a_j}^{w_j}}{\prod_{j=1}^{k}x_{a_j}^{w_j}+\prod_{j=1}^{k}y_{a_j}^{w_j}}\times\frac{x_{a_{j+1}}^{w_{j+1}}-y_{a_{j+1}}^{w_{j+1}}}{x_{a_{j+1}}^{w_{j+1}}+y_{a_{j+1}}^{w_{j+1}}}},\right.\right.$$

$$\left.\frac{\frac{\prod_{j=1}^{k} x_{b_j}^{w_j}-\prod_{j=1}^{k} y_{b_j}^{w_j}}{\prod_{j=1}^{k} x_{b_j}^{w_j}+\prod_{j=1}^{k} y_{b_j}^{w_j}}+\frac{x_{b_{j+1}}^{w_{j+1}}-y_{b_{j+1}}^{w_{j+1}}}{x_{b_{j+1}}^{w_{j+1}}+y_{b_{j+1}}^{w_{j+1}}}}{1+\frac{\prod_{j=1}^{k} x_{b_j}^{w_j}-\prod_{j=1}^{k} y_{b_j}^{w_j}}{\prod_{j=1}^{k} x_{b_j}^{w_j}+\prod_{j=1}^{k} y_{b_j}^{w_j}}\times\frac{x_{b_{j+1}}^{w_{j+1}}-y_{b_{j+1}}^{w_{j+1}}}{x_{b_{j+1}}^{w_{j+1}}+y_{b_{j+1}}^{w_{j+1}}}}\right],$$

$$\left[\frac{\frac{2\prod_{j=1}^{k} t_{c_j}^{w_j}}{\prod_{j=1}^{k} z_{c_j}^{w_j}+\prod_{j=1}^{k} t_{c_j}^{w_j}}\times\frac{2t_{c_{j+1}}^{w_{j+1}}}{z_{c_{j+1}}^{w_{j+1}}+t_{c_{j+1}}^{w_{j+1}}}}{1+\left(1-\frac{2\prod_{j=1}^{k} t_{c_j}^{w_j}}{\prod_{j=1}^{k} z_{c_j}^{w_j}+\prod_{j=1}^{k} t_{c_j}^{w_j}}\right)\times\left(1-\frac{2t_{c_{j+1}}^{w_{j+1}}}{z_{c_{j+1}}^{w_{j+1}}+t_{c_{j+1}}^{w_{j+1}}}\right)},\right.$$

$$\left.\left.\frac{\frac{2\prod_{j=1}^{k} t_{d_j}^{w_j}}{\prod_{j=1}^{k} z_{d_j}^{w_j}+\prod_{j=1}^{k} t_{d_j}^{w_j}}\times\frac{2t_{d_{j+1}}^{w_{j+1}}}{z_{d_{j+1}}^{w_{j+1}}+t_{d_{j+1}}^{w_{j+1}}}}{1+\left(1-\frac{2\prod_{j=1}^{k} t_{d_j}^{w_j}}{\prod_{j=1}^{k} z_{d_j}^{w_j}+\prod_{j=1}^{k} t_{d_j}^{w_j}}\right)\times\left(1-\frac{2t_{d_{j+1}}^{w_{j+1}}}{z_{d_{j+1}}^{w_{j+1}}+t_{d_{j+1}}^{w_{j+1}}}\right)}\right]\right)$$

$$=\left(\left[\frac{\prod_{j=1}^{k+1} x_{a_j}^{w_j}-\prod_{j=1}^{k+1} y_{a_j}^{w_j}}{\prod_{j=1}^{k+1} x_{a_j}^{w_j}+\prod_{j=1}^{k+1} y_{a_j}^{w_j}},\frac{\prod_{j=1}^{k+1} x_{b_j}^{w_j}-\prod_{j=1}^{k+1} y_{b_j}^{w_j}}{\prod_{j=1}^{k+1} x_{b_j}^{w_j}+\prod_{j=1}^{k+1} y_{b_j}^{w_j}}\right],\right.$$

$$\left.\left[\frac{2\prod_{j=1}^{k+1} t_{c_j}^{w_j}}{\prod_{j=1}^{k+1} z_{c_j}^{w_j}+\prod_{j=1}^{k+1} t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{k+1} t_{d_j}^{w_j}}{\prod_{j=1}^{k+1} z_{d_j}^{w_j}+\prod_{j=1}^{k+1} t_{d_j}^{w_j}}\right]\right).$$

Therefore, when $n=k+1$, Eq. (34) holds.

(c) According to steps (a) and (b), we can deduce that Eq. (34) holds for any n. Therefore:

$$\overset{n}{\underset{j=1}{\oplus_E}}\left(w_j\tilde{a}_j^{\lambda}\right)$$

$$=\left(\left[\frac{\prod_{j=1}^{n} x_{a_j}^{w_j}-\prod_{j=1}^{n} y_{a_j}^{w_j}}{\prod_{j=1}^{n} x_{a_j}^{w_j}+\prod_{j=1}^{n} y_{a_j}^{w_j}},\frac{\prod_{j=1}^{n} x_{b_j}^{w_j}-\prod_{j=1}^{n} y_{b_j}^{w_j}}{\prod_{j=1}^{n} x_{b_j}^{w_j}+\prod_{j=1}^{n} y_{b_j}^{w_j}}\right],\right.$$

$$\left.\left[\frac{2\prod_{j=1}^{n} t_{c_j}^{w_j}}{\prod_{j=1}^{n} z_{c_j}^{w_j}+\prod_{j=1}^{n} t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{n} t_{d_j}^{w_j}}{\prod_{j=1}^{n} z_{d_j}^{w_j}+\prod_{j=1}^{n} t_{d_j}^{w_j}}\right]\right).$$

4. Calculate $(\overset{n}{\underset{j=1}{\oplus_E}}(w_j\tilde{a}_j^{\lambda}))^{1/\lambda}$; we can obtain the equation as shown in Box II.

