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Comparison of solutions of systems of delay differential equations using Taylor collocation method, Lambert W function and variational iteration method

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Abstract. In this paper, solution of systems of delay differential equations, with initial conditions, using numerical methods, including the Taylor collocation method, the Lambert W function and the variational iteration method, is considered. We have endeavored to show the most appropriate method by comparing the solutions of this system of equations with different types of methods. All numerical computations have been performed on the computer algebraic system, Matlab.

1. Introduction

Initial value problems are used frequently in mathematical modeling when solving problems in everyday life, such that:

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \geq t_0 \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where t_0 is the starting point and y_0 is the initial value. For instance, suppose we wish to estimate the amount of population growth in a community. Firstly, we assume that there is no kind of external influence in this group, as if isolated in a closed box. Let $y(t)$ show the amount of the population at time t, and that the speed of growth is proportional to the current population at that moment. We denote this rate with "k" constant. In this case, if the change of population is shown by $y'(t)$, we can rewrite System (1) as follows [1]:

$$\begin{cases} y'(t) = k.y(t), \quad t \geq t_0 \\ y(t_0) = y_0 \end{cases}$$

The delays are always ignored in the systems to be modeled using ordinary differential equations. However, very small amounts of delay in the system can cause large changes in the current case of the system. So, while modeling the majority of encountered problems, the use of delay differential equations is more real [2-4].

In previous modeling, in order to determine population growth, it was accepted that the rate is only proportional to the current population. But, generally, the previous state of the system can significantly affect its future status. We use amounts of delay to indicate the status of systems in the past, and, thus, when modeling the systems, we also take into account the dependencies of systems on the past. In this case, when we accept that population change in the community is commensurate with the previous population at a certain period of time, (τ), rather than the current population, we obtain the delay differential equation [1], as follows:

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$$\begin{cases} y'(t) = k.y(t-\tau), & t \geq t_0, \quad \tau > 0 \\ y(t_0) = \varphi(t), & t_0 - \tau \leq t \leq t_0 \end{cases}$$

2. Description of methods

2.1. Lambert W function

In this section, we examine the first order (scalar), linear and homogeneous delay differential equation system, such that:

$$y'(t) + A(t)y(t-\tau) + B(t)y(t) = 0, \qquad \tau > 0. \quad (2)$$

In this system, A and B are $n \times n$ type matrices of real value functions, depending on the t variable, and $\tau > 0$ is a real value constant. If A and B are constant, real-valued matrices in Eq. (2), then:

$$y'(t) + Ay(t-\tau) + By(t) = 0, \qquad \tau > 0. \quad (3)$$

Here, in order to obtain the characteristic equation of System (2), we assume that $y = e^{st}$ is the solution of Eq. (2). This solution provides the given equality. In that case:

$$se^{st} + Ae^{s(t-\tau)} + Be^{st} = 0.$$

Dividing both sides of the equation by e^{st} yields:

$$sI + Ae^{-s\tau} + B = 0.$$

Rearranging the equation, we get:

$$sI = -Ae^{-s\tau} - B.$$

Multiplying by $e^{s\tau}$, τ and $e^{B\tau}$, respectively, we obtain:

$$sIe^{s\tau} = -A - Be^{s\tau},$$

$$(sI)\tau e^{s\tau} = (-A)\tau - B\tau e^{s\tau},$$

$$(s\tau)Ie^{(s\tau)I} = (-A)\tau - B\tau e^{s\tau I},$$

$$(s\tau)Ie^{(s\tau)I} + B\tau e^{s\tau I} = (-A)\tau,$$

$$(sI+B)\tau e^{(s\tau)I} = (-A)\tau,$$

$$(sI+B)\tau e^{(sI+B)\tau} = (-A)\tau e^{B\tau}.$$

By the definition of the Lambert W Function, we obtain the characteristic equation as:

$$W((sI+B)\tau)e^{W((sI+B)\tau)} = (sI+B)\tau.$$

Rearranging the equation:

$$(sI+B)\tau = W\left(-A\tau e^{B\tau}\right),$$

$$sI = \frac{1}{\tau}W\left(-A\tau e^{B\tau}\right) - B.$$

Particularly for $B = 0$, we can get:

$$sI = \frac{1}{\tau}W(-A\tau).$$

In this case, the general solution of System (2) is determined as:

$$y(t) = \sum_{-\infty}^{\infty} c_k e^{\left[\frac{1}{\tau}W_k(-A\tau e^{B\tau})-B\right]t}.$$

In this equation, the matrix of the coefficients, c_k, is an $n \times 1$ type, and it is calculated by means of the initial function [5-8].

