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Analytical solution to nonlinear behavior of electrostatically actuated nanobeams incorporating van der Waals and Casimir forces

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Abstract. In this research, influences of intermolecular interactions on the behavior of nanobeams are studied. Suddenly applied voltages actuates the clamped-clamped nanobeam. The effects of electrostatic actuation, intermolecular forces, midplane stretching, the fringing field effect and residual stress are considered. Initially, the governing equation is non-dimensionalized, and the partial differential equation of motion is converted to a nonlinear ordinary differential equation by means of the Galerkin method. Afterwards, the nonlinear ordinary differential equation of motion is solved using the homotopy analysis method. To validate the model, the response of a sample beam was compared with that in the relevant literature. Finally, the effects of various parameters on the nonlinear frequency of the response are studied. The results indicate that the nonlinear frequency of oscillations significantly decreases by increasing intermolecular effects.

1. Introduction

Nanoelectromechanical (NEMS) and microelectromechanical (MEMS) systems are used in nano or microscaled devices. Some applications of NEMS and MEMS are nano or microscaled resonators, switches, valves, grippers, tweezers and sensors. Identification of the behavior of these systems would help in their correct design. For example, in a nanoresonator, a DC voltage is applied to deform the beam. Next, an AC voltage, with a frequency close to the fundamental natural frequency of the beam, is applied to make it vibrate [1]. The purpose is to identify the fundamental natural frequency of the beam. By decreasing the dimensions of devices, intermolecular force effects become significant [2]. Casimir and van der Waals effects are considered in this study. Van der Waals force is an electrostatic force among polar molecules or even plates [3]. While the Casimir effect is generated by the presence of two parallel plates of solids [4]. This effect even produces a force in macroscopic scale [5]. The Casimir force is effective at longer distances in comparison with the van der Waals force, thus, they cannot be considered simultaneously [6].

The effects of van der Waals and Casimir forces in the oscillation of nanobeams are considered in some literature. Lin and Zhao [7] involved the van der Waals force in the stability of nanoscale actuators. They showed a mass-spring model to exhibit bifurcations in equilibrium points. They also considered the influence of the Casimir force on the stability of those actuators by using a one-degree-of-freedom mass-spring model [8]. Moghimi Zand and Ahmadian [9] studied the dynamic pull-in instability of a microbeam under intermolecular effects. They used a finite element model to discretize the governing equations and

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employed Newmark time discretization to solve the discretized equations.

Liao [10] proposed a method to analytically solve strong nonlinear problems; the Homotopy Analysis Method (HAM). The advantage of HAM is in the freedom for the user to choose different initial approximations, auxiliary linear operators and auxiliary functions [11].

HAM has been used to solve the motion equations of MEMS and NEMS. Moghimi Zand and Ahmadian [12] used HAM to predict the dynamic pull-in instability of MEMS. The results of HAM were in good agreement with numerical and experimental data. Qian and et al. [13] obtained analytical solutions for a strong nonlinear problem using HAM. The nonlinearity of the problem was due to the large-amplitude vibrations of the microbeam. Moghimi Zand and Ahmadian [14] investigated the nonlinear frequency of vibration in an electrostatically actuated microbeam by means of HAM. For further study on other methods of solution, please see [15-19].

The present study investigates the effects of various parameters on the nonlinear frequency of clamped-clamped nanobeams incorporating intermolecular forces such as van der Waals and Casimir. Different sources of nonlinearity are considered. Galerkin's method is used to convert the nonlinear partial differential equation of motion to a nonlinear ordinary differential equation. HAM solves the nonlinear governing equation. An oscillatory response for a sample microbeam is obtained to compare with the literature and validate the model. Effects of parameters, such as voltage, fringing field, residual stress, intermolecular forces and midplane stretching on the nonlinear frequency, are studied.