The proof ends.

It is easy to prove that the IVIFGEWA operator has the following properties.

1. **Theorem 3 (monotonicity)**
Let $(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n')$ and $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$ be two collections of IVIFNs, if $\tilde{a}_j'\leq\tilde{a}_j$ for all $j=1,2,\cdots,n$, then:

$$\text{IVIFGEWA}\ (\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n')\leq\text{IVIFGEWA}\ (\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n).$$

2. **Theorem 4 (idempotency)**
Let $\tilde{a}_j=\tilde{a}$, $j=1,2,\cdots,n$, then IVIFGEWA $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}$.

3. **Theorem 5 (bounded)**
The IVIFGEWA operator lies between the max and min operators.

$$\min(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\leq\text{IVIFGEWA}\ (\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\leq\max\ (\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n).$$

Now we can discuss some special cases of the IVIFGEWA operator with respect to the parameter λ.

1. If $\lambda=1$, then the IVIFGEWA operator (Eq. (32)) will be reduced to the Interval-Valued Intuitionistic Fuzzy Einstein Weighted Averaging (IVIFEWA) operator, which is shown as follows:

$$\text{IVIFEWA}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=w_1\tilde{a}_1\oplus_E w_2\tilde{a}_2\oplus_E\cdots\oplus_E w_n\tilde{a}_n.$$

According to Eq. (33), we can obtain:

$$\text{IVIFEWA}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$

$$=\left(\left[\frac{\prod_{j=1}^{n}(1+a_j)^{w_j}-\prod_{j=1}^{n}(1-a_j)^{w_j}}{\prod_{j=1}^{n}(1+a_j)^{w_j}+\prod_{j=1}^{n}(1-a_j)^{w_j}},\frac{\prod_{j=1}^{n}(1+b_j)^{w_j}-\prod_{j=1}^{n}(1-b_j)^{w_j}}{\prod_{j=1}^{n}(1+b_j)^{w_j}+\prod_{j=1}^{n}(1-b_j)^{w_j}}\right],\right.$$

$$\left.\left[\frac{2\prod_{j=1}^{n}c_j^{w_j}}{\prod_{j=1}^{n}(2-c_j)^{w_j}+\prod_{j=1}^{n}c_j^{w_j}},\frac{2\prod_{j=1}^{n}d_j^{w_j}}{\prod_{j=1}^{n}(2-d_j)^{w_j}+\prod_{j=1}^{n}d_j^{w_j}}\right]\right). \tag{35}$$

$$\left(\underset{j=1}{\overset{n}{\oplus}}{}_E\left(w_j\tilde{a}_j^{\lambda}\right)\right)^{1/\lambda}=\left(\left[\frac{\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}}{\prod_{j=1}^{n}x_{a_j}^{w_j}+\prod_{j=1}^{n}y_{a_j}^{w_j}},\frac{\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}}{\prod_{j=1}^{n}x_{b_j}^{w_j}+\prod_{j=1}^{n}y_{b_j}^{w_j}}\right],\left[\frac{2\prod_{j=1}^{n}t_{c_j}^{w_j}}{\prod_{j=1}^{n}z_{c_j}^{w_j}+\prod_{j=1}^{n}t_{c_j}^{w_j}},\frac{2\prod_{j=1}^{n}t_{d_j}^{w_j}}{\prod_{j=1}^{n}z_{d_j}^{w_j}+\prod_{j=1}^{n}t_{d_j}^{w_j}}\right]\right)^{1/\lambda}$$

$$=\left(\left[\frac{2\left(\frac{\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}}{\prod_{j=1}^{n}x_{a_j}^{w_j}+\prod_{j=1}^{n}y_{a_j}^{w_j}}\right)^{1/\lambda}}{\left(2-\frac{\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}}{\prod_{j=1}^{n}x_{a_j}^{w_j}+\prod_{j=1}^{n}y_{a_j}^{w_j}}\right)^{1/\lambda}+\left(\frac{\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}}{\prod_{j=1}^{n}x_{a_j}^{w_j}+\prod_{j=1}^{n}y_{a_j}^{w_j}}\right)^{1/\lambda}},\frac{2\left(\frac{\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}}{\prod_{j=1}^{n}x_{b_j}^{w_j}+\prod_{j=1}^{n}y_{b_j}^{w_j}}\right)^{1/\lambda}}{\left(2-\frac{\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}}{\prod_{j=1}^{n}x_{b_j}^{w_j}+\prod_{j=1}^{n}y_{b_j}^{w_j}}\right)^{1/\lambda}+\left(\frac{\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}}{\prod_{j=1}^{n}x_{b_j}^{w_j}+\prod_{j=1}^{n}y_{b_j}^{w_j}}\right)^{1/\lambda}}\right],\right.$$

