Scientia Iranica E (2015) 22(6), 2644–2663

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
www.scientiairanica.com

Hedging strategies for multi-period portfolio optimization

H. Davari-Ardakani, M. Aminnayeri* and A. Seifi

Department of Industrial Engineering and Management Systems, Amirkabir University of Technology, Tehran, P.O. Box 15875-4413, Iran.

Received 3 February 2014; received in revised form 30 August 2014; accepted 1 December 2014

KEYWORDS
Multi-period portfolio optimization;
European options;
Hedging strategies;
Greek letters;
Scenario generation.

Abstract. This paper develops a multi-period portfolio optimization model that utilizes hedging decisions in a dynamic setting. In this regard, a portfolio of options and underlying stocks is constructed and different time-varying Greek letters are utilized to mitigate the market risk. The presented model considers rebalancing decisions during the planning horizon. It assumes an investor is aiming to maximize his/her wealth at the end of the planning horizon, while controlling the investor's regret during the planning horizon. The uncertainty of asset prices is represented in terms of a scenario tree. In addition, a scenario generation method is presented that characterizes the temporal correlations and dependence structure of asset returns. Also, it preserves marginal distributions of asset returns. To investigate the effect of hedging strategies, we first implement the scenario generation method on a set of stocks selected from the New York Stock Exchange (NYSE). Numerical results show the high performance of the scenario generation method. Then, the multi-period portfolio optimization model is implemented via the generated scenario tree. Results show that incorporation of options remarkably reduces investor risk. Finally, different hedging strategies are assessed by imposing bounds on the values of Greek letters and a discussion about numerical results is presented.

© 2015 Sharif University of Technology. All rights reserved.

1. Introduction

The traditional single period portfolio selection problem often fails to be efficient in long-term investment problems. This arises from the fact that in real world applications, transaction costs are not zero; returns are temporally correlated and the investor can borrow money to invest [1]. Hence, multi-period portfolio selection models were developed. These models act in response to an evolving information structure over time. Option contracts are financial instruments that are increasingly used by investors and speculators. Due to their asymmetric and nonlinear payoffs, they can protect investor wealth in response to highly negative variations in financial markets. Also, some traders utilize them as speculative instruments [2].

Merton et al. [3] investigated different investment strategies involving options over a significant period of time. Harrison and Pliska [4] and Follmer and Sondermann [5] were pioneering researchers contributing to the development of general hedging methods. Afterward, a remarkable number of studies included derivatives, especially options, in their portfolio, to hedge exposure to currency and market risks. The main studies in this area are [6-19].

Topaloglou et al. [2] extended the previous work and optimized an international portfolio of stock and bond indices, as well as currency forward contracts, in a multi-asset structure and dynamic setting. Topaloglou et al. [20] used options and currency forward contracts

*. *Corresponding author. Tel.: +98 21 66413034; Fax: +98 21 66954569
E-mail address: mjnayeri@aut.ac.ir (M. Aminnayeri)*

to hedge exposures to market and currency risks in a static manner.

Yin and Han [21] generalized the work of Topaloglou et al [20] through adaptation of their method in a dynamic setting.

There are alternative hedging approaches, referred to as Greek letters, which measure different dimensions to the risk in an option position. The trader can manage the Greek letters (Greeks), while ensuring that all aspects of risk are acceptable. These hedging strategies make the investor more flexible in hedging exposure to different aspects of risk.

Wu and Sen [22] proposed a stochastic programming approach to develop currency option hedging strategies to address multiple random factors in an imperfect market. Their study incorporated constraints on sensitivity measures, such as Delta and Gamma. Papahristodoulou [23] formulated a linear programming model and devised hedging strategies on all Greek letters.

Horasanli [24] extended Papahristodoulou [23] to a multi-asset structure to deal with a portfolio of options and underlying assets. Gao [25] proposed a linear programming model with some flexible risk bounds on all Greek letters. These bounds can be adjusted by the investor to suit the needs of market change.

In a recent study, Yin and Han [26] presented a stochastic programming model for international portfolio management. Moreover, they used stock options with overall control on time-varying Greek letters to achieve an efficient hedging level. Compared to previous studies that used Greeks for hedging exposures to different aspects of risk, their improvement was twofold. First, they constructed an international portfolio with a multi-period and multi-currency structure. Second, they extended previous studies on optimal options strategies into a dynamic and nondeterministic framework.

Although Yin and Han [26] incorporated hedging decisions in the multi-period portfolio selection problem, there exist some important issues that should be necessarily addressed. Their study assumed that options have a one-period time to maturity. Thus, when the investor takes a long position in a call or put option, at the next period, he/she has only the opportunity to exercise the option. Nevertheless, in real world financial markets, derivatives, such as options, are issued with different maturities that are often more than one time period. Thus, they can be traded during their lifetime. Since the time to maturity of an option affects its value, in each time of rebalancing, the pricing process for all options should be performed. This makes the problem more complex. These important features of derivatives were ignored in previous studies.

In addition, options may impose two types of transaction cost to investors that are charged when any option is traded. These types of transaction cost are fixed and proportional. While the former does not depend on the price of traded options, the latter does. To consider both fixed and proportional transaction costs is a matter of particular importance and strongly affects the obtained results. This strong effect will be discussed later. The fixed transaction cost was ignored in [26].

Moreover, although Yin and Han [26] used Greeks in a dynamic and nondeterministic setting, they considered that the volatility of each asset returns to be a constant parameter in all scenarios. However, volatility should be considered a stochastic parameter that is dependent on scenarios.

In this paper, all these important issues are addressed in a multi-period portfolio selection setting. In addition, different from [2,20,21,26] that dealt with international portfolios, a portfolio with domestic diversification is considered. This approach seems to be more appropriate for risk taking investors, since it enables them to invest in a wider variety of assets, rather than a limited set of indices. In addition, those risks associated with currency exchange rates are not imposed on investor wealth. Of course, the investor can assign a part of his/her capital to less risky assets that have a relatively guaranteed payoff.

In addition, the most recent work in this area [2,20,21,26] utilized moment matching [27] to generate scenarios of asset returns. This may have two main drawbacks. First, the temporal correlations of data series, e.g. asset returns, are ignored. Second, the dependence structure of different data series is modelled via a covariance matrix that illustrates the linear dependence of the data series. This assumption holds when the time series data are normally distributed. However, as these studies mentioned, in most cases, the distributions of financial parameters, e.g. asset returns, are not normal. In this paper, these important issues of scenario generation are addressed.

Briefly speaking, this paper considers the dynamic hedging of options in a multi-period portfolio selection problem. In fact, we incorporate options with different maturities into a portfolio that is regularly rebalanced during the planning horizon. Moreover, in addition to transaction costs pertaining to trading financial assets, those pertaining to trading, as well as exercising options, are carefully considered. Furthermore, the temporal correlations and dependence structure of return series are modelled in a careful manner.

The remainder of this paper is organized as follows. Section 2 presents a stochastic programming model for multi-period portfolio selection with options and bounds on Greek letters. Section 3 discusses the presented scenario generation method. In Section 4, a practical application of the proposed framework, the presented model, as well as the scenario generation method, on the New York Stock Exchange (NYSE) is provided, and computational results are discussed. Finally, Section 5 concludes the paper.

2. Multi-period portfolio model

To incorporate hedging decisions in the multi-period portfolio selection problem, we use a stochastic programming paradigm. In addition to determining the optimal decisions for rebalancing financial assets, this model can make optimal hedging decisions against future uncertainties to provide overall risk management. Greek letters play an important role in hedging exposure to different aspects of risk. Each Greek letter measures the sensitivity of an option to a specific variable.

2.1. Model assumptions

We consider an investor that has an initial wealth and wants to maximize his/her wealth at the end of a specified period through investment in financial markets. Moreover, he/she is going to control the bankruptcy risk during the planning horizon. In fact, this problem has a dynamic structure that involves portfolio rebalancing decisions at periodic intervals in response to new information on market conditions. We assume that borrowing and short selling are not allowed during the lifetime of the investment. Also, the investor wants to exploit European style options to mitigate market risk. We assume that options are issued with different maturities that are often more than a one time period. Hence, they can be traded during their lifetime. Note that when the clock advances one period, the time to maturity of options decreases one period. Therefore, the process of option pricing should be performed at the beginning of each period.

Since transaction costs have a key role in rebalancing decisions, proportional transaction costs are charged for trading assets and taking long positions in call and put options. Also, fixed transaction costs are charged for taking long positions in call and put options. The uncertainty of asset returns is represented in terms of a scenario tree.

In Section 3, we elaborate on the proposed scenario generation method and related concepts.

2.2. Greek letters for call and put options

Generally, Greek letters measure the sensitivity of option prices to different parameters. Different hedging strategies can be designed by controlling different aspects of risk with Greek letters. For a detailed discussion about Greek letters, one can refer to Hull [28]. We adapt the definitions to the problem under consideration.

For a European call option, the five basic Greek letters are defined as follows:

$$\delta^s_{it\tau k} = N(d^s_{1,it\tau k}), \qquad \gamma^s_{it\tau k} = \frac{N'(d^s_{1,it\tau k})}{P^s_{it}\sigma^s_i\sqrt{\tau}},$$

$$\theta^s_{it\tau k} = -\frac{P^s_{it}\sigma^s_i N'(d^s_{1,it\tau k})}{2\sqrt{\tau}} - rke^{-r\tau}N(d^s_{2,it\tau k}),$$

$$\rho^s_{it\tau k} = k\tau e^{-r\tau}N(d^s_{2,it\tau k}),$$

$$\kappa^s_{it\tau k} = P^s_{it}\sqrt{\tau}N'(d^s_{1,it\tau k}).$$

Also, they can be defined for a European put option as follows:

$$\delta'^s_{it\tau k} = N(d^s_{1,it\tau k}) - 1, \qquad \gamma'^s_{it\tau k} = \frac{N'(d^s_{1,it\tau k})}{P^s_{it}\sigma^s_i\sqrt{\tau}},$$

$$\theta'^s_{it\tau k} = -\frac{P^s_{it}\sigma^s_i N'(d^s_{1,it\tau k})}{2\sqrt{\tau}} + rke^{-r\tau}N(-d^s_{2,it\tau k}),$$

$$\rho'^s_{it\tau k} = -k\tau e^{-rT}N(d^s_{2,it\tau k}),$$

$$\kappa'^s_{it\tau k} = P^s_{it}\sqrt{\tau}N'(d^s_{1,it\tau k}),$$

where τ, P^s_{it}, k, r and σ^s_i denote time to maturity, price of asset i at time t under scenario s, exercise price of option, interest rate, and volatility of asset i under scenario s, respectively. Also:

$$N(x) = \int_{-\infty}^{x} \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{2}} dz,$$

denotes the cumulative probability distribution function for a standard normal variable:

$$d_{1,it\tau k} = \frac{\ln\left(\frac{P^s_{it}}{k}\right) + \left(r + \frac{(\sigma^s_i)^2}{2}\right)\tau}{\sigma^s_i\sqrt{\tau}},$$

and:

$$d_{2,it\tau k} = d_{1,it\tau k} - \sigma^s_i\sqrt{\tau}.$$

Moreover, $\delta^s_{it\tau k}$, $\gamma^s_{it\tau k}$, $\theta^s_{it\tau k}$, $\rho^s_{it\tau k}$ and $\kappa^s_{it\tau k}$ denote Delta, Gamma, Theta, Rho and Vega of a call option on underlying asset i at time t, with time to maturity τ and exercise price, k, respectively. Also, $\delta'^s_{it\tau k}$, $\gamma'^s_{it\tau k}$, $\theta'^s_{it\tau k}$, $\rho'^s_{it\tau k}$ and $\kappa'^s_{it\tau k}$ denote Delta, Gamma, Theta, Rho and Vega of a put option on underlying asset i at time t, with time to maturity τ, and exercise price, k, respectively. We can derive the values of all Greek letters for the portfolio, as described later in the mathematical programming formulation. As mentioned above, in this study, we do not assume volatility to be constant for all scenarios. In other words, for each asset in any scenario, a different volatility, according to the asset prices in that scenario, is considered. We replace σ with σ^s_i, which denotes the volatility of asset i under scenario s.

