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Positive solutions for eigenvalue problems of fractional differential equations with p-Laplacian

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Abstract. In this paper, we investigate the existence of positive solutions for the eigenvalue problem of nonlinear fractional differential equation with p-Laplacian operator:

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = \lambda f(u(t)),\ 0 < t < 1,$$

$$u(0) = u'(0) = u'(1) = 0,\ \phi_p(D_{0+}^{\alpha}u(0)) = (\phi_p(D_{0+}^{\alpha}u(1)))' = 0,$$

where $2 < \alpha \leqslant 3$, and $1 < \beta \leqslant 2$ are real numbers, D_{0+}^{α} and D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives, $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, 1/p + 1/q = 1, \lambda > 0$ is a parameter, and $f : (0, +\infty) \to (0, +\infty)$ is continuous. By using the properties of the Green function and the Guo–Krasnosel'skii fixed-point theorem on cones, several new existence results of at least one or two positive solutions, in terms of different eigenvalue intervals, are obtained. Moreover, the nonexistence of positive solution, in terms of parameter λ, is also considered.

1. Introduction

In recent years, many researchers have shown their interest in fractional differential equations. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and applications, such as in economics, engineering and other fields. Many papers and books on fractional calculus, including fractional differential equations, have appeared [1-6].

Recently, much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary value problems of fractional differential equations using techniques of nonlinear analysis (fixed-point theorems [7-24], upper and lower solution methods [25-27], the fixed-point index [28-29], coincidence theory [30], etc.).

Xu et al. [21] considered the existence of positive solutions for the following problem:

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $3 < \alpha \leqslant 4$ is a real number, $f \in C([0,1] \times [0, +\infty), (0, +\infty))$ and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. As an application of the Green function, they obtained some existence criteria for one or two positive solutions for singular and nonsingular boundary value problems by means of the Leray-Schauder nonlinear alternative, the Guo-Krasnosel'skii fixed-point theorem and a mixed monotone method.

Zhao et al. [17] studied the existence of positive solutions for the boundary value problem of a nonlinear

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fractional differential equation:

$$D_{0+}^{\alpha}u(t) + \lambda f(u(t)) = 0,\ 0 < t < 1,$$

$$u(0) = u(1) = u'(0) = 0,$$

where $2 < \alpha \leqslant 3$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, λ is a positive parameter, and $f : (0,+\infty) \to (0,+\infty)$ is continuous. Using properties of the Green function and the Guo-Krasnosel'skii fixed-point theorem on cones, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

El-Shahed [24] studied the existence and nonexistence of positive solutions to the nonlinear fractional boundary value problem:

$$D_{0+}^{\alpha}u(t) + \lambda a(t) f(u(t)) = 0,\ 0 < t < 1,\ 2 < \alpha < 3,$$

$$u(0) = u'(0) = u'(1) = 0,$$

where λ is a positive parameter, $a : (0,1) \to [0,+\infty)$ and $f : [0,+\infty) \to [0,+\infty)$ are continuous. D_{0+}^{α} is the standard Riemann–Liouville fractional derivative. Using the Guo–Krasnosel'skii fixed-point theorem on cones, some existence and nonexistence results for positive solutions on parameter λ are obtained.

To the best of our knowledge, some results were obtained dealing with the existence of positive solutions for the eigenvalue problem of fractional differential equations (see [27,31-33]), but very little is found in the literature regarding eigenvalue problems of fractional differential equations with a p-Laplacian operator. Its theories and applications seem to have only just been initiated. Therefore, in order to enrich the theoretical knowledge of the above, in this paper, we investigate the following p-Laplacian fractional differential equation boundary value problem:

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = \lambda f(u(t)),\ 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_{0+}^{\alpha}u(0)) = (\phi_p(D_{0+}^{\alpha}u(1)))' = 0, \tag{2}$$

where $2 < \alpha \leqslant 3, 1 < \beta \leqslant 2$, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Riemann–Liouville fractional derivatives, and $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, 1/p + 1/q = 1, \lambda > 0$ is a parameter. $f : (0,+\infty) \to (0,+\infty)$ is continuous. Using properties of the Green function and the Guo-Krasnosel'skii fixed-point theorem on cones, several new existence and nonexistence results for positive solutions, in terms of different values of parameter λ, are obtained. Moreover, the existence of two positive solutions on the boundary value problem, (Eqs. (1) and (2)), is also considered. As applications, examples are presented to illustrate the main results.