$$\left.\left[\frac{\left(1+\frac{2\prod_{j=1}^{n}t_{c_j}^{w_j}}{\prod_{j=1}^{n}z_{c_j}^{w_j}+\prod_{j=1}^{n}t_{c_j}^{w_j}}\right)^{1/\lambda}-\left(1-\frac{2\prod_{j=1}^{n}t_{c_j}^{w_j}}{\prod_{j=1}^{n}z_{c_j}^{w_j}+\prod_{j=1}^{n}t_{c_j}^{w_j}}\right)^{1/\lambda}}{\left(1+\frac{2\prod_{j=1}^{n}t_{c_j}^{w_j}}{\prod_{j=1}^{n}z_{c_j}^{w_j}+\prod_{j=1}^{n}t_{c_j}^{w_j}}\right)^{1/\lambda}+\left(1-\frac{2\prod_{j=1}^{n}t_{c_j}^{w_j}}{\prod_{j=1}^{n}z_{c_j}^{w_j}+\prod_{j=1}^{n}t_{c_j}^{w_j}}\right)^{1/\lambda}},\frac{\left(1+\frac{2\prod_{j=1}^{n}t_{d_j}^{w_j}}{\prod_{j=1}^{n}z_{d_j}^{w_j}+\prod_{j=1}^{n}t_{d_j}^{w_j}}\right)^{1/\lambda}-\left(1-\frac{2\prod_{j=1}^{n}t_{d_j}^{w_j}}{\prod_{j=1}^{n}z_{d_j}^{w_j}+\prod_{j=1}^{n}t_{d_j}^{w_j}}\right)^{1/\lambda}}{\left(1+\frac{2\prod_{j=1}^{n}t_{d_j}^{w_j}}{\prod_{j=1}^{n}z_{d_j}^{w_j}+\prod_{j=1}^{n}t_{d_j}^{w_j}}\right)^{1/\lambda}+\left(1-\frac{2\prod_{j=1}^{n}t_{d_j}^{w_j}}{\prod_{j=1}^{n}z_{d_j}^{w_j}+\prod_{j=1}^{n}t_{d_j}^{w_j}}\right)^{1/\lambda}}\right]\right)$$

$$=\left(\left[\frac{2\left(\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}x_{a_j}^{w_j}+3\prod_{j=1}^{n}y_{a_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}x_{a_j}^{w_j}-\prod_{j=1}^{n}y_{a_j}^{w_j}\right)^{1/\lambda}},\frac{2\left(\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}x_{b_j}^{w_j}+3\prod_{j=1}^{n}y_{b_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}x_{b_j}^{w_j}-\prod_{j=1}^{n}y_{b_j}^{w_j}\right)^{1/\lambda}}\right],\right.$$

$$\left.\left[\frac{\left(\prod_{j=1}^{n}z_{c_j}^{w_j}+3\prod_{j=1}^{n}t_{c_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}z_{c_j}^{w_j}-\prod_{j=1}^{n}t_{c_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}z_{c_j}^{w_j}+3\prod_{j=1}^{n}t_{c_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}z_{c_j}^{w_j}-\prod_{j=1}^{n}t_{c_j}^{w_j}\right)^{1/\lambda}},\frac{\left(\prod_{j=1}^{n}z_{d_j}^{w_j}+3\prod_{j=1}^{n}t_{d_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}z_{d_j}^{w_j}-\prod_{j=1}^{n}t_{d_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}z_{d_j}^{w_j}+3\prod_{j=1}^{n}t_{d_j}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}z_{d_j}^{w_j}-\prod_{j=1}^{n}t_{d_j}^{w_j}\right)^{1/\lambda}}\right]\right).$$

Box II

2. If $\lambda \to 0$, then the IVIFGEWA operator (Eq. (32)) will be reduced to the Interval-Valued Intuitionistic Fuzzy Einstein Weighted Geometric (IVIFEWG) operator, which is shown as follows:

$$\text{IVIFEWG}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}_1^{w_1}\otimes_E\tilde{a}_2^{w_2}\otimes_E\cdots\otimes_E\tilde{a}_n^{w_n}.$$

According to Eq. (33), we can obtain:

$$\text{IVIFEWG}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\left[\frac{2\prod_{j=1}^{n}a_j^{w_j}}{\prod_{j=1}^{n}(2-a_j)^{w_j}+\prod_{j=1}^{n}a_j^{w_j}},\frac{2\prod_{j=1}^{n}b_j^{w_j}}{\prod_{j=1}^{n}(2-b_j)^{w_j}+\prod_{j=1}^{n}b_j^{w_j}}\right],\right.$$
$$\left.\left[\frac{\prod_{j=1}^{n}(1+c_j)^{w_j}-\prod_{j=1}^{n}(1-c_j)^{w_j}}{\prod_{j=1}^{n}(1+c_j)^{w_j}+\prod_{j=1}^{n}(1-c_j)^{w_j}},\frac{\prod_{j=1}^{n}(1+d_j)^{w_j}-\prod_{j=1}^{n}(1-d_j)^{w_j}}{\prod_{j=1}^{n}(1+d_j)^{w_j}+\prod_{j=1}^{n}(1-d_j)^{w_j}}\right]\right). \quad (36)$$

Definition 12. Let $\tilde{a}_j=([a_j,b_j],[c_j,d_j])$ $(j=$

$1, 2 \cdots, n$) be a collection of the IVIFNs, and IVIFGEOWA : $\Omega^n \to \Omega$, if:

$$\text{IVIFGEOWA}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \underset{j=1}{\overset{n}{\oplus_E}} \left(\omega_j \tilde{a}^{\lambda}_{\sigma(j)}\right)^{1/\lambda}, \tag{37}$$

where Ω is the set of all IVIFNs, and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with IVIFGEOWA, such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for any j. Then IVIFGEOWA is called the Interval-Valued Intuitionistic Fuzzy Generalized Einstein Ordered Weighted Averaging (IVIFGEOWA) operator.

Based on the Einstein operational rules of the IVIFNs, we can derive the result shown as Theorem 6.

Theorem 6. Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2 \cdots, n)$ be a collection of the IVIFNs, and $\lambda > 0$, then, the result aggregated from Definition 12 is still an IVIFN, and can be expressed in Eq. (38) as shown in Box III. The proof is similar to Theorem 2, and it is not repeated here.