2. Modeling and formulation

A schematic of a clamped-clamped nanobeam is shown in Figure 1. The voltage, V, is applied between the nanobeam and the substrate. The coordinates are assumed as in Figure 1. $\hat{W}$ is the deflection along the $\hat{z}$ coordinate, and b and h are the width and thickness of the beam, respectively. The electrostatic force per unit length between the nanobeam and the substrate due to voltage V is [20]:

$$F_e = \frac{1}{2}\frac{b\varepsilon V^2}{[d_0 - \widehat{W}]^2}\left[1 + \hat{f}\frac{d_0 - \widehat{W}}{b}\right], \tag{1}$$

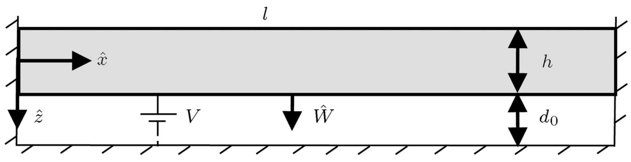

Figure 1. Schematic view of a double clamped nanobeam.

where ε is the vacuum permittivity. The term $\hat{f}$ represents the fringing field effect, which is 0.65 for a clamped-clamped nanobeam [9]. The value of the van der Waals intermolecular force per unit length can be obtained from Eq. (2), where A is the Hamaker constant. According to Eq. (2), it is obvious that the van der Waals force depends on both the geometric parameters and the materials [21]:

$$F_3 = \frac{Ab}{6\pi\left[d_0 - \hat{W}\right]^3}. \tag{2}$$

The value of the Casimir intermolecular force per unit can be obtained from:

$$F_4 = \frac{\pi^3 \bar{h} c b}{240\left[d_0 - \hat{W}\right]^4}. \tag{3}$$

Here, $\bar{h}$ is the Plank's constant divided by 2π, and c is the speed of light. Unlike the van der Waals force, the Casimir force just depends on geometrical parameters [22].

The motion equation of a nanobeam, considering intermolecular and electrostatic forces, von Kármán nonlinearity, due to midplane stretching and residual stress, is given by Eq. (4). Here, ρ, E, I and $\hat{N}$ are density, Young's modulus, the moment of the cross-section about the $\hat{y}$-axis and axial load due to residual stress. The index, n, is 3 and 4 for the van der Waals and Casimir forces, respectively:

$$\rho b h \hat{W}_{\hat{t}\hat{t}} + EI\hat{W}_{\hat{x}\hat{x}\hat{x}\hat{x}} - \left[\hat{N} + \frac{Ebh}{2l}\int_0^l \hat{W}_{\hat{x}}^2 d\hat{x}\right]\hat{W}_{\hat{x}\hat{x}} - F_e - F_n = 0. \tag{4}$$

The boundary conditions of the nanobeam are:

$$\hat{W}(0,\hat{t}) = 0, \qquad \hat{W}_{\hat{x}}(0,\hat{t}) = 0,$$
$$\hat{W}(l,\hat{t}) = 0, \qquad \hat{W}_{\hat{x}}(l,\hat{t}) = 0. \tag{5}$$

To non-dimensionalize the motion equation, the following non-dimensional parameters are chosen:

$$x = \frac{\hat{x}}{l}, \qquad w = \frac{\hat{w}}{d_0}, \qquad N = \frac{\hat{N}l^2}{EI},$$
$$f = \frac{\hat{f}d_0}{b}, \qquad t = \hat{t}\sqrt{\frac{Eh^2}{12\rho l^4}}, \qquad \alpha = 6\left(\frac{d_0}{h}\right)^2,$$
$$\beta = \frac{\varepsilon b V^2 l^4}{2EId_0^3}, \qquad \lambda_3 = \frac{Abl^4}{6EI\pi d_0^4}, \qquad \lambda_4 = \frac{\pi^2\bar{h}cbl^4}{240EId_0^5}. \tag{6}$$

The non-dimensional parameters, N, α, β, λ_3 and λ_4, represent axial load, midplane stretching, voltage, the van der Waals and Casimir intermolecular forces, respectively. The non-dimensionalized motion equation and the relevant boundary conditions are shown in Eqs. (7) and (8):

$$W_{tt} + W_{xxxx} - \left(N + \alpha \int_0^1 W_x^2 dx\right) W_{xx} - \frac{f\beta}{1-W} - \frac{\beta}{(1-W)^2} - \frac{\lambda_n}{(1-W)^n} = 0, \tag{7}$$