2.3. Notations

Let I and S be the set of available assets and scenarios, respectively. We use the following notations:

Deterministic input data:

W_0	Investor initial wealth;
η	Proportional transaction cost for sales or purchases of asset $i \in I$;
η'	Proportional transaction cost for taking long positions in options issued on asset $i \in I$;
η''	Fixed transaction cost for taking long positions in options issued on asset $i \in I$;
ψ_{i0}	Initial market price of asset $i \in I$;
μ_t	Investor level of risk aversion at time t $(t = 1, \cdots, T)$;
λ_t	Investor target wealth at time t $(t = 1, \cdots, T)$;
r	Risk free interest rate;
$\pi_{i0\tau k}$	Price of call option on asset $i \in I$ at $t = 0$, with time maturity τ and exercise price $k \in K_i$;
$\pi'_{i0\tau k}$	Price of put option on asset $i \in I$ at $t = 0$, with time maturity τ and exercise price $k \in K_i$;
Δ_u	Upper risk bound for Delta of the portfolio;
Δ_l	Lower risk bound for Delta of the portfolio;
Γ_u	Upper risk bound for Gamma of the portfolio;
Γ_l	Lower risk bound for Gamma of the portfolio;
Θ_u	Upper risk bound for Theta of the portfolio;
Θ_l	Lower risk bound for Theta of the portfolio;
P_u	Upper risk bound for Rho of the portfolio;
P_l	Lower risk bound for Rho of the portfolio;
K_u	Upper risk bound for Kappa of the portfolio;
K_l	Lower risk bound for Kappa of the portfolio.

Scenario dependent parameters:

ψ_{it}^s	Market price of asset $i \in I$ at period t $(t = 1, \cdots, T)$ under scenario s;
$\pi_{i,t,\tau,k}^s$	Price of call option on asset $i \in I$, with time maturity τ and exercise price $k \in K_i$ at time period t $(t = 1, \cdots, T-1$ and $\tau = t+1, \cdots, T)$ under scenario $s \in S$;
${\pi'}_{i,t,\tau,k}^s$	Price of put option on asset $i \in I$, with time maturity τ and exercise price $k \in K_i$ at time period t $(t = 1, \cdots, T-1$ and $\tau = t+1, \cdots, T)$ under scenario $s \in S$.

Decision variables:

x_{it}^s	Amounts of asset $i \in I$ purchased at t $(t = 0, 1, \cdots, T-1)$ under scenario $s \in S$;
x_{ft}^s	Amounts of investment in the risk free asset at t $(t = 0, 1, \cdots, T-1)$ under scenario $s \in S$;
y_{it}^s	Amounts of asset $i \in I$ sold at t $(t = 1, \cdots, T-1)$ under scenario $s \in S$;
h_{it}^s	Amounts of asset $i \in I$ held at t $(t = 0, 1, \cdots, T-1)$ under scenario $s \in S$;
$v_{it\tau k}^s$	Amounts of call option on asset $i \in I$, with time maturity τ and exercise price $k \in K_i$ held at time period t $(t = 0, 1, \cdots, T-1$ and $\tau > t)$ under scenario $s \in S$;
$w_{it\tau k}^s$	Amounts of put option on asset $i \in I$, with time maturity τ and exercise price $k \in K_i$ held at time period t $(t = 0, 1, \cdots, T$ and $\tau > t)$ under scenario $s \in S$;
$u_{it\tau k}^s$	Number of long positions in call options on asset $i \in I$ taken at period t $(t = 0, 1, \cdots, T-1)$, with time maturity τ $(\tau > t)$ and exercise price $k \in K_i$ under scenario $s \in S$;
$c_{it\tau k}^s$	Number of long positions in put options on asset $i \in I$ taken at period t $(t = 0, 1, \cdots, T-1)$, with time maturity $\tau(\tau > t)$ and exercise price $k \in K_i$ under scenario $s \in S$;
m_{itk}^s	Number of long positions in call options on asset $i \in I$, with exercise price $k \in K_i$ exercised at time period t $(t = 1, \cdots, T)$ under scenario $s \in S$;
q_{itk}^s	Number of long positions in put options on asset $i \in I$, with exercise price $k \in K_i$ exercised at time period t $(t = 1, \cdots, T)$ under scenario $s \in S$;
V_t^s	Investor wealth at time t $(t = 1, \cdots, T)$ under scenario $s \in S$;

Δ_t^s Delta of portfolio at period t ($t = 1, \cdots, T-1$) under scenario $s \in S$;

Γ_t^s Gamma of portfolio at period t ($t = 1, \cdots, T-1$) under scenario $s \in S$;

Θ_t^s Theta of portfolio at period t ($t = 1, \cdots, T-1$) under scenario $s \in S$;

P_t^s Rho of portfolio at period t ($t = 1, \cdots, T-1$) under scenario $s \in S$;

K_t^s Kappa of portfolio at period t ($t = 1, \cdots, T-1$) under scenario $s \in S$.

Auxiliary variables:

C_t Auxiliary variable to make the objective function linear ($t = 1, \cdots, T$).

2.4. Computing values of Greek letters for the portfolio

Eqs. (1) to (5) can be used to compute the values of Greek letters:

$$\Delta_t^s = \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} v_{it\tau k}^s \delta_{it\tau k}^s + \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} w_{it\tau k}^s {\delta'}_{it\tau k}^s, \quad s \in S, \quad t = 1, \cdots, T-1, \tag{1}$$

$$\Gamma_t^s = \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} v_{it\tau k}^s \gamma_{it\tau k}^s + \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} w_{it\tau k}^s {\gamma'}_{it\tau k}^s, \quad s \in S, \quad t = 1, \cdots, T-1, \tag{2}$$

$$\Theta_t^s = \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} v_{it\tau k}^s \theta_{it\tau k}^s + \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} w_{it\tau k}^s {\theta'}_{it\tau k}^s, \quad s \in S, \quad t = 1, \cdots, T-1, \tag{3}$$

$$P_t^s = \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} v_{it\tau k}^s \rho_{it\tau k}^s + \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} w_{it\tau k}^s {\rho'}_{it\tau k}^s, \quad s \in S, \quad t = 1, \cdots, T-1, \tag{4}$$

$$K_t^s = \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} v_{it\tau k}^s \kappa_{it\tau k}^s + \sum_{i=1}^{n} \sum_{\tau=t+1}^{T} \sum_{k \in K_i} w_{it\tau k}^s {\kappa'}_{it\tau k}^s, \quad s \in S, \quad t = 1, \cdots, T-1. \tag{5}$$

2.5. Model formulation

In this paper, we are going to maximize investor expected wealth at the end of the planning horizon, while controlling the risk of investor bankruptcy during the planning horizon. Zhu et al. [29] emphasized the necessity of controlling investor bankruptcy risk in intermediate periods of a long-term investment, since it reduces the possibility that due to fluctuations in investor wealth, he/she becomes bankrupt. To control the risk of investor bankruptcy, we use the expected regret of investor wealth. A comparative discussion between expected regret and Conditional Value-at-Risk (CVaR) [30,31], a coherent and most popular risk measure, was made by Testuri and Uryasev [32]. They prove that an optimal portfolio in the CVaR sense is also optimal in the expected regret sense for a given target, and vice versa. In addition, expected regret can be well adapted to dynamic stochastic programming models. Ji et al. [33] minimized the expected regret of investor wealth during the planning horizon. However, the main objective of the investor, i.e. maximizing terminal wealth, was neglected. In this paper, we use a combination of expected final wealth and expected regret of wealth, not only during the end-of-horizon period, but also over the entire planning horizon. The objective function of the proposed model is as follows:

$$\max \left(\frac{\sum\limits_{s=1}^{S} p^s V_T^s}{\prod\limits_{p=1}^{T} (1+r_p)} - \sum_{s=1}^{S} \sum_{t=1}^{T} \mu_t \frac{p^s \max\{\lambda_t - V_t^s, 0\}}{\prod\limits_{p=1}^{t} (1+r_p)} \right). \tag{6}$$

In fact, the objective function makes a tradeoff between the present value of the expected terminal wealth (first part) discounted by the risk free interest rate, and the present value of investor expected regret, with a target level of λ_t (second part) during the planning horizon. The objective function is in line with the definition of robust optimization introduced by Mulvey et al. [34]. The "*max*" operator in the second part of the objective function will be linearized with an auxiliary variable, C_t.

The set of constraints for the proposed model are as follows:

$$x_{f0}^s + \sum_{i=1}^{n} x_{i0}^s \psi_{i0}(1+\eta) + \sum_{i=1}^{n}\sum_{\tau=1}^{T}\sum_{k\in K_i} u_{i0\tau k}^s \pi_{i0\tau k}(1+\eta') + \sum_{i=1}^{n}\sum_{\tau=1}^{T}\sum_{k\in K_i} c_{i0\tau k}^s \pi'_{i0\tau k}(1+\eta') = W_0$$

$$s=1,\cdots,S, \tag{7}$$

$$h_{i,t-1}^s + x_{it}^s - y_{it}^s + \sum_{k\in K_i} m_{itk}^s - \sum_{k\in K_i} q_{itk}^s = h_{it}^s$$

$$i=1,\cdots,n, \qquad t=1,\cdots,T, \qquad s=1,\cdots,S, \tag{8}$$

$$v_{it\tau k}^s = v_{i,t-1,\tau,k}^s + u_{it\tau k}^s \qquad i=1,\cdots,n,$$

$$s=1,\cdots,S, \quad t=0,1,\cdots,T-1, \quad k\in K_i, \tag{9}$$

$$w_{it\tau k}^s = w_{i,t-1,\tau,k}^s + c_{it\tau k}^s \qquad i=1,\cdots,n,$$

$$s=1,\cdots,S, \quad t=0,1,\cdots,T-1, \quad k\in K_i, \tag{10}$$

$$m_{itk}^s \le v_{i,t-1,\tau,k}^s \qquad i=1,\cdots,n, \quad s=1,\cdots,S,$$

$$t=1,\cdots,T, \qquad \tau=t, \qquad k\in K_i, \tag{11}$$

$$q_{itk}^s \le w_{i,t-1,\tau,k}^s \qquad i=1,\cdots,n, \quad s=1,\cdots,S,$$

$$t=1,\cdots,T, \qquad \tau=t, \qquad k\in K_i, \tag{12}$$

$$x_{f,t-1}^s(1+r_t-\omega) + \sum_{i=1}^{n}\sum_{k\in K_i} q_{itk}^s k + \sum_{i=1}^{n} \psi_{it}^s y_{it}^s (1-\eta_i) = \sum_{i=1}^{n} \psi_{it}^s x_{it}^s (1+\eta) + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} u_{it\tau k}^s \pi_{it\tau k}^s (1+\eta') + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} u_{it\tau k}^s \eta'' + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} c_{it\tau k}^s {\pi'}_{it\tau k}^s (1+\eta') + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} c_{it\tau k}^s \eta'' + \sum_{i=1}^{n}\sum_{k\in K_i} m_{itk}^s k + x_{ft}^s,$$

$$s=1,\cdots,S, \qquad t=1,\cdots,T, \tag{13}$$

$$V_t^s = x_{ft}^s + \sum_{i=1}^{n} h_{it}^s \psi_{it}^s + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} \frac{v_{it\tau k}^s \max(\psi_{i\tau}^s - k, 0)}{\prod_{p=t+1}^{\tau}(1+r_p)} + \sum_{i=1}^{n}\sum_{\tau=t+1}^{T}\sum_{k\in K_i} \frac{w_{it\tau k}^s \max(k-\psi_{i\tau}^s, 0)}{\prod_{p=t+1}^{\tau}(1+r_p)}$$