The rest of this paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, some sufficient conditions for the existence of at least one or two positive solutions for boundary value problems (Eqs. (1) and (2)) are investigated. In Section 4, we derive several nonexistence results for positive solutions on parameter λ. As applications, examples are presented to illustrate our main results in Section 5.

2. Preliminaries and lemmas

For the convenience of the reader, we give some background on fractional calculus theory to facilitate analysis of the problem (Eqs. (1) and (2)). This information can be found in recent literature (see [4,24,34-36]).

Definition 1 [4]. *The fractional integral of order $\alpha > 0$ of function $y : (0,+\infty) \to \mathbb{R}$ is given by:*

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2 [4]. *The $R-L$ fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,+\infty) \to \mathbb{R}$ is given by:*

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Lemma 1 [4]. *Let $\alpha > 0$. Then the fractional differential equation;*

$$D_{0+}^{\alpha}u(t) = 0,$$

has solutions of the form:

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

$$c_i \in \mathbb{R}, i = 1,2,\cdots,n$$

where n is the smallest integer greater than, or equal to, α.

Lemma 2 [4]. *Let $\alpha > 0$. Assume that $u, D_{0+}^{\alpha}u \in L(0,1)$. Then,*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

holds for some $c_i \in \mathbb{R}, i = 1,2,\cdots,n$, where n is the smallest integer greater than, or equal to, α.

Lemma 3 [27]. *Let $y \in C[0,1]$ and $2 < \alpha \leqslant 3$. Then, the fractional differential equation boundary value problem:*

$$D_{0+}^{\alpha}u(t) + y(t) = 0, \qquad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = u'(1) = 0, \tag{4}$$

has a unique solution:

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where the Green function is:

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \leqslant t, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & t \leqslant s. \end{cases} \tag{5}$$

Lemma 4. *Let $2 < \alpha \leqslant 3, 1 < \beta \leqslant 2$. Then the fractional differential equation boundary value problem:*

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = \lambda f(u(t)),\ 0 < t < 1, \tag{6}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_{0+}^{\alpha}u(0)) = (\phi_p(D_{0+}^{\alpha},u(1)))' = 0, \tag{7}$$

has a unique solution:

$$u(t) = \int_0^1 G(t,s)\phi_q\Big(\lambda\int_0^1 H(s,\tau)f(u(\tau))d\tau\Big)ds,$$

where:

$$H(s,\tau) = \begin{cases} \frac{s^{\beta-1}(1-\tau)^{\beta-2}-(s-\tau)^{\beta-1}}{\Gamma(\beta)}, & \tau \leqslant s, \\ \frac{s^{\beta-1}(1-\tau)^{\beta-2}}{\Gamma(\beta)}, & s \leqslant \tau, \end{cases} \tag{8}$$

$G(t,s)$ is defined as Eq. (5).

Proof. From Lemma 2 and $1 < \beta \leqslant 2$, we have:

$$I_0^{\beta} + D_0^{\beta} + (\phi_p(D_0^{\alpha} + u(t)))$$

$$= \phi_p(D_0^{\alpha} + u(t)) + c_1t^{\beta-1} + c_2t^{\beta-2}$$

for some $c_1, c_2 \in \mathbb{R}$.