2.2. Taylor collocation method

The Taylor Collocation Method is an effective method for finding approximate solutions of systems of linear, high-order delay, differential equations, in the form:

$$\sum_{r=0}^{m}\sum_{i=1}^{k} P_{ji}^{r}(t)y_i^{(r)}(\lambda t + \mu) = f_j(t),$$

$$j = 1, 2, \cdots, k, \quad (4)$$

under mixed conditions, defined as:

$$\sum_{j=0}^{m-1} a_{rj}^{n}y_n^{(j)}(a) + b_{rj}^{n}y_n^{(j)}(b) + c_{rj}^{n}y_n^{(j)}(c) = \lambda_{nr},$$

$$a \leq c \leq b,$$

$$r = 0, 1, 2, \cdots, m-1,$$

$$n = 1, 2, \cdots, k,$$

where $y_i(t)$ is an unknown function; known functions, $P_{ji}^{n}(t)$ and $f_j(t)$, are defined on interval $a \leq t \leq b$, and also, a_{rj}, b_{rj}, c_{rj} and λ_{nr} are appropriate constants.

Our main purpose is to find the approximate solutions of system (4) expressed in the truncated Taylor series form:

$$y_i(t) = \sum_{n=0}^{N} y_{in}(t-c)^n,$$

$$y_{in} = \frac{y_i^{(n)}(c)}{n!},$$

$$i = 1, 2, \cdots k,$$

$$a \leq t \leq b, \quad (5)$$

where y_{in} ($n = 0, 1, \cdots, N$ and $i = 1, 2, \cdots, k$) are unknown coefficients, and N is any positive integer, such that $N \geq m$.

For fundamental relations and methods of solutions, we refer to [9].

2.3. Variational iteration method

According to the variational iteration method, we consider the following differential equation:

$$Lu + Nu = f(x),$$

where L is a linear operator, N is a non-linear operator and $f(x)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(Lu_n(s)+N\tilde{u}_n(s)-f(s))ds,$$

where, λ is a general Lagrangian multiplier, which can be identified optimally via variational theory. The second term on the right is called the correction, and $\tilde{u}_n$ is considered a restricted variation, i.e. $\delta\tilde{u}_n = 0$ [10-12].

3. Numerical examples

Example 1

$$y_1' = -y_2(t-1),$$

$$y_2' = 2y_1(t-2) + y_3(t-2),$$

$$y_3' = 3y_2(t-1).$$

Let the delay differential equation system be solved using the Lambert W function.

Solution: We write:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix},$$

where:

$$y'(t) = \frac{dy}{dt} = \sum_{j=1}^{3} A_j y(t-\tau_j) = A_1 y(t-\tau_1)$$

$$+ A_2 y(t-\tau_2) + A_3(t-\tau_3).$$

Particular solutions of this system of equations are types of $y(t) = ce^{st}$, and it should be:

$$\det\left(sI - \sum_{j=1}^{3} A_j y(t-\tau_j)\right) = 0,$$

to get non-zero solutions. Thus, this particular solution is calculated as:

$$y(t) = ce^{st}, \qquad y_1(t-2) = ce^{s(t-2)},$$

$$y_2(t-1) = ce^{s(t-1)}, \qquad y_3(t-2) = ce^{s(t-2)}.$$

When the matrices of these equations are set up, we obtain:

$$\det\begin{pmatrix} sI - A_1 y_1(t-\tau_1) \\ sI - A_2 y_2(t-\tau_2) \\ sI - A_3 y_3(t-\tau_3) \end{pmatrix} = 0.$$

So:

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \left(A_1 ce^{s(t-2)} + A_2 ce^{s(t-1)}\right.$$

$$\left. + A_3 ce^{s(t-2)}\right),$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \left(A_1 ce^{st}.e^{-2s} + A_2 ce^{st}.e^{-s}\right.$$

$$\left. + A_3 ce^{st}.e^{-2s}\right),$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}_{3x1} \begin{pmatrix} e^{-2s} & e^{-s} & e^{-2s} \end{pmatrix}_{1x3},$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} A_1 e^{-2s} & A_1 e^{-s} & A_1 e^{-2s} \\ A_2 e^{-2s} & A_2 e^{-s} & A_2 e^{-2s} \\ A_3 e^{-2s} & A_3 e^{-s} & A_3 e^{-2s} \end{pmatrix},$$

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0.e^{-2s} & -1.e^{-s} & 0.e^{-2s} \\ 2.e^{-2s} & 0.e^{-s} & 1.e^{-2s} \\ 0.e^{-2s} & 3.e^{-s} & 0.e^{-2s} \end{pmatrix},$$

are found. Thus:

$$\det\begin{pmatrix} s & e^{-s} & 0 \\ -2e^{-2s} & s & -e^{-2s} \\ 0 & -3e^{-s} & s \end{pmatrix} = 0.$$