$$s=1,\cdots,S, \qquad t=0,1,\cdots,T, \tag{14}$$

$$\Delta^l \le \Delta_t^s \le \Delta^u, \qquad \Gamma^l \le \Gamma_t^s \le \Gamma^u,$$

$$\Theta^l \le \Theta_t^s \le \Theta^u, \qquad P^l \le P_t^s \le P^u,$$

$$K^l \le K_t^s \le K^u,$$

$$s=1,\cdots,S, \qquad t=1,\cdots,T-1, \tag{15}$$

$$x_{it}^s = x_{it}^{s'} \qquad \forall s,s' \quad \text{for which} \quad B_t^s = B_t^{s'}$$

$$i=1,\cdots,n, \qquad t=0,\cdots,T-1,$$

$$s,s'=1,\cdots,S, \tag{16}$$

$$y_{it}^s = y_{it}^{s'} \qquad \forall s,s' \quad \text{for which} \quad B_t^s = B_t^{s'}$$

$$i=1,\cdots,n, \qquad t=0,\cdots,T-1,$$

$$s,s'=1,\cdots,S, \tag{17}$$

$$x_{ft}^s = x_{ft}^{s'} \qquad \forall s,s' \quad \text{for which} \quad B_t^s = B_t^{s'}$$

$$t=0,\cdots,T-1, \qquad s,s'=1,\cdots,S, \tag{18}$$

$$u_{it\tau k}^s = u_{it\tau k}^{s'} \qquad \forall s,s' \quad \text{for which} \quad B_t^s = B_t^{s'}$$

$$i=1,\cdots,n, \qquad t=0,\cdots,T-1, \qquad \tau>t,$$

$$s,s'=1,\cdots,S, \qquad k\in K_i, \tag{19}$$

$$c_{it\tau k}^{s}=c_{it\tau k}^{s'}\qquad \forall s,s' \quad \text{for which} \quad B_t^s=B_t^{s'}$$
$$i=1,\cdots,n,\qquad t=0,\cdots,T-1,\qquad \tau>t,$$
$$s,s'=1,\cdots,S,\qquad k\in K_i, \tag{20}$$

$$v_{it\tau k}^{s}=v_{it\tau k}^{s'}\qquad \forall s,s' \quad \text{for which} \quad B_t^s=B_t^{s'}$$
$$i=1,\cdots,n,\qquad t=0,\cdots,T-1,\qquad \tau>t,$$
$$s,s'=1,\cdots,S,\qquad k\in K_i, \tag{21}$$

$$w_{it\tau k}^{s}=w_{it\tau k}^{s'}\qquad \forall s,s' \quad \text{for which} \quad B_t^s=B_t^{s'}$$
$$i=1,\cdots,n,\qquad t=0,\cdots,T-1,\qquad \tau>t,$$
$$s,s'=1,\cdots,S,\qquad k\in K_i, \tag{22}$$

$$m_{itk}^{s}=m_{itk}^{s'}\qquad \forall s,s' \quad \text{for which} \quad B_t^s=B_t^{s'}$$
$$i=1,\cdots,n,\qquad t=1,\cdots,T,$$
$$s,s'=1,\cdots,S,\qquad k\in K_i, \tag{23}$$

$$q_{itk}^{s}=q_{itk}^{s'}\qquad \forall s,s' \quad \text{for which} \quad B_t^s=B_t^{s'}$$
$$i=1,\cdots,n,\qquad t=1,\cdots,T,$$
$$s,s'=1,\cdots,S,\qquad k\in K_i. \tag{24}$$

First, the investor's initial wealth is utilized to construct an initial portfolio. The construction of the initial portfolio is performed regarding Eq. (7). We assume that the initial wealth can be invested in risk free and risky assets, as well as call and put options on risky assets.

Also, we have an inventory equation for all assets. Eq. (8) ensures the balance for quantities of all risky assets during the planning horizon. The amount of exercised call and put options at the current period are also considered in Eq. (8).

Since we assume that the lifetime of each option can be more than one period, information about amounts of call and put options held at all periods is required. Eqs. (9) and (10) calculate these holding amounts for different call and put options, respectively. The investor is able to take long positions on different call and put options. In other words, different options can be purchased during their lifetime. This assumption, considered in Eqs. (9) and (10), is in conformance with real conditions in financial markets.

Eqs. (11) and (12) ensure that at each time period, the amounts of call and put options exercised do not exceed the holding amounts of these options in the previous time period.

In the cash balance Eq. (13), the principal and profits of risk free investment and the funds provided by exercising put options on hand and selling risky assets are used to cover the expenditure for the purchase of risky assets, taking long positions on call and put options, exercising call options on hand, and investment in risk free assets. Linear transaction costs are charged for purchases and sales of risky assets, as well as call and put options. In addition, fixed transaction costs are charged for purchases of call and put options.

Eq. (14) computes the portfolio value at the end of time period t $(t=1,\cdots,T)$ under scenario $s\in S$. The portfolio value represents the value of the risk free asset, the market value of all risky asset holdings, as well as the value of call and put option holdings.

Eq. (15) sets lower and upper bounds on values of all Greek letters of the portfolio at time period t $(t=1,\cdots,T)$ under scenario $s\in S$. The risk taking investor can relax bounds on Greek letters and, if the bounds approach infinity, the associated constraints will be redundant. Contrarily, if the bounds approach zero, the investor will be risk neutral in terms of the associated Greek letter.

Nonanticipativity constraints are important parts of the stochastic programming models that should be necessarily addressed. If two scenarios, s and s', are indistinguishable from the beginning to time t, on the basis of information available at time t, then the decision rendered for the two scenarios must be identical from the beginning to time t. Let B_t^s denotes a decision part of scenario $s\in S$ from the beginning to time t. Eqs. (16) to (24) represent constraints that are used to satisfy the nonanticipative condition.

In addition, all decision variables are considered to be nonnegative. The constraints associated with this assumption, as well as those pertaining to the amount of holdings in the initial investment, are not mentioned for reasons of brevity.

2.6. Pricing options on the scenario tree

A key part of incorporating option decisions into multi-period portfolio optimization problems is the appropriate valuation of call and put options. We use the Black-Scholes model for pricing European call and put options. The option pricing is done according to the following formulas:

$$\pi_{it\tau k}^{s}=\psi_{it}^{s}N(a_1)-ke^{-r(\tau-t)}N(a_2), \tag{25}$$

$${\pi'}_{it\tau k}^{s}=ke^{-r(\tau-t)}N(-a_2)-\psi_{it}^{s}N(-a_2), \tag{26}$$

$$a_1=\frac{\ln(\psi_{it}^{s}/k)+(r+(\sigma_i^s)^2/2)(\tau-t)}{\sigma_i^n\sqrt{\tau-t}}, \tag{27}$$

$$a_2=\frac{\ln(\psi_{it}^{s}/k)+(r-(\sigma_i^s)^2/2)(\tau-t)}{\sigma_{it}^{s}\sqrt{\tau-t}}=a_1-\sigma_i^n\sqrt{\tau-t}, \tag{28}$$

where, $\pi^s_{it\tau k}$ and $\pi'^s_{it\tau k}$ denote the prices of call and put options issued on asset i ($i = 1, \cdots, n$), with time to maturity $\tau(\tau > t)$, and strike price $k \in K_i$ at time period t ($t = 0, 1, \cdots, T-1$), under scenario s ($s = 1, \cdots, S$), respectively. In addition, $N(.)$ denotes the cumulative distribution function of the standard normal distribution.

3. Scenario generation

Scenario generation is a key step in dealing with the uncertainty of parameters in stochastic programming models. A good scenario set provides the investor with realistic future information about the market, with which he/she can make better investment and hedging decisions. Although we do not intend to make a detailed discussion about scenario generation techniques, some explanation regarding previous studies, at least in the area of portfolio optimization, seem to be unavoidable.

One popular scenario generation method used in financial applications is moment matching, which minimizes a distance measure between specified statistical moments and statistical moments of generated scenarios. This method was proposed by Høyland and Wallace [35] and used widely in different studies in the area of finance [27,36,37]. Also, some studies used the moment matching in combination with other methods. For instance, we can mention vector autoregression (VAR) [33], K-means clustering [38] and scenario reduction [39]. Regardless of its advantages, moment matching has the important drawback of utilizing a covariance matrix for modelling the dependence structure of returns. The covariance matrix is a useful tool for modelling the dependence structure of normally distributed data, since it only measures the linear dependence of the data series. Thus, when the data series are not normally distributed, there is not any guarantee of appropriately capturing the dependence structure of the data series with the moment matching method. This issue is more important for financial data series, since experience shows that financial data have often fatter tails than normal distribution. This feature is referred to as leptokurtic behavior.

Regardless of the important property of the financial data series, some studies either explicitly assume that they are normally distributed [33] or, without any discussion about normality, use the special properties of normal distributions for the data set under consideration [40].

Temporal correlation is another important feature of the data series that should be modelled properly. This task is usually performed with time series models. Numbers of research work on applications of time series in scenario generation can be found in the literature, e.g. the VAR method [33,41] and the hidden Markov model [42]. Generally, in financial time series, a large change tends to be followed by large changes, and vice versa. This characteristic is referred to as volatility clustering or heteroskedasticity [43]. Thus, classic time series models often fail to model temporal correlations of financial time series. To deal with this specific feature of financial time series data, the autoregressive conditional heteroskedastic (ARCH) model of volatility was presented by Engle [44] and further developed to the Generalized ARCH (GARCH) model by Bollerslev [45]. A detailed discussion about conditional heteroskedastic models and their extensions can be found in Tsay [46]. A limited number of studies use heteroskedastic time series models. Chen and Yuen [47], Chen et al. [48] and Chen [49] used the GARCH-type process and conditional sampling to generate scenarios of risky asset returns. Yet, most financial scenario generation procedures either do not consider the temporal correlations of data series [27,39,50] or use traditional homoskedastic time series models to capture the temporal correlations of data series [33,41]. In addition, there are some studies that use reduction methods to provide an accurate and computationally tractable set of scenarios. Beraldi and Bruni [51] presented a reduction method based on cluster analysis to obtain a set of scenarios in an efficacious, computationally efficient manner.

In this study, a scenario generation method is adopted that:

1. Uses Johnson transformation [52] to make asset returns normally distributed. This enables the decision maker to use especial properties of normal distributions, e.g. a linear dependence structure of normal distributions;
2. Utilizes ARMA/GARCH type models to properly model temporal correlations of asset returns;
3. Utilizes Cholesky decomposition to generate a set of scenarios, such that the dependence structure of historical returns is preserved in the set of scenarios;
4. Preserves marginal distributions of historical data in the set of scenarios.

The steps of the proposed scenario generation method are as follows:

1. Transform all series of asset prices to asset returns to provide a stable data set. Eq. (29) transforms a price series into a return series:

$$R_{it} = \log\left(\frac{P_{i,t+1}}{P_{it}}\right), \tag{29}$$

where P_{it} denotes the price of asset i in period t, and R_{it} denotes the return of risky asset i in period t.

2. Use Johnson transformation to transform marginal distributions of random variables, which are often nonnormal, to standard normal distribution. This helps to simply model the dependence structures of different return series.

   Johnson transformation has the main advantage of enabling one to transform nonnormal data into normal ones without identification of marginal distributions of the data series. This makes the proposed methodology more accurate, since it eliminates the errors associated with fitting marginal distributions into the univariate data series.

3. Fit ARMA/GARCH type models to the return series. In this paper, ARMA/GARCH, ARMA /EGARCH [53] and ARMA/GJR GARCH [54] models are fitted to the return series and the best fitted model is selected based on (AIC) and Bayesian Information Criteria (BIC) [55]. Here, Johnson transformation helps to fit time series models with Gaussian innovations.