In view of Eq. (6), we obtain:

$$I_0^{\beta} + D_0^{\beta} + (\phi_p(D_0^{\alpha} + u(t))) = \lambda I_0^{\beta} + f(u(t)).$$

Therefore:

$$\phi_p(D_0^{\alpha} + u(t)) = \lambda I_0^{\beta} + f(u(t)) + c_1t^{\beta-1} + c_2t^{\beta-2}$$

for some $c_1, c_2 \in \mathbb{R}$,

that is:

$$\phi_p(D_0^{\alpha} + u(t)) = \lambda\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}f(u(\tau))d\tau + c_1t^{\beta-1} + c_2t^{\beta-2}.$$

By the boundary conditions:

$$\phi_p(D_0^{\alpha} + u(0)) = (\phi_p(D_0^{\alpha} + u(1)))' = 0,$$

we have:

$$c_2 = 0, c_1 = -\lambda\int_0^1 \frac{(1-\tau)^{\beta-2}}{\Gamma(\beta)}f(u(\tau))d\tau.$$

Therefore, the solution, $u(t)$, of the fractional differential equation boundary value problem, Eqs. (6) and (7), satisfies:

$$\begin{aligned}\phi_p(D_0^{\alpha} + u(t)) =& \lambda\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}f(u(\tau))d\tau \\ &- \lambda\int_0^1 \frac{t^{\beta-1}(1-\tau)^{\beta-2}}{\Gamma(\beta)}f(u(\tau))d\tau \\ &= -\lambda\int_0^1 H(t,\tau)f(u(\tau))d\tau.\end{aligned}$$

Consequently, $D_{0+}^{\alpha}u(t) + \phi_q\big(\lambda\int_0^1 H(t,\tau)f(u(\tau))d\tau\big) = 0$. Thus, the fractional differential equation boundary value problem, Eqs. (6) and (7), is equivalent to the following problem:

$$D_0^{\alpha} + u(t) + \phi_q\Big(\lambda\int_0^1 H(t,\tau)f(u(\tau))d\tau\Big) = 0,$$

$$0 < t < 1,$$

$$u(0) = u'(0) = u'(1) = 0.$$

Lemma 3 implies that the fractional differential equation boundary value problem, Eqs. (6) and (7), has a unique solution:

$$u(t) = \int_0^1 G(t,s)\phi_q\Big(\lambda\int_0^1 H(s,\tau)f(u(\tau))d\tau\Big)ds.$$

The proof is completed.

Lemma 5 [27]. *Let $2 < \alpha \leqslant 3, 1 < \beta \leqslant 2$. The functions $G(t,s)$ and $H(s,\tau)$ defined by Eqs. (5) and (6), respectively, are continuous on $[0,1] \times [0,1]$ and satisfy:*

1. $G(t,s) \geqslant 0,\ H(s,\tau) \geqslant 0,$ *for* $t,s,\tau \in [0,1]$;
2. $G(t,s) \leqslant G(1,s),\ H(s,\tau) \leqslant H(\tau,\tau),$ *for* $t,s,\tau \in [0,1]$;
3. $G(t,s) \geqslant k(t)G(1,s),\ H(s,\tau) \geqslant s^{\beta-1}H(1,\tau),$ *for* $t,s,\tau \in (0,1)$, *where* $k(t) = t^{\alpha-1}$.

Lemma 6 [35]. *Let X be a Banach space, and $P \subset X$ be a cone in X. Assume Ω_1, and Ω_2 are open subsets of X, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator, such that, either:*

(C1) $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_2$, or
(C2) $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_2$

holds. Then, S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence

In this section, we investigate the existence of at least one or two positive solutions for the nonlinear fractional differential equation boundary value problem, Eqs. (1) and (2).

Let the Banach space $E = C[0,1]$ be endowed with $||u|| = \max\limits_{0\leqslant t\leqslant 1} |u(t)|$. Define the cone $P \subset E$ by:

$$P = \{u \in E \mid u(t) \geqslant k(t)||u||, t \in [0,1]\}.$$

Lemma 7. *Let $T_\lambda : P \to E$ be the operator defined by:*

$$T_\lambda u(t) := \int_0^1 G(t,s)\phi_q\Big(\lambda \int_0^1 H(s,\tau)f(u(\tau))d\tau\Big)ds.$$

Then, $T_\lambda : P \to P$ is completely continuous.