So:

$$s(s^2 - 3e^{-3s} + 2e^{-3s}) = 0, \qquad s_1 = 0,$$

and:

$$s^2 = e^{-3s},$$

is calculated. Hence, we also get:

$$s_2 = \frac{2}{3}W\left(\frac{3}{2}\right), \qquad s_3 = \frac{2}{3}W\left(-\frac{3}{2}\right).$$

So, the general solution to s_2 is:

$$y(t) = \cdots + c_{-1}e^{\frac{2}{3}W_{-1}\left(\frac{3}{2}\right)t} + c_0 e^{\frac{2}{3}W_0\left(\frac{3}{2}\right)t}$$

$$+ c_1 e^{\frac{2}{3}W_1\left(\frac{3}{2}\right)t} + \cdots = \cdots$$

$$+ c_{-1}e^{\frac{2}{3}(-1.12168-4.46634i)t} + c_0 e^{\frac{2}{3}(0.725861)t}$$

$$+ c_1 e^{\frac{2}{3}(-1.12168+4.46634i)t} + \cdots$$

and the general solution to s_3 is also:

$$y(t)=\cdots+c_{-1}e^{\frac{2}{3}W_{-1}(-\frac{3}{2})t}+c_0e^{\frac{2}{3}W_0(-\frac{3}{2})t}$$
$$+c_1e^{\frac{2}{3}W_1(-\frac{3}{2})t}+\cdots=\cdots$$
$$+c_{-1}e^{\frac{2}{3}(-1.65090-7.64120i)t}$$
$$+c_0e^{\frac{2}{3}(-0.03278+1.54964i)t}$$
$$+c_1e^{\frac{2}{3}(-1.65090+7.64120i)t}+\cdots$$

3.1. Example 2

$$y_1'=-y_2(t-1),$$
$$y_1(0)=0,$$
$$y_2'=2y_1(t-2)+y_3(t-2),$$
$$y_2(0)=-2,$$
$$y_3'=2y_2(t-1),$$
$$y_3(0)=-2.$$

Let the delay differential equation system be solved using the Taylor collocation method for $-2\leq t\leq 4$.

Solution: Once the set of the equation is set as follows:

$$y_1'+y_2(t-1)=0,$$
$$y_2'-2y_1(t-2)-y_3(t-2)=0,$$
$$y_3'-2y_2(t-1)=0,$$

the equation may be written in matrix form as:

$$\begin{bmatrix}0&1&0\\0&0&0\\0&-2&0\end{bmatrix}\begin{bmatrix}y_1(t-1)\\y_2(t-1)\\y_3(t-1)\end{bmatrix}$$
$$+\begin{bmatrix}0&0&0\\-2&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}y_1(t-2)\\y_2(t-2)\\y_3(t-2)\end{bmatrix}$$
$$+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_1'(t)\\y_2'(t)\\y_3'(t)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus, P_1 and P_2 are obtained, as shown in Box I.

If we take $N=3$ for $-2\leq t\leq 4$, we get collocation points as $t_0=-2$, $t_1=0$, $t_2=2$ and $t_3=4$.

Similarly, we write the equation as shown in Box II. So, we get the equation:

$$WA=F.$$

Here the equations, as shown in Box III, are obtained. Now, let $\bar{W}$ and $\bar{F}$ be written using initial conditions as shown in Box IV.

Now, by equation $\bar{W}A=\bar{F}$, we find:

$$A=\mathrm{inv}\left(\bar{W}\right)*\bar{F}=\begin{pmatrix}0\\0.0000\\1.0000\\0.0000\\-2.0000\\-2.0000\\-0.0000\\-0.0000\\-2.0000\\-0.0000\\-2.0000\\-0.0000\end{pmatrix},$$

using Matlab computer programming. Consequently,

$$P_1=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\-2&0&-1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&-2&0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-2&0&-1&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&-2&0&-1\\0&0&0&0&0&0&0&0&0&0&0&0\end{pmatrix}$$

and $P_2=I_{12\times 12}$.