4. Determine the variance-covariance matrix of innovations generated via fitting ARMA/GARCH type models in step 3. This helps one to generate scenarios that preserve the dependence structure of the historical data series.

   The variance-covariance matrix is a symmetric, square matrix composed of a set of blocks. Each block demonstrates the dependence structure between two innovation series, based on a definite number of lags. In fact, the size of each block is $(N_L + N_T)$, where N_L denotes the number of positive and negative time lags whose cross correlation is considered to be significant. N_T denotes the number of periods for which the scenario generation process is performed. The following matrix demonstrates the general structure of a variance-covariance matrix for five innovation series:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} & \mathbf{\Lambda}_{13} & \mathbf{\Lambda}_{14} & \mathbf{\Lambda}_{15} \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} & \mathbf{\Lambda}_{23} & \mathbf{\Lambda}_{24} & \mathbf{\Lambda}_{25} \\ \mathbf{\Lambda}_{31} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{33} & \mathbf{\Lambda}_{34} & \mathbf{\Lambda}_{35} \\ \mathbf{\Lambda}_{41} & \mathbf{\Lambda}_{42} & \mathbf{\Lambda}_{43} & \mathbf{\Lambda}_{44} & \mathbf{\Lambda}_{45} \\ \mathbf{\Lambda}_{51} & \mathbf{\Lambda}_{52} & \mathbf{\Lambda}_{53} & \mathbf{\Lambda}_{54} & \mathbf{\Lambda}_{55} \end{bmatrix},$$

   where, each $\mathbf{\Lambda}_{ij}$, $i,j \in \{1,2,3,4,5\}$, is a submatrix, with the size $N_L + N_T$. Each entry of $\mathbf{\Lambda}_{ij}$ represents the covariance of the ith and jth innovation series, based on a time lag equal to $k-l$, where k and l represent the row and column indices in $\mathbf{\Lambda}_{ij}$. When the absolute value of $k-l$ is greater than the maximum defined time lag of significant correlation, the associated entry of $\mathbf{\Lambda}_{ij}$ will be zero. For instance, suppose that $\mathbf{\Lambda}_{ij}$ be an 8×8 matrix, and, also, let the maximum time lag of significant correlation be 3. Matrix $\mathbf{\Lambda}_{ij}$ will be as follows:

$$\mathbf{\Lambda}_{ij} = \begin{bmatrix} C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} & C_{ij}^{3} & 0 & 0 & 0 & 0 \\ C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} & C_{ij}^{3} & 0 & 0 & 0 \\ C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} & C_{ij}^{3} & 0 & 0 \\ C_{ij}^{-3} & C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} & C_{ij}^{3} & 0 \\ 0 & C_{ij}^{-3} & C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} & C_{ij}^{3} \\ 0 & 0 & C_{ij}^{-3} & C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} & C_{ij}^{2} \\ 0 & 0 & 0 & C_{ij}^{-3} & C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} & C_{ij}^{1} \\ 0 & 0 & 0 & 0 & C_{ij}^{-3} & C_{ij}^{-2} & C_{ij}^{-1} & C_{ij}^{0} \end{bmatrix},$$

where C_{ij}^{m} denotes the time lag-m covariance of the ith and jth innovation series:

$$m \in \{-3,-2,-1,0,1,2,3\}.$$

Here, we wish to describe how variance-covariance matrix $\mathbf{\Lambda}$ can be used to generate the cross correlated innovation series.

Suppose that there is a transformation matrix, $\mathbf{M}$, that makes a vector of independent standard normal innovations, ς, cross correlated, as Eq. (30):

$$\varepsilon = M\varsigma, \tag{30}$$

where:

$$\varsigma_{nk\times 1} = \begin{bmatrix} \underline{\varsigma_1^{tr}} & \underline{\varsigma_2^{tr}} & \cdots & \underline{\varsigma_n^{tr}} \end{bmatrix}^{tr},$$

denotes the vector of independent standard normal innovations (white noise), and:

$$\varepsilon_{nk\times 1} = \begin{bmatrix} \underline{\varepsilon_1^{tr}} & \underline{\varepsilon_2^{tr}} & \cdots & \underline{\varepsilon_n^{tr}} \end{bmatrix}^{tr},$$

denotes the vector of cross correlated innovations with the symmetric variance-covariance matrix, $\mathbf{\Lambda}$.

Taking variance from both sides of Eq. (30), we get:

$$\mathbf{\Lambda} = \mathbf{M}\mathrm{var}(\varsigma)\mathbf{M}^{tr}, \tag{31}$$

where, var(.) denotes the variance of input data.

We know that ς is the vector of independent standard normal innovations. Hence, var(ς) is equal to the identity matrix $\mathbf{I}$. Therefore, Eq. (31) becomes $\mathbf{\Lambda} = \mathbf{MIM}^{tr}$, or:

$$\mathbf{\Lambda} = \mathbf{MM}^{tr}. \tag{32}$$

The variance-covariance matrix, $\mathbf{\Lambda}$, is positive-definite. Thus, it can be decomposed by Cholesky decomposition, such that:

$$\mathbf{\Lambda} = \mathbf{LL}^{tr}. \tag{33}$$

A comparison of Eqs. (32) and (33) leads to $\mathbf{LL}^{tr} = \mathbf{MM}^{tr}$ and, consequently, $\mathbf{L} = \mathbf{M}$.□

5. Use Cholesky decomposition on the variance-covariance matrix $\mathbf{\Lambda}$ to obtain the lower triangular matrix, $\mathbf{M}$.
6. Construct the vector of innovations, ς, of which the number of coordinates is the same as the size of the variance-covariance matrix. The first N_L coordinates of the vector are historical innovations, obtained from fitting the appropriate ARMA/GARCH type model to the first series of returns. The next N_T coordinates of the vector are standard normal innovations that have been generated independently. The next N_L coordinates of the vector are historical innovations, obtained from fitting the appropriate ARMA/GARCH type model to the second series of returns. Again, the next N_T coordinates of the vector are standard normal innovations that have been generated independently. This procedure is repeated for other return series until the vector of innovations is completed.
7. Use the lower triangular matrix, $\mathbf{M}$, obtained in step 5, as well as the vector of innovations, ς, constructed in step 6, to generate cross correlated innovations via Eq. (30).
8. Use cross correlated innovations, obtained in step 7, and the best fitted ARMA/GARCH type models, obtained in step 3, to simulate returns of assets.
9. Use the inverse of the Johnson transformations, used in step 2, to provide the scenario set regarding the original marginal distributions of asset returns.
10. Repeat steps 6-9 until the preferred number of scenarios is generated.
11. Use scenario reduction to convert the scenario fan to a scenario tree that reasonably approximates the original set of scenarios. In this paper, this task is performed using the GAMS/ScenRed tool, which is a prominent, widely used scenario reduction tool. For a more detailed discussion about the algorithm of reducing a scenario fan to a scenario tree, one can refer to Gröwe-Kuska et al. [56].

Figure 1 shows a schematic representation of the presented scenario tree generation method.

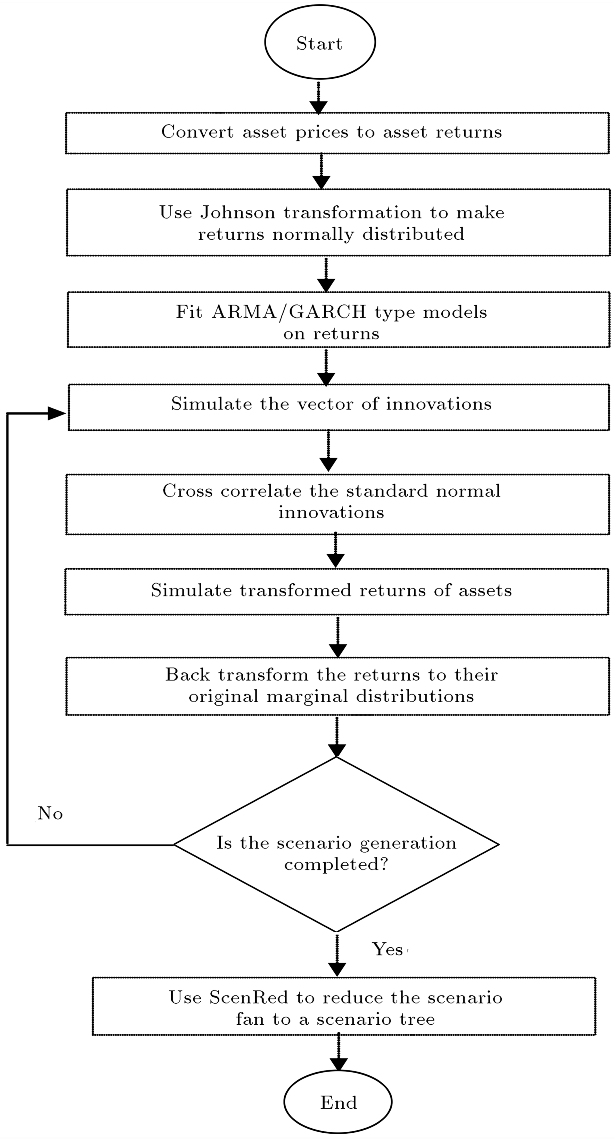

Figure 1. The schematic representation of the proposed scenario tree generation method.

4. Computational results

In this section, the proposed model is implemented with five stocks selected from the New York Stock Exchange (NYSE) to assess the performance of different hedging strategies. First of all, the scenarios of asset returns are generated. Then, the scenario set is utilized to solve the proposed scenario based stochastic programming model and assess the performance of different hedging strategies.

4.1. Generating scenarios of asset prices

We use the prices of five stocks of different sectors, AT&T, Inc. (T), The Boeing Company (BA), Bank of America Corporation (BAC), Caterpillar Inc. (CAT) and Citigroup, Inc. (C), from May 1, 1995 to May 1, 2013. All data are provided from finance.yahoo.com, and MATLAB 7.9 and Minitab 16 are used on a computer with Intel C2D 2 GHz CPU and 2 GB RAM to generate scenarios of asset returns.

To generate scenarios of asset returns, Eq. (29) is used to convert asset prices to asset returns. Then, the Johnson transformation is utilized to transform marginal distributions of asset returns to the standard normal distribution. Eq. (34) shows the unbounded system of the Johnson transformation used in this

Table 1. Johnson transformations used to make return series normally distributed.

Stock	Johnson transformation	p-value (Anderson-Darling normality test)
T	$0.0138785 + 1.18890 \times \text{arcsinh}\ ((R - 0.000246482)/0.00637721)$	0.391
BA	$0.0289165 + 1.38872 \times \text{arcsinh}\ ((R - 0.000369799)/0.00925504)$	0.961
BAC	$0.0327179 + 0.911502 \times \text{arcsinh}\ ((R - 0.000369799)/0.00581721)$	0.936
CAT	$-0.0254359 + 1.42996 \times \text{arcsinh}\ ((R + 0.000003624)/0.0104243)$	0.968
C	$0.00164829 + 0.984505 \times \text{arcsinh}\ ((R - 0.0000541525)/0.00696959)$	0.305

study:

$$z = \gamma + \delta \text{arcsinh}\left(\frac{x-\theta}{\eta}\right), \tag{34}$$

where z denotes the transformed value, γ and δ denote shape parameters, and θ and η denote location and scale parameters. The parameters of these transformations are optimized by Minitab 16 software. Table 1 shows the optimum parameters of the Johnson transformation used to make all return series normally distributed.

The p-values of Anderson-Darling normality tests confirm the appropriate performance of Johnson transformations.