$$W(0,t) = 0, \qquad W_x(0,t) = 0,$$

$$W(1,t) = 0, \qquad W_x(1,t) = 0. \tag{8}$$

Before using HAM, the terms of forces in Eq. (7) should be changed into polynomial forms using Taylor's series. These forms are shown in Appendix A. According to Galerkin method, the deflection of the nanobeam is assumed to be the product of two independent functions as follows:

$$W(x,t) = u(t)w(x), \tag{9}$$

where function $w(x)$ is chosen properly to satisfy the boundary conditions in Eq. (8):

$$w(x) = x^2(1-x)^2. \tag{10}$$

Eq. (7) can be transformed into a nonlinear ordinary differential equation by multiplying it by $w(x)$ and integrating it along the period [0,1]. This procedure results in Eq. (11):

$$M\ddot{u} + c_0 + c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 + c_5 u^5 + c_6 u^6 + \cdots = 0. \tag{11}$$

In this study, the voltage is applied suddenly, so, the initial conditions are as follows:

$$u(0) = 0, \qquad \dot{u}(t) = 0. \tag{12}$$

The first seven coefficients of Eq. (11) are shown in Appendix B. the next step is to use HAM to solve Eq. (11).

3. Applying HAM

HAM is used to solve strongly nonlinear differential equations. The homotopy function is shown in Eq. (13):

$$\mathcal{H}(\phi(t;q); q, \hbar, H(t)) = (1-q)\mathcal{L}\left\{\phi(t;q) - u_0(t)\right\} - q\hbar H(t)\mathcal{N}\left\{\phi(t;q), \Lambda(q)\right\}. \tag{13}$$

q is an auxiliary parameter that varies from 0 to 1. $\hbar$, $H(t)$ and $\mathcal{L}\{\}$ are an auxiliary parameter, function and linear operator, respectively. $\mathcal{N}\{\}$ is a nonlinear operator obtained from the main nonlinear equation. HAM maps $u(t)$ to $\phi(t;q)$ continuously, in such a way that when q varies from 0 to 1, $\phi(t;q)$ varies from the initial guess, $u_0(t)$, to the exact solution, $u(t)$. In this method, we have freedom to choose the auxiliary parameter, $\hbar$, auxiliary function, $H(t)$, auxiliary operator, $\mathcal{L}\{\}$, and the initial guess that satisfies the initial conditions. This freedom establishes the validity and flexibility of HAM. For this study, $H(t)$ can be chosen as 1. $u_0(t)$ is the initial guess for $u(t)$ that satisfies the initial conditions in Eq. (12). One can choose it as:

$$u_0(t) = 0. \tag{14}$$

Linear operator $\mathcal{L}\{\}$ is chosen as:

$$\mathcal{L}\{\phi\} = \phi_{tt} + \omega^2\phi. \tag{15}$$

The nonlinear operator, $\mathcal{N}\{\}$, is defined according to Eq. (11):

$$\mathcal{N}\{\phi(t;q), \Lambda(q)\} = \phi_{tt} + \Lambda\phi + \frac{c_0 + c_2\phi^2 + c_3\phi^3 + c_4\phi^4 + c_5\phi^5 + c_6\phi^6 + \cdots}{M}. \tag{16}$$

For $q = 1$, Eq. (16) should be the same as Eq. (11), so, $\Lambda(1) = \frac{c_1}{M}$. To find the zero-order deformation equation (Eq. (17)), one equates the homotopy function to zero.

$$(1-q)\mathcal{L}\{\phi(t;q) - u_0(t)\} = q\hbar H(t)\mathcal{N}\{\phi(t;q), \Lambda(q)\}. \tag{17}$$

Subjected to initial conditions:

$$\phi(0;q) = 0, \qquad \phi_t(0;q) = 0. \tag{18}$$

To find the zero approximation of $u(t)$, one equates q in Eq. (17) to zero:

$$\mathcal{L}\{\phi(t;0) - u_0(t)\} = 0. \tag{19}$$

From Eq. (19), it is clear that:

$$\phi(t;0) = u_0(t). \tag{20}$$

$\phi(t;q)$ can be expanded in a power series of q around $\phi(t;0)$. It is assumed that $\hbar$ is chosen properly such that solution $\phi(t;q)$ of Eq. (18) exists for $0 \le q \le 1$. Besides, its m-derivative in Eq. (20), with respect to q, can be obtained as follows:

$$u_0^m(t) = \left.\frac{\partial^m \phi(t;q)}{\partial q^m}\right|_{q=0}. \tag{21}$$