Box I

$$\left(P_0T\tilde{B}(1,-1)+P_1T\tilde{B}(1,-2)+P_2T\tilde{B}(1,0)\tilde{B}\right)A=F, \qquad \text{where:}$$

$$T=\begin{pmatrix}
1 & -2 & 4 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8\\
1 & 4 & 16 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 4 & 16 & 64 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 16 & 64
\end{pmatrix},$$

$$\tilde{B}(1,-1)=\begin{pmatrix}
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\tilde{B}(1,-2)=\begin{pmatrix}
1 & -2 & 4 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -4 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -4 & 12 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 12\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\tilde{B}(1,0)=I_{12\times 12} \qquad \text{and:} \qquad \tilde{B}=\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Box II

$$W = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -2 & 6 & -18 & 54 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -2 & 2 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 & 0 & 1 & 4 & 12 \\ 0 & 1 & 8 & 48 & 1 & 3 & 9 & 27 & 0 & 0 & 0 & 0 \\ -2 & -4 & -8 & -16 & 0 & 1 & 8 & 48 & -1 & -2 & -4 & -8 \\ 0 & 0 & 0 & 0 & -2 & -6 & -18 & -54 & 0 & 1 & 8 & 48 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Box III

$$\bar{W} = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -2 & 6 & -18 & 54 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -2 & 2 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 & 0 & 1 & 4 & 12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \bar{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ -2 \end{pmatrix}.$$

Box IV

we write:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} -2 & -2 & 0 & 0 \end{bmatrix}^T,$$

$$A_3 = \begin{bmatrix} -2 & 0 & -2 & 0 \end{bmatrix}^T.$$

Hence, the system of exact solutions is obtained as:

$$y_1(t) = t^2,$$

$$y_2(t) = -2t - 2,$$

$$y_3(t) = -2t^2 - 2.$$

Example 3

$$y_1' = -y_2(t-1), \qquad y_1(0) = 0,$$

$$y_2' = 2y_1(t-2) + y_3(t-2), \qquad y_2(0) = -2,$$

$$y_3' = 3y_2(t-1), \qquad y_3(0) = -2.$$

Let the delay differential equation system be solved using the Taylor Collocation Method for $-2 \leq t \leq 4$.

Solution: If the same process is used as in the previous example, we write:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ y_3(t-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-2) \\ y_2(t-2) \\ y_3(t-2) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, we find the equation shown in Box V.

$$W = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -3 & 9 & -27 & 81 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -3 & 3 & -3 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 & 0 & 1 & 4 & 12 \\ 0 & 1 & 8 & 48 & 1 & 3 & 9 & 27 & 0 & 0 & 0 & 0 \\ -2 & -4 & -8 & -16 & 0 & 1 & 8 & 48 & -1 & -2 & -4 & -8 \\ 0 & 0 & 0 & 0 & -3 & -9 & -27 & -81 & 0 & 1 & 8 & 4 \end{pmatrix}.$$

Box V

$$\bar{W} = \begin{pmatrix} 0 & 1 & -4 & 12 & 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ -2 & 8 & -32 & 128 & 0 & 1 & -4 & 12 & -1 & 4 & -16 & 64 \\ 0 & 0 & 0 & 0 & -2 & 6 & -18 & 54 & 0 & 1 & -4 & 12 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & -8 & 16 & 0 & 1 & 0 & 0 & -1 & 2 & -4 & 8 \\ 0 & 0 & 0 & 0 & -2 & 2 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 12 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & -2 & 0 & 1 & 4 & 12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Box VI

By using initial conditions the equation as shown in Box VI is obtained. Thus, we get the coefficients matrix as:

$$A = \operatorname{inv}\left(\bar{W}\right) * \bar{F} = \begin{pmatrix} 0 \\ 0.8910 \\ 0.4265 \\ 0.0758 \\ -2.0000 \\ -1.3175 \\ -0.1991 \\ 0.0095 \\ -2.0000 \\ -2.6730 \\ -1.2796 \\ -0.2275 \end{pmatrix}.$$

Therefore, the solutions for $-2 \leq t \leq 4$ and $N = 3$ are:

$$y_1(t) = 0.0758t^3 + 0.4265t^2 + 0.8910,$$

$$y_2(t) = 0.0095t^3 - 0.1991t^2 - 1.3175t - 2,$$

$$y_3(t) = -0.2275t^3 - 1.2796t^2 - 2.6730t - 2.$$

Example 4

$$y_1'(t-1) + y_2'(t-1) = 2t, \qquad y_1(0) = 0,$$

$$y_1'(t-1) - y_3'(t-1) = 2t - 1, \qquad y_2(0) = 0,$$

$$y_1'(t-1) + y_3'(t-1) = t - 1, \qquad y_3(0) = 0.$$

Let the delay differential equation system be solved using VIM for $-3 \leq t \leq 4$.