Afterward, ARMA/GARCH, ARMA/EGARCH and ARMA/GJR GARCH models are used to model the dynamic behavior of the historical return series. Then, the best fitted model is selected, based on two penalized model selection criteria, the Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC) [57]. Table 2 shows the best fitted time series models, as well as their corresponding AIC and BIC values.

Next, the residuals generated from fitting ARMA/GARCH type models that are cross correlated, but not auto correlated, are used to calculate the variance-covariance matrix, $\mathbf{\Lambda}$. Here, the maximum number of time lags for significant correlation, N_L, and the number of periods for scenario generation, N_T, are considered to be 10 and 50 periods, respectively. Hence, the variance-covariance matrix of innovations $\mathbf{\Lambda}$ is a 300×300 matrix. After constructing the variance-covariance matrix, $\mathbf{\Lambda}$, Cholesky decomposition is used to provide the lower triangular matrix, $\mathbf{M}$. Then, regarding step 6 of the scenario generation method, the vector of innovations, ς, is generated and pre-multiplied by matrix $\mathbf{M}$ to make the innovations cross correlated.

Afterward, cross correlated innovations and formerly fitted ARMA/GARCH type models, mentioned in Table 2, are used to simulate transformed returns of stocks. Finally, simulated returns are transformed to the return series with marginal distributions of historical data. This step is performed using the inverse of Johnson transformations, displayed in Table 1.

We repeat the above procedure providing 1000 scenarios of asset returns. Figures 1 and 2 compare cross correlations of historical and simulated returns for two random couples of stocks (T and C, BA and BAC). Figures 2 and 3 show the identical dependence structure of historical and simulated return series in the case of these two couples of stocks. Thus, the great performance of the scenario generation method in preserving the dependence structure of the historical return series is confirmed. This important issue matters for other couples of stocks, whose figures are ignored for conciseness.

Table 2. The preferred time series models for modeling dynamic behavior of stock returns.

Stock	Selected ARMA/GARCH type model	Parameters (p, q, r, s)	AIC	BIC
T	ARMA/GJR-GARCH (Gaussian)	(1,1,1,1)	12175.73	12220.66
BA	ARMA/GJR-GARCH (Gaussian)	(0,0,1,1)	12406.05	12438.14
BAC	ARMA/GJR-GARCH (Gaussian)	(0,0,1,1)	11559.91	11592.01
CAT	ARMA/GJR-GARCH (Gaussian)	(0,0,1,1)	12499.19	12531.28
C	ARMA/GJR-GARCH (Gaussian)	(0,1,1,1)	11833.06	11871.57

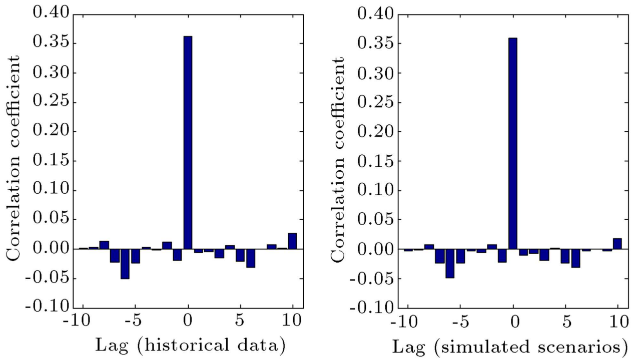

Figure 2. Cross correlations of (historical vs. simulated) returns (T and C).

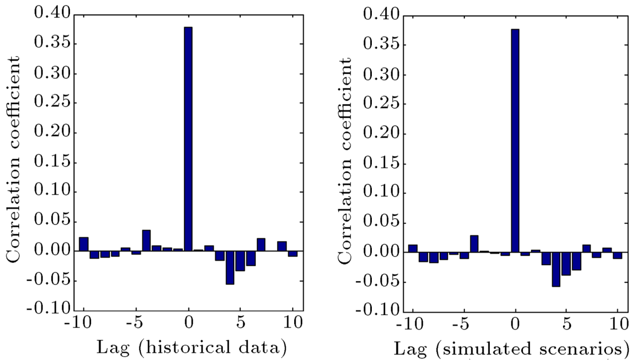

Figure 3. Cross correlations of (historical vs. simulated) returns (BA and BAC).

Figures 4, 5 and 6 compare the estimates of the Probability Density Function (PDF) for historical and simulated returns of all stocks.

Although errors of transformation may affect the results of the scenario generation method, Figures 4, 5 and 6 show the great performance of the method in terms of preserving marginal distributions of the return series.

After generating the large-sized scenario fan, the GAMS/ScenRed tool is applied to reduce the set of scenarios. This task may have two important advantages. First, it can reduce the size of the scenario set and speed up the calculation process, with a reasonably good approximation of the original set. Second, it helps to convert the scenario fan to a scenario tree, which can be used to solve the presented multistage stochastic programming model.

4.2. Model implementation

After implementing the scenario tree generation method, the scenario tree is utilized to implement the multistage portfolio optimization model with stocks and options, and assess different hedging strategies. This task is performed by a GAMS 22.2, CPLEX solver.

The investor is assumed to have $W_0 = 100$ units of money and wants to invest during the multi-period planning horizon. Also, the target wealth in time t is set as $\lambda_t = (1 + 0.01t)W_0 = 100 + t$. Other parameter values are set as $\eta = 0.005$, $\eta' = 0.01$, $\eta'' = 0.1$, $r = 0.01$ and $\mu_t = 1$ in the computational tests.

4.2.1. *The assessment of stochastic modelling of the problem*

In this part, we utilize a prominent measure to assess the performance of the stochastic model compared to its deterministic counterpart. This measure is referred to as Value of Stochastic Solution (VSS). To calculate this measure, we consider a single-stage problem, in which, the scenario tree acts like a scenario fan. Consider the following stochastic program:

$$\max c^T x + E_\xi Q(x,\xi),$$

$$Ax = b,$$

$$x \geq 0,$$

where, ξ is a random variable whose realizations correspond to various scenarios.

Here, all random variables are replaced by their expected values. This is called the Expected Value problem (EV) and simply defined as follows:

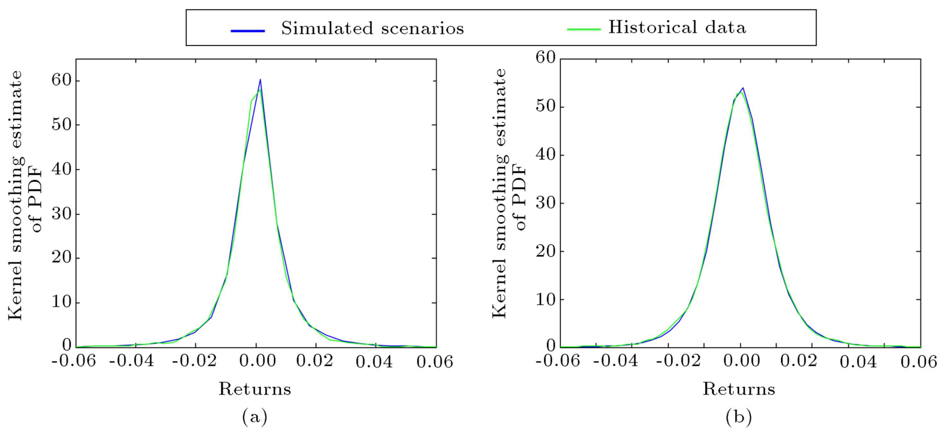

Figure 4. Historical vs. simulated returns for BAC (a) and CAT (b).

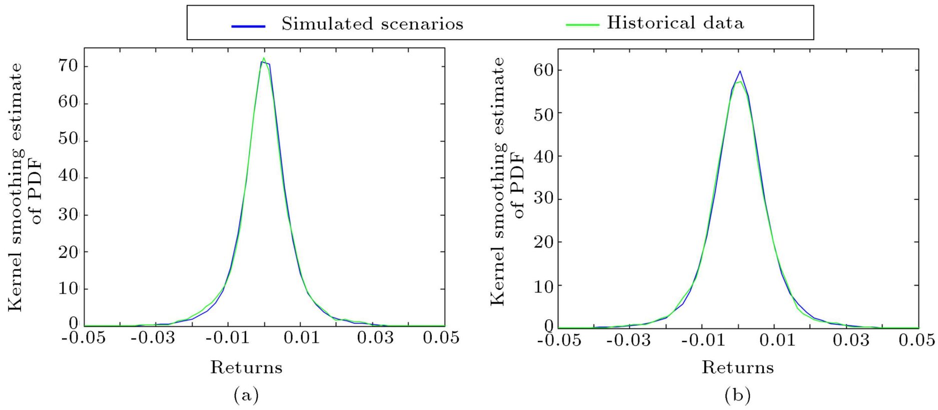

Figure 5. Historical vs. simulated returns for T (a) and BA (b).

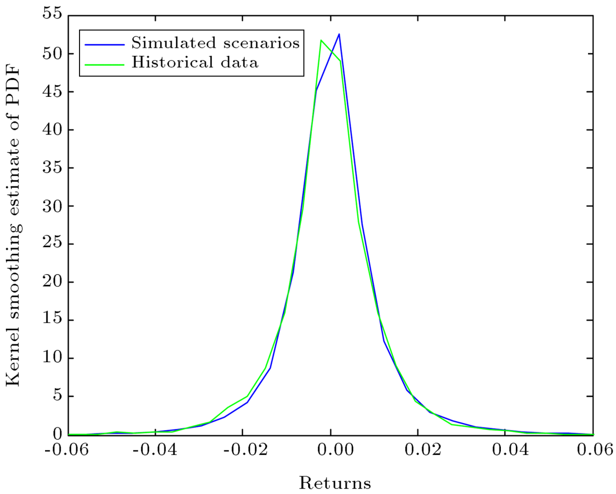

Figure 6. Historical vs. simulated returns for C.

$$\text{EV} = \max_x z(x, \bar{\zeta}), \tag{35}$$

where, $\bar{\zeta} = E(\boldsymbol{\zeta})$ denotes the expectation of $\boldsymbol{\zeta}$. Let us denote an optimal solution to Eq. (35) by $\bar{x}(\bar{\zeta})$. The Value of the Stochastic Solution (VSS) measures how good a decision $\bar{x}(\bar{\zeta})$ is in terms of Eq. (35). The expected result of using the EV Solution (EEV) is defined as follows:

$$\text{EEV} = E_{\boldsymbol{\zeta}}\left(z\left(\bar{x}\left(\bar{\zeta}\right), \boldsymbol{\zeta}\right)\right). \tag{36}$$

Also, we define the so-called here and now solution as Eq. (37):

$$\text{RP} = \max_x E_{\xi} z(x, \xi). \tag{37}$$

The value of the stochastic solution is defined as Eq. (38):

$$\text{VSS} = \text{RP} - \text{EEV}. \tag{38}$$

We considered different problems with 2, 3, 4, and 5 stocks, and generated single-stage scenario trees with 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 scenarios. The parameters of the model were set as $\eta = 0.005$, $\eta' = 0.01$, $\eta'' = 0.01$, and $r = 0.01$. Table 3 presents EEV, RP, and VSS values of these problems for an investor with 100 USD initial cash holding.

The VSS results show that ignoring uncertainty in the multistage portfolio optimization problem imposes a remarkable cost to the investor. Also, they show that when the size of the scenario set is changed, VSS does not change a lot. Hence, VSS is stable with respect to the size of the scenario set.

4.2.2. In-sample stability

Di Domenica et al. [58] provided a comprehensive study regarding the issue of evaluation in scenario based stochastic programming models. From now on, we utilize a number of tests to assess the stability and performance of the proposed scenario generation method. The initial cash endowment is considered to be 10000 USD.

Figure 6 shows the in-sample stability [59] of the scenario generation method. In this regard, different scenario fans with three time periods are generated and reduced to scenario trees with the same structures. The procedure is utilized to generate 10 different scenario trees. Then, for each set, the stochastic programming model is solved and the optimum objective value is computed. As Figure 7 illustrates, optimum objective values are rather close and do not change a lot. Thus, the in-sample stability of the scenario generation method is confirmed.