One defines $u_m(t)$ as:

$$u_m(t) = \frac{u_0^m(t)}{m!}. \tag{22}$$

$\phi(t;q)$'s expansion by Taylor's series is mentioned in Eq. (23):

$$\phi(t;q) = \phi(t;0) + \sum_{m=1}^{+\infty} \frac{1}{m!} \left.\frac{\partial^m \phi(t;q)}{\partial q^m}\right|_{q=0} q^m. \tag{23}$$

Applying Eq. (22) to Eq. (23) leads to the following expansion of $\phi(t;q)$:

$$\begin{aligned}\phi(t;q) =& u_0 + u_1 q + u_2 q^2 + u_3 q^3 + u_4 q^4 + u_5 q^5 \\ &+ u_6 q^6 + u_7 q^7 + u_8 q^8 + \cdots\end{aligned} \tag{24}$$

Furthermore, Eq. (25) is the expansion of $\Lambda(q)$:

$$\begin{aligned}\Lambda(q) =& \omega^2 + \omega_1 q + \omega_2 q^2 + \omega_3 q^3 + \omega_4 q^4 + \omega_5 q^5 \\ &+ \omega_6 q^6 + \cdots\end{aligned} \tag{25}$$

To obtain higher-order approximations, Eq. (17) should be differentiated, with respect to q. Afterwards, q should be set to zero. Eq. (26) shows the higher-order approximations of $u_j(t)$ in brief:

$$\begin{aligned}&\mathcal{L}\{u_j(t) - \chi_j u_{j-1}(t)\} \\ &= \frac{1}{(j-1)!} \hbar \left.\frac{\partial^{j-1} \mathcal{N}\{\phi(t;q), \Lambda(q)\}}{\partial q^{j-1}}\right|_{q=0},\end{aligned} \tag{26}$$

where:

$$\chi_j = \begin{cases} 0 & j \le 1 \\ 1 & \text{otherwise} \end{cases} \tag{27}$$

The terms of $\omega_{(j-2)}$ in Eq. (16) can be obtained by setting the coefficient of $\cos(\omega t)$ in Eq. (26) to zero to prevent it from producing secular terms causing unacceptable solutions. This procedure finds ω_{j-2} as a function of ω. After finding coefficients ω_{j-2} and u_j, to find the nonlinear frequency of oscillation, one can set q in Eq. (25) to 1, and obtain the following equation:

$$\Lambda(1) = \omega^2 + \omega_1 + \omega_2 + \omega_3 + \omega_4 + \cdots = \frac{c_1}{M}. \tag{28}$$

ω can be computed from Eq. (28). Substituting ω from Eq. (28) to Eq. (24) and setting q to 1, assigns the exact solution of $u(t)$ as follows:

$$\begin{aligned}u(t) &= \phi(t;1) \\ &= u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + \cdots\end{aligned} \tag{29}$$

In this study, the force terms are expanded by the order of 6 (i.e. W^6), as presented in Appendix A. As a rule, to find ω_m, $\phi(t;q)$ in Eq. (24) should be expanded till $u_{(m+2)} q^{(m+2)}$. Hence, $\phi(t;q)$ and $\Lambda(q)$ in Eqs. (24) and (25) are expanded till $u_8 q^8$ and $\omega_6 q^6$, respectively.

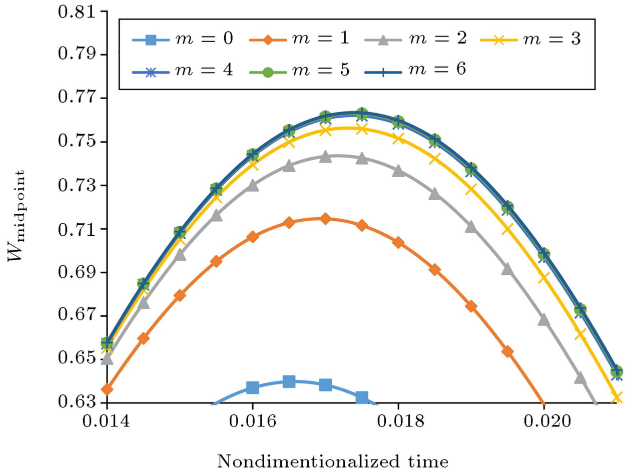

Figure 2. Time history of the midpoint of the sample beam for different orders of expansion in Appendix A.