The IVIFGEOWA operator can weigh the input data according to the data position in ranking from largest to smallest for all data. We can call ω the position weighted vector. In general, position weighted vector ω can be assigned according to the real decision-making needs. In some cases, it can be determined by some mathematical methods. Two examples are shown as follows:

1. The method proposed by Herrera et al. [41] is as follows:

$$\omega_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \qquad j = 1, 2, \cdots, n, \tag{39}$$

where quantitative fuzzy semantics operator, Q, can be given by the following formula:

$$Q(r) = \begin{cases} 0, & r < \alpha \\ \frac{r-\alpha}{\beta-\alpha}, & \alpha \leq r \leq \beta \\ 1, & r > \beta \end{cases} \qquad \alpha, \beta, r \in [0, 1]. \tag{40}$$

The parameters α and β in function Q are determined by the fuzzy linguistic quantitative rules according to Table 1.

2. The method proposed by Wang and Xu [42] is as follows:

$$\omega_{i+1} = \frac{C^i_{n-1}}{2^{n-1}} \qquad i = 0, 1, \cdots, n-1. \tag{41}$$

Table 1. Relation between the fuzzy linguistic quantitative rules and values of the parameters.

Quantitative criteria for fuzzy semantic	Most	At least half	As much as possible
Values of (α, β)	(0.3,0.8)	(0,0.5)	(0.5,1.0)

The IVIFGEOWA operator has the following properties:

1. **Theorem 7 (monotonicity)**
Let $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n)$ and $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ be two collections of IVIFNs. If $\tilde{a}'_j \leq \tilde{a}_j$ for all $j = 1, 2, \cdots, n$, then:

$$\text{IVIFGEOWA}(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n) \leq \text{IVIFGEOWA}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

2. **Theorem 8 (idempotency)**
Let $\tilde{a}_j = \tilde{a}$, $j = 1, 2, \cdots, n$, then IVIFGEOWA $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}$.

3. **Theorem 9 (bounded)**
The IVIFGEOWA operator lies between the max and min operators.

$$\min(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \text{IVIFGEOWA} \leq \max(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

4. **Theorem 10 (commutativity)**
Let $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n)$ and $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ be two collections of IVIFNs, and $(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, then:

$$\text{IVIFGEOWA } (\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n) = \text{IVIFGEOWA}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Similarly, some special cases of the IVIFGEOWA operator with respect to the parameter λ are discussed as follows:

1. If $\lambda = 1$, then the IVIFGEOWA operator (Eq. (37)) will be reduced to the Interval-Valued Intuitionistic Fuzzy Einstein Ordered Weighted Averaging (IVIFEOWA) operator, which is shown as follows:

$$\text{IVIFEOWA } (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \omega_1 \tilde{a}_{\sigma(1)} \oplus_E \omega_2 \tilde{a}_{\sigma(2)} \oplus_E \cdots \oplus_E \omega_n \tilde{a}_{\sigma(n)}.$$

According to Eq. (38), we can obtain:

$$\text{IVIFEOWA}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$\text{IVIFGEOWA}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\left[\frac{2\left(\prod_{j=1}^n x_{a_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n y_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n x_{a_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n y_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n x_{a_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n y_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}},\right.\right.$$
$$\left.\frac{2\left(\prod_{j=1}^n x_{b_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n y_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n x_{b_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n y_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n x_{b_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n y_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}\right],$$
$$\left[\frac{\left(\prod_{j=1}^n z_{c_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n t_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n z_{c_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n t_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n z_{c_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n t_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n z_{c_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n t_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}},\right.$$
$$\left.\left.\frac{\left(\prod_{j=1}^n z_{d_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n t_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n z_{d_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n t_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n z_{d_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^n t_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n z_{d_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^n t_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}\right]\right), \quad (38)$$

where:

$$x_{a_j}=(2-a_j)^\lambda+3a_j^\lambda,\quad y_{a_j}=(2-a_j)^\lambda-a_j^\lambda,\quad x_{b_j}=(2-b_j)^\lambda+3b_j^\lambda,\quad y_{b_j}=(2-b_j)^\lambda-b_j^\lambda,$$
$$z_{c_j}=(1+c_j)^\lambda+3(1-c_j)^\lambda,\quad t_{c_j}=(1+c_j)^\lambda-(1-c_j)^\lambda,\quad z_{d_j}=(1+d_j)^\lambda+3(1-d_j)^\lambda,$$
$$t_{d_j}=(1+d_j)^\lambda-(1-d_j)^\lambda.$$

Box III

$$=\left(\left[\frac{\prod_{j=1}^n(1+a_{\sigma(j)})^{\omega_j}-\prod_{j=1}^n(1-a_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^n(1+a_{\sigma(j)})^{\omega_j}+\prod_{j=1}^n(1-a_{\sigma(j)})^{\omega_j}},\frac{\prod_{j=1}^n(1+b_{\sigma(j)})^{\omega_j}-\prod_{j=1}^n(1-b_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^n(1+b_{\sigma(j)})^{\omega_j}+\prod_{j=1}^n(1-b_{\sigma(j)})^{\omega_j}}\right],\right.$$
$$\left.\left[\frac{2\prod_{j=1}^n c_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^n(2-c_{\sigma(j)})^{\omega_j}+\prod_{j=1}^n c_{\sigma(j)}^{\omega_j}},\frac{2\prod_{j=1}^n d_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^n(2-d_{\sigma(j)})^{\omega_j}+\prod_{j=1}^n d_{\sigma(j)}^{\omega_j}}\right]\right). \quad (42)$$