Solution: We know that $y_1''(t-1)+y_3'(t-1) = 1$ and $y_1'(t-1) - y_3'(t-1) = 2t - 1$. They are summed up and we get:

$$y_1''(t-1) + y_1'(t-1) = 2t.$$

Now, let the variational iteration method be applied. In order to implement this method, we start the iteration by choosing $y_0 = t^2$:

$$y_{n+1}(t)=y_n(t)+\int_0^t \lambda(t,s)\{y_n''(t-1)+y_n'(t-1)-2s\}ds.$$

Hence, we find the Lagrange multiplier as $\lambda(s,t) = s-t$. If this value is substituted and the iteration is continued, we get:

$$y_0 = t^2,$$

$$y_1(t) = t^2 + \int_0^t (s-t)\{2+2(s-1)-2s\}ds = t^2,$$

$$y_2(t) = y_1(t) + \int_0^t (s-t)\{2+2(s-1)-2s\}ds = t^2,$$

$$\vdots$$

Consequently, $y_n(t) = t^2$ and $y(t) = t^2$ are found. If solution $y_1(t) = t^2$ is used, $y_2(t)$ and $y_3(t)$ can be found. Finally, all solutions are obtained as:

$$y_1(t) = t^2, \qquad y_2(t) = 2t, \qquad y_3(t) = -t.$$

Example 5

$$y_1'(t-1) + y_2'(t-1) = 2t, \qquad y_1(0) = 0,$$

$$y_1'(t-1) - y_3'(t-1) = 2t-1, \qquad y_2(0) = 0,$$

$$y_1'(t-1) + y_3'(t-1) = t-1, \qquad y_3(0) = 0.$$

Let the delay differential equation system be solved using the Taylor Collocation Method for $-3 \le t \le 4$.

Solution: First, we assume that the solution is in the form of:

$$y_i(t) = \sum_{n=0}^{3} \frac{y_i^n(0)}{n!} t^n.$$

In this system:

$$P_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_1(t) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$f(t) = \begin{bmatrix} 2t \\ 2t-1 \\ t-1 \end{bmatrix}.$$

And the Taylor collocation points are obtained as $t_0 = -3$, $t_1 = -2/3$, $t_2 = 5/3$, $t_3 = 4$.

So, we obtain the equations shown in Box VII.

By equation $\bar{W}A = \bar{F}$, we get $A = [0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 0\ -1\ 0\ 0]^T$.

Consequently, the solutions of system are $y_1(t) = t^2$, $y_2(t) = 2t$, $y_3(t) = -t$, as in the previous example [9].

$$\bar{W} = \begin{pmatrix}
0 & 1 & -8 & 48 & 0 & 1 & -8 & 48 & 0 & 0 & 0 & 0 \\
0 & 1 & -8 & 48 & 0 & 0 & 0 & 0 & 0 & -1 & 8 & -48 \\
0 & 1 & -8 & 48 & 0 & 0 & 0 & 0 & 1 & -4 & 16 & -64 \\
0 & 1 & \frac{-10}{3} & \frac{25}{3} & 0 & 1 & \frac{-10}{3} & \frac{25}{3} & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{-10}{3} & \frac{25}{3} & 0 & 0 & 0 & 0 & -1 & \frac{-5}{3} & \frac{10}{3} & \frac{-25}{3} \\
0 & 1 & \frac{-10}{3} & \frac{25}{3} & 0 & 0 & 0 & 0 & 1 & \frac{-5}{3} & \frac{25}{9} & \frac{-125}{27} \\
0 & 1 & \frac{4}{3} & \frac{4}{3} & 0 & 1 & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{4}{3} & \frac{4}{3} & 0 & 1 & 0 & 0 & 0 & -1 & -\frac{4}{3} & -\frac{4}{3} \\
0 & 1 & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

$$\bar{F} = \begin{bmatrix} -6 & -7 & -4 & \frac{-4}{3} & \frac{-7}{3} & \frac{-5}{3} & \frac{10}{3} & \frac{7}{3} & \frac{2}{3} & 0 & 0 & 0 \end{bmatrix}^T.$$

Box VII

4. Conclusions

The Lambert W function is a very useful method to obtain general solutions of systems of delay differential equations effectively and easily. It is found that the Lambert W function is faster than the Taylor collocation method but it is not enough to find the exact solutions. The Taylor collocation method, is a more appropriate method than Lambert W when we have initial conditions. But, in the Taylor collocation method, as the number of collocation points are increasing, it takes more time to obtain solutions. Using the variational iteration method, when the amounts of delay are equal in the system, and by choosing a suitable initial function, the solution can easily be obtained in a short time. But, when the system has different delays, it is hard to obtain the solutions due to the difficulty of finding the Lagrange multiplier. Thus, the variational iteration method is the most appropriate method for finding a general formula for the Lagrange multiplier.

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