Proof. By Lemma 5, we have:

$$||T_\lambda u(t)|| \leqslant! \int_0^1 G(1,s)\phi_q\Big(\lambda \int_0^1 H(s,\tau)f(u(\tau))d\tau\Big)ds.$$

$$(T_\lambda u)(t) \geqslant \int_0^1 t^{\alpha-1}G(1,s)$$

$$\phi_q\big(\lambda \int_0^1 H(s,\tau)f(u(\tau))d\tau\big)ds \geqslant k(t)||T_\lambda u||.$$

Thus, $T_\lambda(P) \subset P$. In view of the nonnegativeness and continuity of $G(t,s)$, $H(s,\tau)$ and $f(u(t))$, $T_\lambda : P \to P$ is continuous.

Let $\Omega \subset P$ be bounded, i.e. there exists a positive constant, $M > 0$, such that $||u|| \leqslant M$ for all $u \in \Omega$. Let $L = \max\limits_{0\leqslant u\leqslant M} |f(u)| + 1$, then, for $u \in \Omega$, we have:

$$|T_\lambda u(t)| = \Big|\int_0^1 G(t,s)\phi_q\big(\lambda \int_0^1 H(s,\tau)f(u(\tau))d\tau\big)ds\Big|$$

$$\leqslant \phi_q(\lambda L)\int_0^1 G(t,s)\phi_q\big(\int_0^1 H(s,\tau)d\tau\big)ds$$

$$\leqslant \phi_q(\lambda L)\int_0^1 G(1,s)\phi_q\big(\int_0^1 H(\tau,\tau)d\tau\big)ds < +\infty.$$

Hence, $T_\lambda(\Omega)$ is uniformly bounded.

On the other hand, since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exists a constant, $\delta > 0$, such that $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$ imply:

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{\phi_q(\lambda L)\phi_q\big(\int_0^1 H(\tau,\tau)d\tau\big)}.$$

Then, for all $u \in \Omega$:

$$|T_\lambda u(t_2) - T_\lambda u(t_1)| \leqslant \int_0^1 |G(t_2,s) - G(t_1,s)|$$

$$\phi_q\big(\lambda \int_0^1 H(s,\tau)f(u(\tau))d\tau\big)ds \leqslant \phi_q(\lambda L)$$

$$\int_0^1 |G(t_2,s) - G(t_1,s)|\phi_q\big(\int_0^1 H(\tau,\tau)d\tau\big)ds$$

$$= \phi_q(\lambda L)\phi_q\big(\int_0^1 H(\tau,\tau)d\tau\big)\int_0^1 |G(t_2,s)$$

$$- G(t_1,s)|ds < \varepsilon,$$

that is to say, $T_\lambda(\Omega)$ is equicontinuous. Using the Arzela-Ascoli theorem, $T_\lambda : P \to P$ is completely continuous. The proof is completed.

For the convenience of the reader, we denote:

$$F_0 = \lim_{u\to 0^+} \sup \frac{f(u)}{\phi_p(u)}, \quad F_\infty = \lim_{u\to +\infty} \sup \frac{f(u)}{\phi_p(u)},$$

$$f_0 = \lim_{u\to 0^+} \inf \frac{f(u)}{\phi_p(u)}, \quad f_\infty = \lim_{u\to +\infty} \inf \frac{f(u)}{\phi_p(u)},$$

$$A_1 = \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds,$$

$$A_2 = \int_0^1 \phi_q(s^{\beta-1})G(1,s)$$

$$\phi_q\Big(\int_0^1 \phi_p(\tau^{\alpha-1})H(1,\tau)d\tau\Big)ds,$$

$$A_3 = \int_0^1 \phi_q(s^{\beta-1})G(1,s)\phi_q\Big(\int_0^1 H(1,\tau)d\tau\Big)ds.$$

Theorem 1. *If there exists $\delta \in (0,1)$ such that $\phi_p(k(\delta))f_\infty\phi_p(A_2) > F_0\phi_p(A_1)$ holds, then, for each;*

$$\lambda \in \big((\phi_p(k(\delta))f_\infty\phi_p(A_2))^{-1},\ (F_0\phi_p(A_1))^{-1}\big), \qquad (9)$$

the fractional differential equation boundary value problem, Eqs. (1) and (2), has at least one positive solution. Here, we impose $(\phi_p(k(\delta))f_\infty\phi_p(A_2))^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0\phi_p(A_1))^{-1} = +\infty$, if $F_0 = 0$.