2. If $\lambda\to 0$, then the IVIFGEOWA operator (Eq. (37)) will be reduced to the Interval-Valued Intuitionistic Fuzzy Einstein Ordered Weighted Geometric (IV-IFEOWG) operator, which is shown as follows:

$$\text{IVIFEOWG}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}_{\sigma(1)}^{\omega_1}\otimes_E\tilde{a}_{\sigma(2)}^{\omega_2}\otimes_E\cdots\otimes_E\tilde{a}_{\sigma(n)}^{\omega_n}.$$

According to Eq. (38), we can obtain:

$$\text{IVIFEOWG}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$

$$= \left(\left[\frac{2\prod_{j=1}^{n} a_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(2-a_{\sigma(j)})^{\omega_j}+\prod_{j=1}^{n} a_{\sigma(j)}^{\omega_j}}, \frac{2\prod_{j=1}^{n} b_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(2-b_{\sigma(j)})^{\omega_j}+\prod_{j=1}^{n} b_{\sigma(j)}^{\omega_j}} \right], \left[\frac{\prod_{j=1}^{n}(1+c_{\sigma(j)})^{\omega_j}-\prod_{j=1}^{n}(1-c_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+c_{\sigma(j)})^{\omega_j}+\prod_{j=1}^{n}(1-c_{\sigma(j)})^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+d_{\sigma(j)})^{\omega_j}-\prod_{j=1}^{n}(1-d_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+d_{\sigma(j)})^{\omega_j}+\prod_{j=1}^{n}(1-d_{\sigma(j)})^{\omega_j}} \right] \right). \quad (43)$$

We have discussed two interval-valued intuitionistic fuzzy Einstein operators, i.e. the IVIFGEWA and IVIFGEOWA. They have their own characteristics; the IVIFGEWA operator weights only the IVIFN, and IVIFGEOWA operator weights only each IVIFN by their ordered positions. In the real decision-making, we need to consider these two weighted modes because they represent different aspects of decision making problems. However, each operator considers only one of them. In order to overcome these shortcomings, hybrid averaging operator based on Einstein operations is given as follows.

Definition 13. Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2\cdots, n)$ be a collection of the IVIFNs, and IVIFGEHWA : $\Omega^n \to \Omega$, if:

$$\text{IVIFGEHWA}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \mathop{\oplus_E}_{j=1}^{n} \left(\omega_j \tilde{s}_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \quad (44)$$

where Ω is the set of all IVIFNs, and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with IVIFGEHWA, such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $w = (w_1, w_2, \cdots, w_n)$ is the weight vector of $\tilde{a}_j (j = 1, 2, \cdots, n)$, and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. Let $\tilde{s}_j = n w_j \tilde{a}_j = \left([\dot{a}_j, \dot{b}_j], [\dot{c}_j, \dot{d}_j]\right)$, n is the adjustment factor. Suppose $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{s}_{\sigma(j-1)} \geq \tilde{s}_{\sigma(j)}$ for any j, and then function IVIFGEHWA is called the Interval-Valued Intuitionistic Fuzzy Generalized Einstein Hybrid Weighted Averaging (IVIFGEHWA) operator.

Based on the Einstein operational rules of the IVIFNs, we can derive the result shown as Theorem 11.

Theorem 11. Let $\tilde{a}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2\cdots, n)$ be a collection of the IVIFNs, and $\lambda > 0$, then, the result aggregated from Definition 13 is still an IVIFN, and even can be expressed in Eq. (45) as shown in Box IV. The proof is similar to Theorem 2, so it is not repeated here.

Theorem 12. The IVIFGEWA and IVIFGEOWA operators are the special cases of the IVIFGEHWA operator.

It is easy to prove that when:

$$W = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right),$$

the IVIFGEHWA operator will reduce to IVIFGEOWA operator, and when:

$$\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right),$$

the IVIFGEHWA operator will reduce to IVIFGEWA operator.

From Definition 13, we can know that the IVIFGEHWA operator firstly weights the given dada, and then reorders the weighted values in descending order and weights these ordered data. Therefore, the IVIFGEHWA operator can reflect the importance degrees of both the given data and their ordered positions.

4. Multiple attribute group decision making methods based on generalized Einstein aggregation operators

In this section, we will use these generalized Einstein aggregation operators to the Multiple Attribute Group Decision Making (MAGDM) problems where the attribute values are the interval-valued intuitionistic fuzzy information.

4.1. Description of the decision making problems

For the MAGDM problems, let $A = \{A_1, A_2, \cdots, A_m\}$ be the collection of alternatives, $C = \{C_1, C_2, \cdots, C_n\}$ be the collection of attributes, and $E = \{e_1, e_2, \cdots, e_q\}$ be the collection of decision-makers. Suppose that $\tilde{a}_{ij}^k = ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k])$ is an attribute value for the alternative A_i with respect to the attribute C_j given by the decision-maker e_k. $w = (w_1, w_2, \cdots, w_n)$ is the weight vector of attribute set $C = \{C_1, C_2, \cdots, C_n\}$, and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_q)$ is the weight vector of decision-makers $\{e_1, e_2, \cdots, e_q\}$, and $\lambda_k \in [0, 1]$, $\sum_{k=1}^{q} \lambda_k = 1$.