4. Results and discussion

Figure 2 shows the time history of the midpoint in a sample nanobeam with different approximations of the force terms. Here, the applied voltage is 30 V. The material of the beam is gold, and its Young's modulus is 80 GPa. The length, width, gap and thickness are 20 μm, 100 nm, 1 μm and 50 nm, respectively. The residual stress of this beam is 100 MPa. From Figure 2, one can conclude that considering six and more order of force approximations is acceptable.

To validate the results of this study, the microbeam studied by Moghimi Zand and Ahmadian [9], is considered. The Young's modulus of the beam is 183.4 GPa. Its length, width, thickness and gap are 300 μm, 20 μm, 2 μm and 2 μm, respectively. The intermolecular effect and residual stress are not considered, and the non-dimentionalized voltage is 60. In Figure 3, it can be seen that there is good compatibility between the results of Moghimi Zand and Ahmadian [9] and this study.

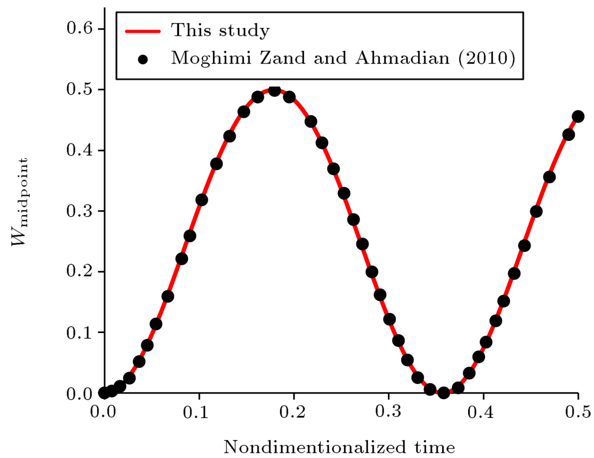

Figure 3. Time history of the midpoint of the sample beam.

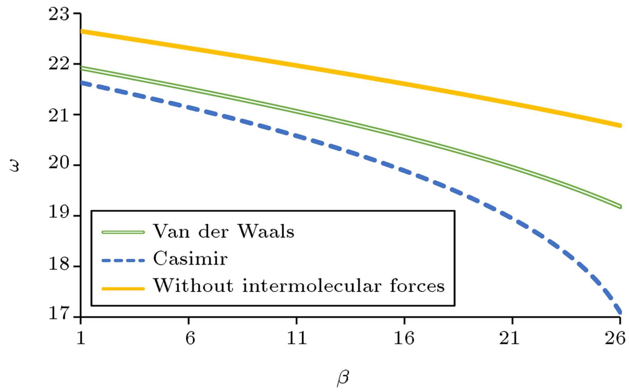

Figure 4. Variation of nonlinear frequency due to voltage: $\lambda 4 = \lambda 3 = 10$, $N = f = 1$, $\alpha = 6$.

Figure 4 shows the variation of nonlinear frequency vs. non-dimensional voltage parameter, β. Intermolecular forces have a noticeable impact on nonlinear frequency. Therefore, they cannot be neglected in nanoscale systems. In Figure 4, in large amounts of β, a decrease in the nonlinear frequency is observed, since the voltage gets closer to the dynamic pull-in voltage. Dynamic pull-in voltage is defined as the critical amount of applied DC voltage in which the nanobeam becomes dynamically instable [9]. For more information about dynamic pull-in instability, see [9,12,15-18,23-29].

In Figure 5, the effect of the value of intermolecular forces on the nonlinear frequency of oscillation is shown. One can say that if the values of λ_3 and λ_4 increase, the nanobeam gets closer to the dynamic pull-in instability.

Figure 6 shows the effect of N on the nonlinear frequency of response. According to this figure, one can say that for large values of N, the effect of intermolecular forces can be neglected.