Proof. Let λ satisfy Relation (9), and $\varepsilon > 0$ be such that:

$$(\phi_p(k(\delta))(f_\infty - \varepsilon)\phi_p(A_2))^{-1} \leqslant \lambda \leqslant ((F_0 + \varepsilon)\phi_p(A_1))^{-1}. \quad (10)$$

In order to apply Lemma 6, we separate the proof into the following two steps:

Step 1. By the definition of F_0, we see that there exists $r_1 > 0$, such that:

$$f(u) \leqslant (F_0 + \varepsilon)\phi_p(u), \quad \text{for } 0 < u \leqslant r_1. \quad (11)$$

So, if $u \in P$ with $||u|| = r_1$, from Relations (10) and (11), we obtain:

$$||T_\lambda u(t)|| \leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)f(u(\tau))d\tau\big)ds$$

$$\leqslant \phi_q(\lambda)\int_0^1 G(1,s)$$

$$\phi_q\big(\int_0^1 H(\tau,\tau)(F_0+\varepsilon)\phi_p(r_1)d\tau\big)ds$$

$$= \phi_q(\lambda)\phi_q(F_0+\varepsilon)r_1 A_1 \leqslant r_1 = ||u||.$$

So, we choose $\Omega_1 = \{u \in E \mid ||u|| < r_1\}$, then:

$$||T_\lambda u|| \leqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_1. \quad (12)$$

Step 2. By the definition of f_∞, let $r_3 > 0$ be such that:

$$f(u) \geqslant (f_\infty - \varepsilon)\phi_p(u), \quad \text{for } u \geqslant r_3. \quad (13)$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1,\ r_3\}$, then by Eqs. (10) and (13), we obtain:

$$||T_\lambda u(t)|| \geqslant T_\lambda u(\delta)$$

$$= \int_0^1 G(\delta,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)f(u(\tau))d\tau\right)ds$$

$$\geqslant \int_0^1 k(\delta)G(1,s)$$

$$\phi_q\left(\lambda\int_0^1 s^{\beta-1}H(1,\tau)f(u(\tau))d\tau\right)ds$$

$$\geqslant \int_0^1 k(\delta)G(1,s)$$

$$\phi_q\left(\lambda\int_0^1 s^{\beta-1}H(1,\tau)(f_\infty-\varepsilon)\phi_p(u(\tau))d\tau\right)ds$$

$$\geqslant \int_0^1 k(\delta)G(1,s)$$

$$\phi_q\big(\lambda\int_0^1 s^{\beta-1}H(1,\tau)(f_\infty-\varepsilon)$$

$$\phi_p(\tau^{\alpha-1}||u||)d\tau\big)ds = \phi_q(\lambda)k(\delta)$$

$$\phi_q(f_\infty-\varepsilon)A_2||u|| \geqslant ||u||.$$

Thus, if we choose $\Omega_2 = \{u \in E \mid ||u|| < r_2\}$, then:

$$||T_\lambda u|| \geqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_2. \quad (14)$$

Now, from Relations (12) and (14) and Lemma 6, we see that T_λ has a fixed point, $u \in P \cap (\overline{\Omega}_2\backslash\Omega_1)$, with $r_1 \leqslant ||u|| \leqslant r_2$, and clearly u is a positive solution of the fractional differential equation boundary value problem, Eqs. (1) and (2). The proof is completed.