In addition to the values of optimal objectives, the stability of decisions should be assessed for all scenario trees that have been generated. To assess the stability of decisions in these scenario trees, the decisions about asset holdings are recorded for each node of each scenario tree. Then, for the last stage of each scenario tree, the average of these holding decisions is calculated. Table 4 provides a comparison between the statistical characteristics of these recorded decisions for all these scenario trees.

Table 3. Value of Stochastic Solution (VSS) for different sample problems.

Number of	2 stocks			3 stocks			4 stocks			5 stocks		
scenarios	EEV	RP	VSS	EEV	RP	VSS	EEV	RP	VSS	EEV	RP	VSS
100	103.66	106.43	2.78	104.81	107.46	2.66	107.29	110.30	3.01	108.38	111.65	3.27
200	103.38	106.31	2.93	104.50	107.45	2.94	107.09	110.23	3.14	108.61	112.00	3.39
300	103.15	106.21	3.06	104.33	107.38	3.05	106.81	110.07	3.26	108.28	111.78	3.50
400	103.24	106.25	3.01	104.40	107.45	3.05	107.01	110.55	3.53	108.49	111.98	3.49
500	103.16	106.21	3.05	104.37	107.42	3.05	106.95	110.20	3.25	108.42	111.90	3.47
600	103.29	106.30	3.01	104.51	107.51	3.01	107.07	110.28	3.21	108.53	111.96	3.43
700	103.24	106.28	3.04	104.48	107.48	3.01	107.01	110.22	3.21	108.47	111.90	3.43
800	103.33	106.37	3.04	104.58	107.60	3.02	107.14	110.38	3.24	108.60	112.05	3.45
900	103.35	106.36	3.01	104.61	107.60	2.98	107.23	110.43	3.20	108.71	112.13	3.41
1000	103.42	106.41	2.99	104.69	107.66	2.96	107.33	110.52	3.19	108.82	112.22	3.41

Table 4. Statistical characteristics (mean and standard deviation) of decisions on amounts of stock holdings for different scenario trees.

Scenario tree		1	2	3	4	5	6	7	8	9	10
T	Mean	55.10	58.82	57.98	54.62	61.34	59.54	55.46	47.82	52.94	59.91
	Standard deviation	4.03	2.37	3.91	3.24	4.22	3.48	2.77	4.25	3.19	3.98
BA	Mean	24.73	21.01	19.81	26.05	21.01	22.06	23.53	38.15	24.17	22.40
	Standard deviation	2.36	2.01	1.97	2.65	2.32	2.01	3.23	3.55	2.87	2.25
BAC	Mean	103.36	105.04	108.76	100.84	103.36	100.96	107.56	80.42	103.36	99.51
	Standard deviation	7.93	6.43	6.35	7.01	6.87	6.15	7.23	5.96	6.34	5.76
CAT	Mean	26.05	29.41	33.61	26.89	30.25	31.37	26.79	15.78	30.25	32.44
	Standard deviation	2.05	1.98	2.33	3.13	2.94	2.87	1.94	2.09	3.06	2.97
C	Mean	37.82	36.97	42.02	43.70	40.34	36.88	43.61	59.06	42.81	40.25
	Standard deviation	3.42	3.25	4.78	4.53	4.21	3.87	4.43	4.98	4.43	3.76

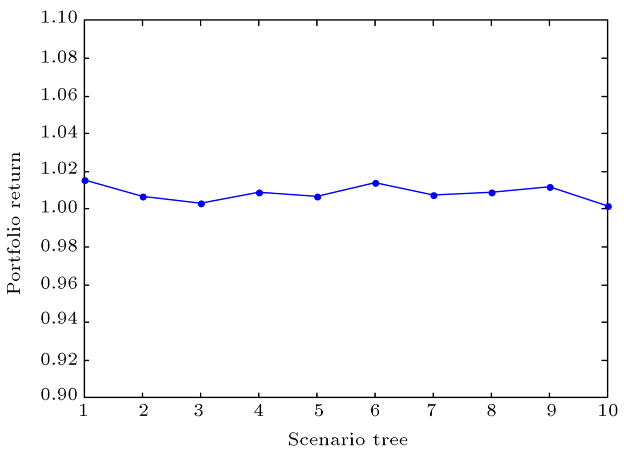

Figure 7. Different objective values obtained by different scenario trees.

Table 4 shows that in all scenario trees, except no. 8, the decisions of holding stocks change infinitesimally. Thus, it is concluded that the proposed scenario tree generation method not only possesses the stability of optimal objectives, but also explains that of optimal decisions.

4.2.3. The role of options in risk reduction

To assess the role of options in the portfolio of underlying assets, the trade-off between investor wealth and expected regret for both hedged and unhedged portfolios is analysed. As mentioned before, we set the target wealth of the investor as $\lambda_t = 100 + t$. Also, the initial cash holding is considered to be 100 USD again.

4.2.3.1. Comparing efficient frontiers

In order to compare hedged and unhedged portfolios, we change the investor's target wealth level through changing the coefficient of the target. For different values of this coefficient, the multistage stochastic programming models are solved, and the investor's wealth and regret for both hedged and unhedged portfolios are recorded. Using the provided results, the efficient frontiers of both hedged and unhedged portfolios are obtained. Figure 8 compares this trade-off, in terms of efficient frontiers, in both hedged and unhedged portfolios.

Such a comparison is referred to as a static one. This arises from the fact that the analyses are performed with one scenario tree in a special time of

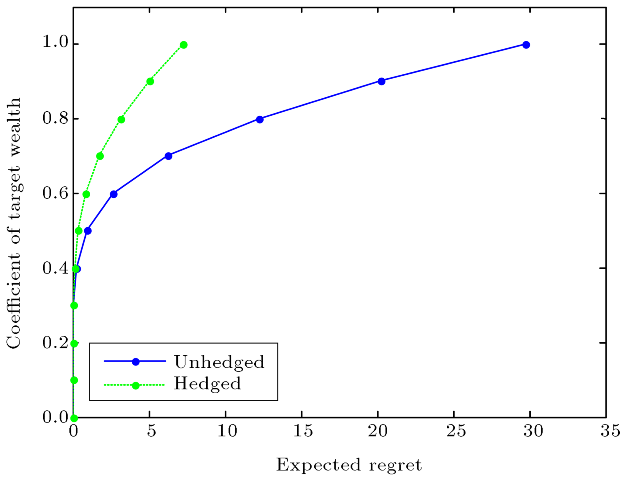

Figure 8. Efficient frontiers in both hedged and unhedged portfolios.

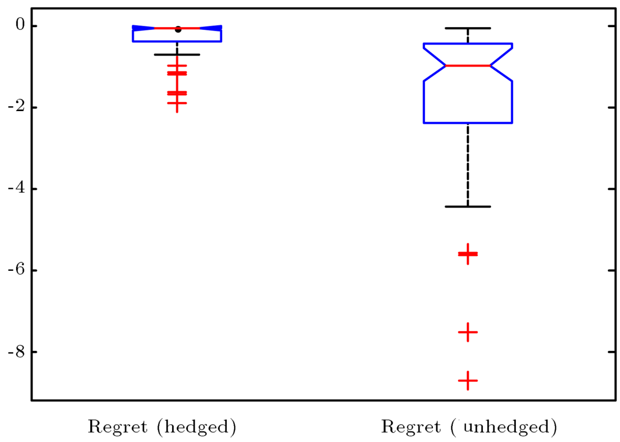

Figure 10. The box plot of investor's regret for both hedged and unhedged portfolios.

the planning horizon. In other words, the role of time is totally ignored. The comparison shows that for an arbitrary target wealth for an investor, the hedged portfolio contains a substantially lower level of risk compared to the unhedged one. Hence, the important role of options in mitigating market risk is confirmed.

4.2.3.2. Investor's risk simulation

To make the decision maker confident about the performance of the scenario generation method, as well as the role of options in reducing investment risk, this risk is simulated via a large out-of-sample scenario tree. This should be performed for both hedged and unhedged portfolios. In this regard, each multistage stochastic programming model is separately solved. Then, the first stage decisions are fixed, and the large out-of-sample scenario tree is used to simulate risk distribution in cases of both models.

Figure 9 presents a comparison between Cumulative Distribution Functions (CDFs) of investor regret for both in-sample and out-of-sample scenarios. In both hedged and unhedged portfolios, the distribution of investor regret, obtained by simulation of decisions, closely replicates that obtained by solving the original stochastic programming model. Furthermore, this figure shows that inclusion of options in the portfolio has a substantial influence on reducing the imposed risk to the investor.

To provide a better view of the role of options in risk reduction, the box plot of the simulated expected regret is illustrated in Figure 10, in cases of both hedged and unhedged portfolios. This figure confirms the substantial influence of options in mitigating imposed risk for the investor.

4.2.4. Assessing hedging strategies via back-testing simulations

In this paper, the performance of different hedging strategies is dynamically assessed using different bounds on Greek letters. In this regard, in the upper and lower bounds, each Greek letter is set to zero during the planning horizon. This approach helps us to dynamically examine Delta neutral, Gamma neutral, Theta neutral, Rho neutral and Vega neutral strategies, as well as a totally unhedged one.

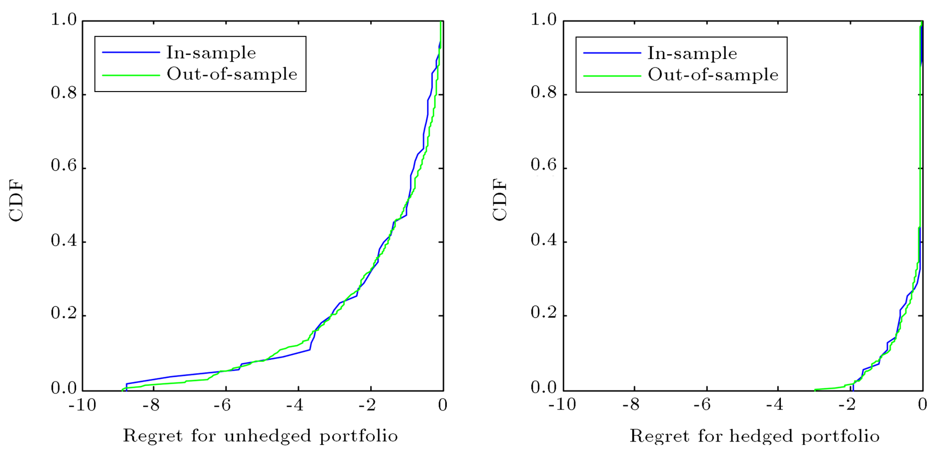

Figure 9. Comparison of regret distributions (in-sample solution and out-of-sample simulation).

None of the static tests can guarantee the appropriate performance of the proposed model, as well as the scenario tree generation method, in a long-term planning horizon. They only provide an image of the performance during a short-term horizon. To dynamically assess different hedging strategies, back-testing simulations should be performed in case of all hedging strategies. In this regard, each model should be run on a rolling horizon basis, at each successive month, in the period 06/2012–04/2014 (i.e. for a total of 23 months). Starting with an initial cash endowment, in June 2012, each model is executed to decide the initial portfolio composition. The clock then advances one period. The realized return of the optimal portfolio is determined on the basis of the revealed marketprices of the assets. A new scenario tree is then generated using the proposed scenario tree generation approach. With the new scenarios as input, and using the portfolio composition resulting from previous decisions as a starting point, the model is solved again. The process is repeated for each successive period, and the ex-post realized returns are recorded. Thus, the back-testing simulations demonstrate the actual returns that would have been realized had the decisions of the models been implemented during the simulation period, 06/2012–04/2014.