Figure 7 shows the effect of a fringing field on the nonlinear frequency of response. In this case, for large values of f, the effects of intermolecular forces are

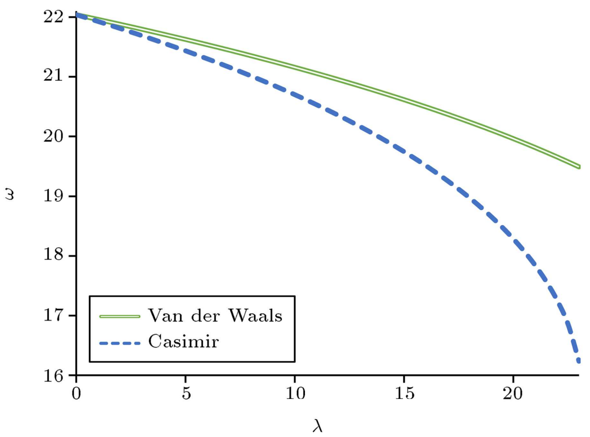

Figure 5. Effect of the value of intermolecular forces: $\beta = 10$, $N = f = 1$, $\alpha = 6$.

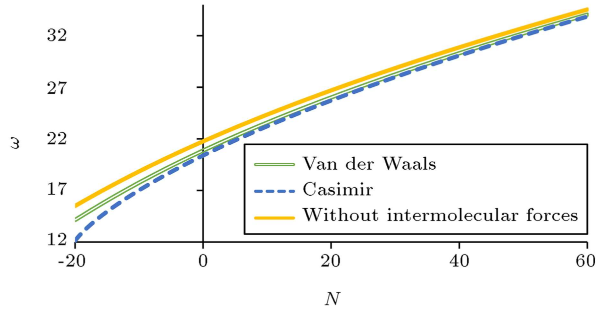

Figure 6. Effect of residual stress: $\lambda 4 = \lambda 3 = \beta = 10$, $f = 1$, $\alpha = 6$.

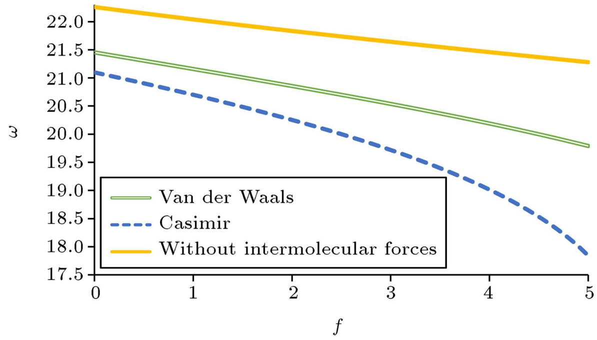

Figure 7. Effect of fringing field: $\lambda 4 = \lambda 3 = \beta = 10$, $N = 1$, $\alpha = 6$.

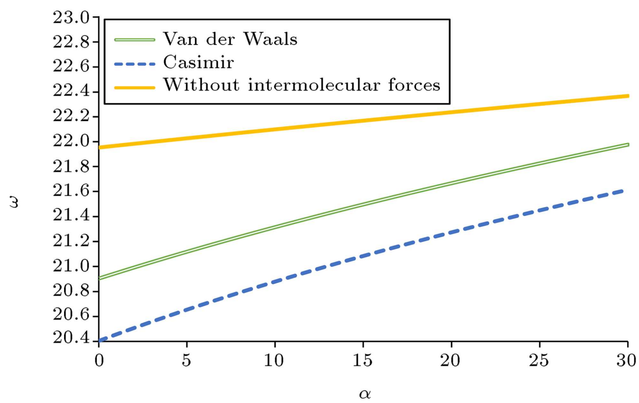

Figure 8. Effect of midplane stretching: $\lambda 4 = \lambda 3 = \beta = 10$, $N = f = 1$.

significant and the response of oscillation gets closer to the pull-in instability.

Figure 8 shows the effect of α on the nonlinear frequency of response. According to this figure, one can say that the intermolecular effects can be omitted in large values of α.