$$\text{IVIFGEHWA}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\left[\frac{2\left(\prod_{j=1}^{n}\dot{x}_{a_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{x}_{a_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{y}_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}\dot{x}_{a_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{a_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}},\right.\right.$$

$$\left.\frac{2\left(\prod_{j=1}^{n}\dot{x}_{b_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{x}_{b_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{y}_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}\dot{x}_{b_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{b_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}\right],$$

$$\left[\frac{\left(\prod_{j=1}^{n}\dot{z}_{c_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{t}_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}\dot{z}_{c_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{z}_{c_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{t}_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}\dot{z}_{c_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{c_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}},\right.$$

$$\left.\left.\frac{\left(\prod_{j=1}^{n}\dot{z}_{d_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{t}_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}\dot{z}_{d_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{z}_{d_{\sigma(j)}}^{\omega_j}+3\prod_{j=1}^{n}\dot{t}_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}\dot{z}_{d_{\sigma(j)}}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{d_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}\right]\right), \tag{45}$$

where:

$$\dot{x}_{a_j}=(2-\dot{a}_j)^{\lambda}+3\dot{a}_j^{\lambda},\qquad \dot{y}_{a_j}=(2-\dot{a}_j)^{\lambda}-\dot{a}_j^{\lambda},\qquad \dot{x}_{b_j}=\left(2-\dot{b}_j\right)^{\lambda}+3\dot{b}_j^{\lambda},$$

$$\dot{y}_{b_j}=\left(2-\dot{b}_j\right)^{\lambda}-\dot{b}_j^{\lambda},\qquad \dot{z}_{c_j}=(1+\dot{c}_j)^{\lambda}+3(1-\dot{c}_j)^{\lambda},\qquad \dot{t}_{c_j}=(1+\dot{c}_j)^{\lambda}-(1-\dot{c}_j)^{\lambda},$$

$$\dot{z}_{d_j}=\left(1+\dot{d}_j\right)^{\lambda}+3(1-\dot{d}_j)^{\lambda},\qquad \dot{t}_{d_j}=\left(1+\dot{d}_j\right)^{\lambda}-(1-\dot{d}_j)^{\lambda},\qquad \dot{a}_j=\frac{(1+a_j)^{nw_j}-(1-a_j)^{nw_j}}{(1+a_j)^{nw_j}+(1-a_j)^{nw_j}},$$

$$\dot{b}_j=\frac{(1+b_j)^{nw_j}-(1-b_j)^{nw_j}}{(1+b_j)^{nw_j}+(1-b_j)^{nw_j}},\qquad \dot{c}_j=\frac{2c_j^{nw_j}}{(2-c_j)^{nw_j}+c_j^{nw_j}},\qquad \dot{d}_j=\frac{2d_j^{nw_j}}{(2-d_j)^{nw_j}+d_j^{nw_j}}.$$

Box IV

4.2. The decision making method based on generalized Einstein hybrid weighted averaging operator

Step 1. Normalize the decision-making information. In general, there are two types for the attribute values: benefit attributes (I_1) (the bigger the attribute value is, the better it is), and cost attributes (I_2) (the bigger the attribute value is, the worse it is). In order to eliminate the impact of different types in the attribute values, it is necessary to normalize the decision-making information.

We may transform the attribute values from cost type to benefit type; in such a case, decision matrices $A^k=[\tilde{a}_{ij}^k]_{m\times n}$ ($k=1,2,\cdots,q$) can be transformed into matrices $R^k=[\tilde{r}_{ij}^k]_{m\times n}$ ($k=1,2,\cdots,q$), where $\tilde{r}_{ij}^k=([\underline{u}_{ij}^k,\bar{u}_{ij}^k],[\underline{f}_{ij}^k,\bar{f}_{ij}^k])$.

$$\tilde{r}_{ij}^k=\left([\underline{u}_{ij}^k,\bar{u}_{ij}^k],[\underline{f}_{ij}^k,\bar{f}_{ij}^k]\right)$$

$$= \begin{cases} ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k]) \\ \quad \text{for benefit attribute } C_j \\ \\ ([c_{ij}^k, d_{ij}^k], [a_{ij}^k, b_{ij}^k]) \\ \quad \text{for cost attribute } C_j \end{cases}$$

$$i = 1, 2, \cdots, m, \qquad j = 1, 2, \cdots, n. \qquad (46)$$

Step 2. Utilize the IVIFGEHWA operator:

$$\tilde{r}_{ij} = \left(\left[\underline{u}_{ij}, \bar{u}_{ij}\right], \left[\underline{f}_{ij}, \bar{f}_{ij}\right]\right)$$
$$= \text{IVIFGEHWA}\left(\tilde{r}_{ij}^1, \tilde{r}_{ij}^2, \cdots, \tilde{r}_{ij}^q\right), \quad (47)$$

to aggregate all the individual interval-valued intuitionistic fuzzy decision matrixes $R^k = [\tilde{r}_{ij}^k]_{m\times n}$ $(k = 1, 2, \cdots, q)$ into the collective decision matrix $R = [\tilde{r}_{ij}]_{m\times n}$.

Step 3. Utilize the IVIFGEHWA operator:

$$\tilde{r}_i = \left([\underline{u}_i^k, \bar{u}_i^k], [\underline{f}_i, \bar{f}_i]\right)$$
$$= \text{IVIFGEHWA}\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}\right), \quad (48)$$

to derive the collective overall preference values $\tilde{r}(i = 1, 2, \cdots, m)$.

Step 4. Calculate the score function $S(\tilde{r}_i)$ $(i = 1, 2, \cdots, m)$ of the collective overall values $\tilde{r}_i(i = 1, 2, \cdots, m)$, and then rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$. When two score functions $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$ are equal, it is necessary to calculate their accuracy functions $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$, then we can rank them by accuracy functions.

Step 5. Rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$ and select the best one(s) by score function $S(\tilde{r}_i)$ and accuracy function $H(\tilde{r}_i)$.