Figure 11 shows the ex-post realized returns of optimal portfolios with different hedging strategies.

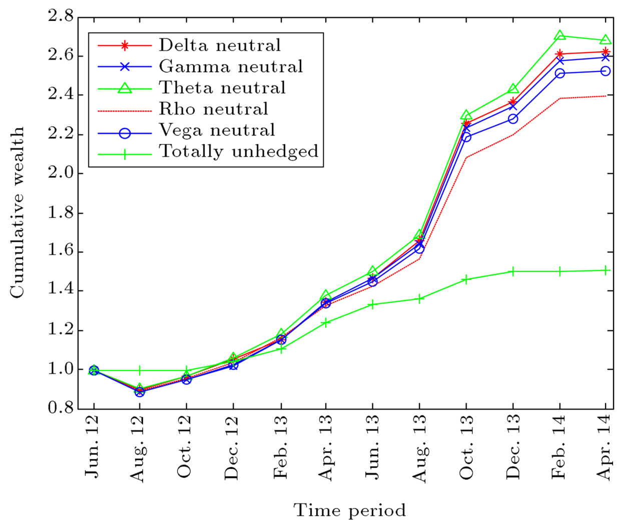

Figure 11. The ex-post realized performance of different hedging strategies in multistage models.

The results clearly illustrate that the totally unhedged approach performs badly compared to strategies that use options for hedging exposure to market risk.

Figure 11 illustrates that the theta neutral strategy is the best among all hedging strategies. It arises from the fact that the market is highly volatile during the planning horizon. The theta neutral strategy could appropriately control these volatilities and perform better than other hedging strategies.

Finally, we compute some measures to compare the overall performance of different hedging strategies. Specifically, we consider the following measures of the ex-postrealized returns over the simulation period: standard deviation, Sharpe ratio and the upside potential and downside risk (UP) ratio proposed by Sortino and van der Meeer [60]. This ratio contrasts the upside potential against as specific target (benchmark) with the shortfall risk against the same target. The U_Pratio is computed as follows. Let r_t be the realized return of a portfolio in month $t = 1, \cdots, k$ of the simulation, where k is the number of periods in the simulation period, 06/2012–04/2014. Let ρ_t be the return of the benchmark (riskless asset) at the same period. Then, the UP_ratio is:

$$\frac{\frac{1}{k}\sum_{t=1}^{k}\max(0, r_t - \rho_t)}{\sqrt{\frac{1}{k}\sum_{t=1}^{k}[\max(0, r_t - \rho_t)]^2}}.$$

The numerator is the average excess return compared to the benchmark, reflecting the upside potential. The denominator is a measure of downside risk, as proposed in Sortino et al. [61], and can be thought of as a shortfall risk compared to the benchmark. The performance measures from the simulation results are reported in Table 5.

Table 5 shows that the Theta neutral hedging strategy is the best, in terms of Sharpe ratio, while it is the 2nd and 3rd best in terms of UP_ratio and standard deviation with slight, insignificant differences. These results are in conformance with those obtained from Figure 10.

5. Conclusions

This paper presents a multi-period portfolio optimization model composed of some options and underlying

Table 5. Statistical characteristics of realized monthly returns.

	Delta neutral	Gamma neutral	Theta neutral	Rho neutral	Vega neutral
Standard deviation	0.265	0.267	0.267	0.271	0.264
Sharpe ratio	0.615	0.597	0.633	0.584	0.587
UP_ratio	0.111	0.114	0.112	0.102	0.111

assets to which the options are issued. In addition, a new method is presented to generate scenario trees of asset returns. In fact, scenarios of asset returns are generated, so that the properties of financial asset returns, e.g. non-constant volatility and non-normal, heavy-tailed distributions, are considered, and their temporal correlations, dependence structure and marginal distributions are also preserved. Different hedging strategies, referred to as Greek letters, are implemented and compared via the proposed model. The study attempts to take real conditions of financial markets into consideration and present a more realistic model than those previously presented. For instance, the investor could purchase option contracts during their lifetime. Furthermore, in addition to transaction costs considered for trading stocks, fixed and proportional ones are also considered for trading options.

Computational results show the high performance of the scenario generation method. Moreover, measuring VSS for different scenario sets shows that ignoring uncertainty imposes a remarkable cost onto the investor. Also, the in-sample stability of the scenario generation method is shown. Then, the important role of options in mitigating the market risk is confirmed through comparing efficient frontiers and simulating the investor's expected regret, in cases of both hedged and unhedged portfolios. Finally, options are utilized to assess different hedging strategies, referred to as Greek letters, during the planning horizon. In this regard, back-testing simulations are used to provide ex-post realized returns of the portfolio through adopting different hedging strategies. According to back-testing simulations, the Theta neutral strategy seems to be the best, since it can take advantage of the high volatilities of the market and perform better than other hedging strategies.

References

1. Mulvey, J.M., Pauling, W.R. and Madey, R.E. "Advantages of multiperiod portfolio models", *The Journal of Portfolio Management*, **29**(2), pp. 35-45 (2003).
2. Topaloglou, N., Vladimirou, H. and Zenios, S.A. "A dynamic stochastic programming model for international portfolio management", *European Journal of Operational Research*, **185**(3), pp. 1501-1524 (2008).
3. Merton, R.C., Scholes, M.S. and Gladstein, M.L. "The returns and risks of alternative put-option portfolio investment strategies", *The Journal of Business*, **55**(1), pp. 1-55 (1982).
4. Harrison, J.M. and Pliska, S.R. "Martingales and stochastic integrals in the theory of continuous trading", *Stochastic Processes and their Applications*, **11**(3), pp. 215-260 (1981).
5. Follmer, H. and Sondermann, D. "Hedging of non-redundant contingent claims", In *Contributions to Mathematical Economics*, W. Hildenbrand and A. Mas-Colell, Eds., pp. 205-223, North-Holland, Amsterdam (1986).
6. Brennan, M. and Cao, H. "Information, trade, and derivative securities", *Review of Financial Studies*, **9**(1), pp. 163-208 (1996).
7. Haugh, M.B. and Lo, A.W. "Asset allocation and derivatives", *Quantitative Finance*, **1**(1), pp. 45-72 (2001).
8. Carr, P., Jin, X. and Madan, D.B. "Optimal investment in derivative securities", *Finance and Stochastics*, **5**(1), pp. 33-59 (2001).
9. Neuberger, A. and Hodges, S. "How large are the benefits from using options?", *Journal of Financial and Quantitative Analysis*, **37**(02), pp. 201-220 (2002).
10. Aliprantis, C.D., Brown, D.J. and Werner, J. "Minimum-cost portfolio insurance", *Journal of Economic Dynamics and Control*, **24**(11-12), pp. 1703-1719 (2000).
11. Blomvall, J. and Lindberg, P.O. "Back-testing the performance of an actively managed option portfolio at the Swedish stock market, 1990-1999", *Journal of Economic Dynamics and Control*, **27**(6), pp. 1099-1112 (2003).
12. Liu, J. and Pan, J. "Dynamic derivative strategies", *Journal of Financial Economics*, **69**(3), pp. 401-430 (2003).
13. Muck, M. "Trading strategies with partial access to the derivatives market", *Journal of Banking & Finance*, **34**(6), pp. 1288-1298 (2010).
14. Scheuenstuhl, G. and Zagst, R. "Integrated portfolio management with options", *European Journal of Operational Research*, **185**(3), pp. 1477-1500 (2008).
15. Liang, J., Zhang, S. and Li, D. "Optioned portfolio selection: models and analysis", *Mathematical Finance*, **18**(4), pp. 569-593 (2008).
16. Liang, J. "Tracking models for optioned portfolio selection", In *Cutting-Edge Research Topics on Multiple Criteria Decision Making*, Y. Shi, S. Wang, Y. Peng, J. Li and Y. Zeng, Eds., pp. 729-736, Springer Berlin Heidelberg (2009).
17. Zymler, S., Rustem, B. and Kuhn, D. "Robust portfolio optimization with derivative insurance guarantees", *European Journal of Operational Research*, **210**(2), pp. 410-424 (2011).
18. Fonseca, R.J. and Rustem, B. "Robust hedging strategies", *Computers & Operations Research*, **39**(11), pp. 2528-2536 (2012).
19. Fonseca, R., Wiesemann, W. and Rustem, B. "Robust international portfolio management", *Computational Management Science*, **9**(1), pp. 31-62 (2012).
20. Topaloglou, N., Vladimirou, H. and Zenios, S.A. "Optimizing international portfolios with options and forwards", *Journal of Banking & Finance*, **35**(12), pp. 3188-3201 (2011).

21. Yin, L. and Han, L. "International assets allocation with risk management via multi-stage stochastic programming", *Computational Economics*, pp. 1-23 (2013).

22. Wu, J. and Sen, S. "A stochastic programming model for currency option hedging", *Annals of Operations Research*, **100**(1-4), pp. 227-249 (2000).

23. Papahristodoulou, C. "Option strategies with linear programming", *European Journal of Operational Research*, **157**(1), pp. 246-256 (2004).

24. Horasanlı, M. "Hedging strategy for a portfolio of options and stocks with linear programming", *Applied Mathematics and Computation*, **199**(2), pp. 804-810 (2008).

25. Gao, P.-W. "Options strategies with the risk adjustment", *European Journal of Operational Research*, **192**(3), pp. 975-980 (2009).

26. Yin, L. and Han, L. "Options strategies for international portfolios with overall risk management via multi-stage stochastic programming", *Annals of Operations Research*, **206**(1), pp. 557-576 (2013).

27. Høyland, K., Kaut, M. and Wallace, S. "A heuristic for moment-matching scenario generation", *Computational Optimization and Applications*, **24**(2-3), pp. 169-185 (2003).

28. Hull, J., *Options, Futures and Other Derivatives*, Prentice Hall (2011).

29. Shu-Shang, Z., Duan, L. and Shou-Yang, W. "Risk control over bankruptcy in dynamic portfolio selection: a generalized mean-variance formulation", *Automatic Control, IEEE Transactions on*, **49**(3), pp. 447-457 (2004).

30. Rockafellar, R.T. and Uryasev, S. "Optimization of conditional value-at-risk", *Journal of Risk*, **2**, pp. 21-41 (2000).

31. Rockafellar, R.T. and Uryasev, S. "Conditional value-at-risk for general loss distributions", *Journal of Banking & Finance*, **26**(7), pp. 1443-1471 (2002).

32. Testuri, C.E. and Uryasev, S. "On relation between expected regret and conditional value at risk", Report 2000-9, University of Florida (2000).

33. Ji, X., Zhu, S., Wang, S. and Zhang, S. "A stochastic linear goal programming approach to multistage portfolio management based on scenario generation via linear programming", *IIE Transactions*, **37**(10), pp. 957-969 (2005).

34. Mulvey, J.M., Vanderbei, R.J. and Zenios, S.A. "Robust optimization of large-scale systems", *Operations Research*, **43**(2), pp. 264-281 (1995).

35. Høyland, K. and Wallace, S.W. "Generating scenario trees for multistage decision problems", *Management Science*, **47**(2), pp. 295-307 (2001).

36. Fleten, S.-E., Høyland, K. and Wallace, S.W. "The performance of stochastic dynamic and fixed mix portfolio models", *European Journal of Operational Research*, **140**(1), pp. 37-49 (2002).

37. Gülpınar, N., Rustem, B. and Settergren, R. "Simulation and optimization approaches to scenario tree generation", *Journal of Economic Dynamics and Control*, **28**(7), pp. 1291-1315 (2004).

38. Chen, Z. and Xu, D. "Knowledge-based scenario tree generation methods and application in multiperiod portfolio selection problem", *Applied Stochastic Models in Business and Industry*, pp. n/a-n/a (2013).

39. Geyer, A., Hanke, M. and Weissensteiner, A. "Scenario tree generation and multi-asset financial optimization problems", *Operations Research Letters*, **41**(5), pp. 494-498 (2013).