5. Conclusion

In this study, the nonlinear behavior of a clamped-clamped nanobeam with the effects of intermolecular forces was investigated by a semi-analytical method, named the homotopy analysis method. In this study, the effects of van der Waals and Casimir intermolecular forces, fringing field, residual stress and midplane stretching are considered. To validate the model, a microbeam studied by Moghimi Zand was used. The effect of intermolecular forces on the nonlinear frequency of response is shown. They decrease this frequency and lead the nanobeam to dynamic pull-in instability. Also it is seen that in equal non-dimensional parameters, the Casimir force has a greater effect than the van der Waals force. It is also indicated that for large amounts of residual stress, intermolecular forces can be neglected. For large amounts of non-dimensional fringing field parameter, the Casimir force has a significant effect on the nonlinear frequency. It is noticed that for large amounts of non-dimensional midplane stretching parameter, like residual stress, intermolecular effects decrease.

Acknowledgment

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Nomenclature

A	Hamaker constant
c	Speed of light
E	Young's modulus
f	Fringing field parameter
F_e	Electrostatic force per unit length
F_3	Van der Waals force per unit length
F_4	Casimir force per unit length
$\bar{h}$	Plank's constant divided by 2π
$\hbar$	Auxiliary parameter
$H(t)$	Auxiliary function
I	Inertia moment of the cross-section
$\mathcal{L}\{\}$	Auxiliary linear operator
$\hat{N}$	Axial load due to residual stresses
$\mathcal{N}\{\}$	Nonlinear operator
q	Auxiliary parameter

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Appendix A

$$\frac{f\beta}{1-W} = f\beta(1+W+W^2+W^3+W^4+W^5 + W^6+\cdots),$$

$$\frac{\beta}{(1-W)^2} = \beta(1+2W+3W^2+4W^3+5W^4 + 6W^5+7W^6+\cdots),$$

$$\frac{\lambda_3}{(1-W)^3} = \lambda_3(1+3W+6W^2+10W^3 + 15W^4+21W^5+28W^6+\cdots),$$

$$\frac{\lambda_4}{(1-W)^4} = \lambda_4(1+4W+10W^2+20W^3 + 35W^4+56W^5+84W^6+\cdots).$$

Appendix B

Considering the effect of van der Waals force:

$$M = \int_0^1 w^2 dx,$$

$$c_0 = -(\beta+\beta f+\lambda_3)\int_0^1 w dx,$$

$$c_1 = -(\beta+\beta f+3\lambda_3)\int_0^1 w^2 dx + \int_0^1 w^{(4)} w dx - N\int_0^1 w^{(2)} w dx,$$

$$c_2 = -(\beta+\beta f+6\lambda_3)\int_0^1 w^3 dx,$$

$$c_3 = -(\beta+\beta f+10\lambda_3)\int_0^1 w^4 dx - \alpha\int_0^1 w^{(2)} w dx \int_0^1 {w^{(1)}}^2 dx,$$

$$c_4 = -(\beta+\beta f+15\lambda_3)\int_0^1 w^5 dx,$$

$$c_5 = -(\beta+\beta f+21\lambda_3)\int_0^1 w^6 dx,$$

$$c_6 = -(\beta+\beta f+28\lambda_3)\int_0^1 w^7 dx.$$

Considering the effect of Casimir force:

$$M = \int_0^1 w^2 dx,$$

$$c_0 = -(\beta+\beta f+\lambda_4)\int_0^1 w dx,$$

$$c_1 = -(\beta+\beta f+4\lambda_4)\int_0^1 w^2 dx + \int_0^1 w^{(4)} w dx - N\int_0^1 w^{(2)} w dx,$$

$$c_2 = -(\beta + \beta f + 10\lambda_4) \int_0^1 w^3 dx,$$

$$c_3 = -\left(\beta + \beta f + 20\lambda_4\right) \int_0^1 w^4 dx$$
$$- \alpha \int_0^1 w^{(2)} w dx \int_0^1 w^{(1)^2} dx,$$

$$c_4 = -(\beta + \beta f + 35\lambda_4) \int_0^1 w^5 dx,$$

$$c_5 = -(\beta + \beta f + 56\lambda_4) \int_0^1 w^6 dx,$$

$$c_6 = -(\beta + \beta f + 84\lambda_4) \int_0^1 w^7 dx.$$

Biographies

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