Step 6. End.

5. An application example

In order to demonstrate the application of the proposed method, we will cite an example about the air quality evaluation (adapted from [43]). To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games held during November 12-27, 2010, the air quality in Guangzhou for the Novembers of 2006, 2007, 2008, and 2009 were collected in order to find out the trends and to forecast the situation in November 2010. There are 3 air-quality monitoring stations (e_1, e_2, e_3) which can be seen as decision-makers, and their weight is $(0.314, 0.355, 0.331)^T$. There are 3 measured indices, namely SO_2 (C_1), NO_2 (C_2), and PM_{10} (C_3), and their weight is $(0.40, 0.20, 0.40)^T$. The measured values from air-quality monitoring stations under these indices are shown in Tables 2-4, and they can be expressed by interval-valued intuitionistic fuzzy numbers. Let (A_1, A_2, A_3, A_4) = {November of 2006, November of

Table 2. Air quality data from station e_1.

	C_1	C_2	C_3
A_1	([0.22, 0.31], [0.23, 0.54])	([0.13, 0.53], [0.20, 0.36])	([0.12, 0.37], [0.40, 0.56])
A_2	([0.28, 0.41], [0.33, 0.49])	([0.33, 0.53], [0.20, 0.36])	([0.12, 0.37], [0.30, 0.46])
A_3	([0.32, 0.41], [0.23, 0.44])	([0.43, 0.53], [0.16, 0.25])	([0.23, 0.45], [0.21, 0.37])
A_4	([0.39, 0.47], [0.18, 0.36])	([0.39, 0.53], [0.27, 0.32])	([0.28, 0.34], [0.11, 0.23])

Table 3. Air quality data from station e_2.

	C_1	C_2	C_3
A_1	([0.04, 0.21], [0.35, 0.46])	([0.10, 0.34], [0.27, 0.45])	([0.32, 0.37], [0.13, 0.20])
A_2	([0.32, 0.39], [0.27, 0.39])	([0.03, 0.57], [0.30, 0.36])	([0.16, 0.25], [0.14, 0.19])
A_3	([0.26, 0.37], [0.21, 0.40])	([0.23, 0.43], [0.06, 0.15])	([0.21, 0.35], [0.11, 0.29])
A_4	([0.30, 0.43], [0.19, 0.35])	([0.28, 0.43], [0.31, 0.34])	([0.39, 0.46], [0.01, 0.17])

Table 4. Air quality data from station e_3.

	C_1	C_2	C_3
A_1	([0.25, 0.27], [0.23, 0.40])	([0.17, 0.27], [0.26, 0.40])	([0.21, 0.30], [0.17, 0.32])
A_2	([0.25, 0.29], [0.33, 0.39])	([0.18, 0.46], [0.43, 0.50])	([0.06, 0.21], [0.28, 0.30])
A_3	([0.22, 0.27], [0.27, 0.31])	([0.13, 0.37], [0.16, 0.20])	([0.11, 0.24], [0.14, 0.19])
A_4	([0.30, 0.48], [0.09, 0.45])	([0.08, 0.53], [0.20, 0.24])	([0.32, 0.61], [0.01, 0.09])

2007, November of 2008, November of 2009} be the set of alternatives. The rank of air quality from 2006 to 2009 can be found as follows.

5.1. Rank the alternatives by the proposed method

To obtain the best alternative(s), the following steps are involved:

Step 1. Normalize the decision-making information. Since all the measured values are of the same type, they do not need normalization.

Step 2. Utilize the IVIFGEHWA operator expressed by Eq. (47) to aggregate all the individual interval-valued intuitionistic fuzzy decision matrices $R^k = [\tilde{r}_{ij}^k]_{4\times 3}$ $(k = 1, 2, 3)$ into the collective decision matrix $R = [\tilde{r}_{ij}]_{4\times 3}$. We can obtain (suppose $\lambda = 2$, $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$):

$$R = \begin{bmatrix} ([0.208, 0.275], [0.264, 0.462]) \\ ([0.141, 0.410], [0.241, 0.400]) \\ ([0.226, 0.350], [0.207, 0.327]) \\ \\ ([0.285, 0.371], [0.308, 0.420]) \\ ([0.241, 0.523], [0.294, 0.400]) \\ ([0.119, 0.295], [0.228, 0.297]) \\ \\ ([0.276, 0.360], [0.235, 0.378]) \\ ([0.311, 0.454], [0.116, 0.196]) \\ ([0.195, 0.366], [0.148, 0.273]) \\ \\ ([0.339, 0.463], [0.146, 0.383]) \\ ([0.295, 0.503], [0.255, 0.296]) \\ ([0.332, 0.490], [0.023, 0.153]) \end{bmatrix}.$$

Step 3. Utilize the IVIFGEHWA operator expressed by Eq. (48) to derive the collective overall preference values (suppose $\lambda = 2$, $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$), then we can obtain:

$$\tilde{r}_1 = ([0.243, 0.426], [0.225, 0.368]),$$

$$\tilde{r}_2 = ([0.313, 0.486], [0.261, 0.348]),$$

$$\tilde{r}_3 = ([0.357, 0.461], [0.153, 0.258]),$$

$$\tilde{r}_4 = ([0.375, 0.525], [0.092, 0.245]).$$

Step 4. Calculate the score function $S(\tilde{r}_i)(i = 1, 2, 3, 4)$ of the collective overall values $\tilde{r}_i(i = 1, 2, 3, 4)$, then we can obtain:

$$S(\tilde{r}_1) = 0.038, \quad S(\tilde{r}_2) = 0.095,$$

$$S(\tilde{r}_3) = 0.203, \quad S(\tilde{r}_4) = 0.282.$$

Step 5. According to the score function $S(\tilde{r}_i)(i = 1, 2, 3, 4)$, rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows:

$$A_4 \succ A_3 \succ A_2 \succ A_1.$$

Therefore, the best alternative is A_4, i.e. the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

5.2. Influence of the parameter λ on decision making result of this example

In order to illustrate the influence of the parameter λ on decision making of this example, we use the different values of λ in Steps 2 and 3 to rank the alternatives. The ranking results are shown in Table 5.