40. Topaloglou, N., Vladimirou, H. and Zenios, S.A. "CVaR models with selective hedging for international asset allocation", *Journal of Banking & Finance*, **26**(7), pp. 1535-1561 (2002).

41. Grebeck, M.J., Rachev, S.T. and Fabozzi, F.J. "Stochastic programming and stable distributions in asset-liability management", *Journal of Risk*, **12**(2), pp. 29-47 (2009).

42. Erlwein, C., Mitra, G. and Roman, D. "HMM based scenario generation for an investment optimisation problem", *Annals of Operations Research*, **193**(1), pp. 173-192 (2012).

43. Mandelbrot, B. "The variation of certain speculative prices", *The Journal of Business*, **36**(4), pp. 394-419 (1963).

44. Engle, R.F. "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflations", *Econometrica*, **50**(4), pp. 987-1007 (1982).

45. Bollerslev, T. "Generalized autoregressive conditional heteroskedasticity", *Journal of Econometrics*, **31**(3), pp. 307-327 (1986).

46. Tsay, R., *Analysis of Financial Time Series*, John Wiley & Sons, Inc., Hoboken, New Jersey (2005).

47. Chen, Z. and Yuen, K.C. "Optimal consumption and investment problems under GARCH with transaction costs", *Mathematical Methods of Operations Research*, **61**(2), pp. 219-237 (2005).

48. Chen, Z.P., Xu, C.X. and Yuen, K.C. "Stochastic programming method for multiperiod consumption and investment problems with transaction costs", *Journal of Systems Science and Complexity*, **17**(1), pp. 39-53 (2004).

49. Chen, Z. "Multiperiod consumption and portfolio decisions under the multivariate GARCH model with transaction costs and CVaR-based risk control", *OR Spectrum*, **27**(4), pp. 603-632 (2005).

50. Hochreiter, R. and Pflug, G. "Financial scenario generation for stochastic multi-stage decision processes as facility location problems", *Annals of Operations Research*, **152**(1), pp. 257-272 (2007).

51. Beraldi, P. and Bruni, M. "A clustering approach for scenario tree reduction: an application to a stochastic programming portfolio optimization problem", *TOP*, pp. 1-16 (2013).

52. Johnson, N.L. "Systems of frequency curves generated by methods of translation", *Biometrika*, **36**(1-2), pp. 149-176 (1949).
53. Nelson, N. "Conditional heteroskedasticity in asset returns: a new approach", *Econometria*, **59**(2), pp. 347-370 (1991).
54. Glosten, L.R., Jagannathan, R. and Runkle, D.E. "On the relation between the expected value and the volatility of the nominal excess return on stocks", *The Journal of Finance*, **48**(5), pp. 1779-1801 (1993).
55. Akaike, H. "A new look at the statistical model identification", *Automatic Control, IEEE Transactions on*, **19**(6), pp. 716-723 (1974).
56. Gr;we-Kuska, N., Heitsch, H. and R;misch, W. "Scenario reduction and scenario tree construction for power management problems", *IEEE Power Tech Conf.*, Bologna (2003).
57. Hajizadeh, E., Seifi, A., Fazel Zarandi, M.H. and Turksen, I.B. "A hybrid modeling approach for forecasting the volatility of S & P 500 index return", *Expert Systems with Applications*, **39**(1), pp. 431-436 (2012).
58. Di Domenica, N., Lucas, C., Mitra, G. and Valente, P. "Scenario generation for stochastic programming and simulation: a modelling perspective", *IMA Journal of Management Mathematics*, **20**(1), pp. 1-38 (2009).
59. Kaut, M. and Wallace, S.W. "Evaluation of scenario generation methods for stochastic programming", *Pacific Journal of Optimization*, **3**(2), pp. 257-271 (2003).
60. Sortino, F.A. and van der Meer, R. "Downside risk-capturing what's at stake in investment situations", *Journal of Portfolio Management*, **17**(4), pp. 27-31 (1991).
61. Sortino, F.A., van der Meer, R. and Plantinga, A. "The Dutch triangle: A framework to measure upside potential relative to downside risk", *Journal of Portfolio Management*, **26**(1), pp. 50-58 (1999).

Appendix A

Johnson transformation

The Johnson system utilizes three families of distribution to transform variables to standard normal distribution. The standard normal variables are generated by transformations of the following form:

$$z = \gamma + \eta k_i(x; \lambda, \varepsilon), \tag{A.1}$$

where, z is a standard normal variable and $k_i(x; \lambda, \varepsilon)$ is chosen to cover a wide range of possible shapes. Johnson suggested the following functions:

$$k_1(x; \lambda, \varepsilon) = \operatorname{arcsinh}\left(\frac{x-\varepsilon}{\lambda}\right), \tag{A.2}$$

$$k_2(x; \lambda, \varepsilon) = \ln\left(\frac{x-\varepsilon}{\lambda+\varepsilon-x}\right), \tag{A.3}$$

$$k_3(x; \lambda, \varepsilon) = \ln\left(\frac{x-\varepsilon}{\lambda}\right). \tag{A.4}$$

These functions are referred to as the S_U distribution, S_B distribution and S_L distribution, respectively.

Consider any of these transformations. For any fixed positive value of z, points $-3z$, $-z$, $+z$ and $+3z$ determine three intervals with equal length. Any of these transformations yields four values of x which are no longer equally spaced. Let x_{-3z}, x_{-z}, x_z and x_{3z} be the values corresponding to $-3z$, $-z$, $+z$ and $+3z$ under any transformation. Let:

$$m = x_{3z} - x_z,$$

$$n = x_{-z} - x_{-3z},$$

$$p = x_z - x_{-z}. \tag{A.5}$$

It can be proved that for any S_U distribution, $\frac{mn}{p^2} > 1$, for any S_B distribution, $\frac{mn}{p^2} < 1$ and for any S_L distribution, $\frac{mn}{p^2} = 1$. This property can be used to discriminate among the three families.

To select the appropriate transformation, a value of z is chosen. Then, from the tables of areas for standard normal distribution, percentages φ_ς, corresponding to $\varsigma = -3z, -z, z$ and $3z$, are determined. For each ς, percentile $x^{(i)}$, corresponding to φ_ς, is obtained using the relationship $\varphi_\varsigma = (i - 1/2)/N$, where N is the number of data points, and x_ς is set equal to $x^{(i)}$. Since i is not necessarily an integer, interpolation may be required. Afterwards, the sample values of m, n and p are computed with Eq. (A.5) and the appropriate transformation is selected. Since the probability that $\frac{mn}{p^2} = 1$ is zero, if one wishes to use S_L distribution, it will be necessary to allow a tolerance interval around 1.

After the selection process is completed, the next problem is to estimate parameters of the chosen distribution. There exist various parameter estimation techniques for the Johnson system. Here, a uniform approach of matching percentiles is introduced. The estimates are given in terms of the chosen value of z and the formerly computed values of m, n and p.

For each family, the formulas are obtained by starting with a given Johnson distribution and fixed positive z, and then solving explicitly for the parameters in terms of z and the population values of m, n and p. It should be mentioned that the parameter values are functions of m, n and p, which, in turn, are functions of x_{-3z}, x_{-z}, x_z and x_{3z}.

For the three families, the estimations are given by the following formulas:

(a) Johnson unbounded system (S_U distribution)

$$z = \gamma + \eta \operatorname{arcsinh}\left(\frac{x-\varepsilon}{\lambda}\right). \tag{A.6}$$

Estimates of the parameters in this case are as follows:

$$\eta = \frac{2z}{\operatorname{arccosh}\left(\frac{m}{2p} + \frac{n}{2p}\right)} \qquad (\eta > 0), \tag{A.7}$$

$$\gamma = \eta \operatorname{arcsinh}\left(\frac{\frac{n}{p} - \frac{m}{p}}{2\left(\frac{mn}{p^2} - 1\right)^{1/2}}\right), \tag{A.8}$$

$$\lambda = \left(\frac{2p\left(\frac{mn}{p^2} - 1\right)^{1/2}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{1/2}}\right), \quad (\lambda > 0), \tag{A.9}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} + \frac{p\left(\frac{n}{p} - \frac{m}{p}\right)}{2\left(\frac{m}{p} + \frac{n}{p} - 2\right)}. \tag{A.10}$$

(b) Johnson bounded system (S_B distribution):

$$z = \gamma + \eta \ln\left(\frac{x - \varepsilon}{\lambda + \varepsilon - x}\right). \tag{A.11}$$

Estimates of the parameters in this case are as follows:

$$\eta = \frac{z}{\operatorname{arccosh}\left(\frac{1}{2}\left[\left(1 + \frac{p}{m}\right)\left(1 + \frac{p}{n}\right)\right]^{1/2}\right)}, \quad (\eta > 0), \tag{A.12}$$

$$\gamma = \eta \operatorname{arcsinh}\left(\frac{\left(\frac{p}{n} - \frac{p}{m}\right)\left[\left(1 + \frac{p}{m}\right)\left(1 + \frac{p}{n}\right) - 4\right]^{1/2}}{2\left(\frac{p^2}{mn} - 1\right)}\right), \tag{A.13}$$

$$\lambda = \frac{p\left\{\left[\left(1 + \frac{p}{m}\right)\left(1 + \frac{p}{n}\right) - 2\right]^2 - 4\right\}^{1/2}}{\frac{p^2}{mn} - 1}, \quad (\lambda > 0), \tag{A.14}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} - \frac{\lambda}{2} + \frac{p\left(\frac{p}{n} - \frac{p}{m}\right)}{2\left(\frac{p^2}{mn} - 1\right)}. \tag{A.15}$$

(c) Johnson log-normal system (S_L distribution):

$$z = \gamma + \eta \ln(x - \varepsilon). \tag{A.16}$$

Note that in case of a log-normal system, we have:

$$\frac{n}{p} = \frac{p}{m}.$$

Estimates of the parameters in this case are as follows:

$$\eta = \frac{2z}{\ln\left(\frac{m}{p}\right)}, \tag{A.17}$$

$$\gamma = \eta \ln\left(\frac{\frac{m}{p} - 1}{p\left(\frac{m}{p}\right)^{1/2}}\right), \tag{A.18}$$

$$\varepsilon = \frac{x_z + x_{-z}}{2} - \frac{p}{2}\frac{\left(\frac{m}{p} + 1\right)}{\left(\frac{m}{p} - 1\right)}. \tag{A.19}$$

Biographies

Hamed Davari-Ardakani received his BS degree in Industrial Engineering from Yazd University, Iran, and his MS degree in Financial Engineering and his PhD degree in Industrial Engineering from Amirkabir University of Technology, Iran. He is currently Assistant Professor of Industrial Engineering at Kharazmi University, Iran. His main research interests include: finance, risk management, multi-criteria decision making and portfolio optimization.

Majid Aminnayeri received his BS degree in Mathematics and MS degree in Mathematical Statistics from Pahlavi University, Iran, in 1973 and 1975, respectively. His PhD degree, double majoring in Statistics and Industrial Engineering, was awarded to him by Iowa State University, USA, in 1981. He is currently employed at Amirkabir University of Technology, Tehran, Iran, where, from 1993 to 1997 and 2009 to 2013, he served as Chairman of the Industrial Engineering Department. His main research interests are: SPC, scheduling, and stochastic modelling.

Abbas Seifi obtained his BASc and MASc degrees in Industrial Engineering from Sharif University of Technology, Tehran, Iran, and his PhD degree in Systems Design Engineering from the University of Waterloo, Canada, where he worked as a postdoctoral research associate for over 2 years. He is currently Associate Professor of Industrial Engineering at Amirkabir University of Technology, Iran. His research interests include stochastic and robust optimization and simulation of operational research problems as well as reliability-based design and manufacturing yield.