Table 5. Ordering the alternatives by utilizing different values of λ.

λ	Score function $S(\tilde{r}_i)$	Ranking
$\lambda = 0.01$	$S(\tilde{r}_1) = -0.037$ $S(\tilde{r}_2) = -0.007$ $S(\tilde{r}_3) = 0.121$ $S(\tilde{r}_4) = 0.240$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 0.5$	$S(\tilde{r}_1) = -0.036$ $S(\tilde{r}_2) = -0.006$ $S(\tilde{r}_3) = 0.120$ $S(\tilde{r}_4) = 0.238$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 1.0$	$S(\tilde{r}_1) = -0.018$ $S(\tilde{r}_2) = 0.020$ $S(\tilde{r}_3) = 0.137$ $S(\tilde{r}_4) = 0.246$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 1.5$	$S(\tilde{r}_1) = 0.008$ $S(\tilde{r}_2) = 0.056$ $S(\tilde{r}_3) = 0.168$ $S(\tilde{r}_4) = 0.262$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 2.0$	$S(\tilde{r}_1) = 0.038$ $S(\tilde{r}_2) = 0.095$ $S(\tilde{r}_3) = 0.203$ $S(\tilde{r}_4) = 0.282$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 5$	$S(\tilde{r}_1) = 0.174$ $S(\tilde{r}_2) = 0.250$ $S(\tilde{r}_3) = 0.361$ $S(\tilde{r}_4) = 0.390$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 30$	$S(\tilde{r}_1) = 0.157$ $S(\tilde{r}_2) = 0.159$ $S(\tilde{r}_3) = 0.250$ $S(\tilde{r}_4) = 0.255$	$A_4 \succ A_3 \succ A_2 \succ A_1$

As we can see from Table 5, the aggregation results using the different aggregation parameter λ are different, but the orderings of the alternatives are the same in this example. In general, we can regard parameter λ as the attitude of decision makers; the more the value of parameter λ is, the more optimistic attitude is. In real decision making, we can select the specific parameter λ according to real decision making problem. Of course, we can take the values of the parameter $\lambda = 1$ for arithmetic aggregation operator, or $\lambda \to 0$ for geometric aggregation operator.

5.3. Comparing the proposed method with the other methods

In order to verify the effectiveness of the proposed method, we can compare it with the method by Yue [43] because the data of this example came from it. Firstly, it is easy to see that there are similar ranking results for two methods. Secondly, the aggregation operators used by Yue [43] are based on algebraic operations, and those in this paper are based on the generalized Einstein operations. Since the generalized Einstein operations for IVIFNs are with parameter λ, the method proposed in this paper is more general and more flexible. In comparison with Einstein operators proposed by Wang and Liu [34,35], they are only the special cases of the proposed operators in this paper. When $\lambda = 0$ and the upper and lower limits of the membership and non-membership degrees in IVIFNs are equal, the generalized Einstein operations for IVIFNs proposed in this paper can be reduced to intuitionistic fuzzy geometric aggregation operators based on Einstein operations introduced by Wang and Liu [34]. Similarly, When $\lambda = 1$ and the upper and lower limits of the membership and non-membership degrees in IVIFNs are equal, the generalized Einstein operations for IVIFNs proposed in this paper can be reduced to intuitionistic fuzzy information aggregation using Einstein operations introduced by Wang and Liu [35], i.e. the operators proposed in this paper are the generalization of those proposed by Wang and Liu [34,35] by two aspects; one extension is from intuitionistic fuzzy numbers to interval-valued intuitionistic fuzzy numbers, the other extension is from the arithmetic aggregation operators or geometric aggregation operators based on Einstein operations to the generalized Einstein operations for IVIFNs. Obviously, the operators and methods proposed in this paper are more general. Of course, superficially, it is more complicated in calculation. However, in real applications, we first need to assign the specific parameter λ; then the calculation will be greatly simplified; for example, when parameter $\lambda = 1$ or $\lambda \to 0$, these operators will be simplified to arithmetic aggregation operator or geometric aggregation operator.

6. Conclusion

In this paper, we explored some generalized Einstein aggregation operators based on IVIFNs and applied them to the multi-attribute group decision making problems where attribute values are the IVIFNs. Firstly, Interval-Valued Intuitionistic Fuzzy Generalized Einstein Weighted Averaging (IVIFGEWA) operator, Interval-Valued Intuitionistic Fuzzy Generalized Einstein Ordered Weighted Averaging (IVIFGEOWA) operator, and Interval-Valued Intuitionistic Fuzzy Generalized Einstein Hybrid Weighted Averaging (IVIFGEHWA) operator were proposed. Some of their general properties such as idempotency, commutativity, monotonicity, and boundedness, were studied, and some special cases of them were analyzed. Furthermore, a method to multi-criteria group decision making based on these operators was developed, and the operational processes were illustrated in detail. Finally, an illustrative example was given to show the decision steps of the proposed method and to demonstrate their effectiveness. In further research, it is necessary and significant to give the applications of these operators to the other domains such as pattern recognition, fuzzy cluster analysis, uncertain programming, etc.

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