Theorem 2. *If there exists $\delta \in (0,1)$, such that $\phi_p(k(\delta))f_0\phi_p(A_2) > F_\infty\phi_p(A_1)$ holds, then, for each;*

$$\lambda \in \left((\phi_p(k(\delta))f_0\phi_p(A_2))^{-1},\ (F_\infty\phi_p(A_1))^{-1}\right), \quad (15)$$

the fractional differential equation boundary value problem, Eqs. (1) and (2), has at least one positive solution. Here, we impose $(\phi_p(k(\delta))f_0\phi_p(A_2))^{-1} = 0$, if $f_0 = +\infty$ and $(F_\infty\phi_p(A_1))^{-1} = +\infty$, if $F_\infty = 0$.

Proof. Let λ satisfy Relation (15), and $\varepsilon > 0$ be such that:

$$(\phi_p(k(\delta))(f_0 - \varepsilon)\phi_p(A_2))^{-1} \leqslant \lambda \leqslant ((F_\infty + \varepsilon)\phi_p(A_1))^{-1}. \quad (16)$$

In order to apply Lemma 6, we separate the proof into the following two steps:

Step 1. By the definition of f_0, there exists $r_1 > 0$, such that:

$$f(u) \geqslant (f_0 - \varepsilon)\phi_p(u), \quad \text{for } 0 < u \leqslant r_1.$$

So, if $u \in P$ with $||u|| = r_1$, then, similar to the second part of Theorem 1, if we choose $\Omega_1 = \{u \in E \mid ||u|| < r_1\}$, then:

$$||T_\lambda u|| \geqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_1. \quad (17)$$

Step 2. We choose $R_1 > 0$, such that:

$$f(u) \leqslant (F_\infty + \varepsilon)\phi_p(u), \quad \text{for } u \geqslant R_1. \quad (18)$$

Next, we consider two cases:

Case 1. Suppose f is bounded. Then, there exists some $N > 0$, such that:

$$f(u) \leqslant N, \quad \text{for } u \in (0, +\infty).$$

We choose $r_3 = \max\{2r_1,\ \phi_q(\lambda N)A_1\}$, and $u \in P$ with $||u|| = r_3$, then:

$$||T_\lambda u(t)|| \leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)f(u(\tau))d\tau\big)ds$$

$$\leqslant \phi_q(\lambda N)\int_0^1 G(1,s)\phi_q\big(\int_0^1 H(\tau,\tau)d\tau\big)ds$$

$$= \phi_q(\lambda N)A_1 \leqslant r_3 = ||u||.$$

So, if we choose $\Omega_3 = \{u \in E \mid ||u|| < r_3\}$, then:

$$||T_\lambda u|| \leqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_3.$$

Case 2. Suppose f is unbounded. Then, there exists some $r_4 > \max\{2r_1,\ R_1\}$, such that:

$$f(u) \leqslant f(r_4), \quad \text{for } 0 < u \leqslant r_4.$$

Let $u \in P$ with $\|u\| = r_4$. Then, by Relations (16) and (18), we have:

$$||T_\lambda u(t)|| \leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)f(u(\tau))d\tau\big)ds$$

$$\leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)(F_\infty+\varepsilon)\phi_p(r_4)d\tau\big)ds$$

$$= \phi_q(\lambda)\phi_q(F_\infty+\varepsilon)r_4A_1 \leqslant r_4 = ||u||.$$

Thus, if we choose $\Omega_4 = \{u \in E \mid ||u|| < r_4\}$, then:

$$||T_\lambda u|| \leqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_4.$$

In view of Cases 1 and 2, if we set $\Omega_2 = \{u \in E \mid ||u|| < r_2 = \max\{r_3,\ r_4\}\}$, then:

$$||T_\lambda u|| \leqslant ||u||, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{19}$$

Now, we obtain Relations (17) and (19) from Lemma 6, where T_λ has a fixed point, $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, with $r_1 \leqslant ||u|| \leqslant r_2$. Clearly, u is a positive solution of the fractional differential equation boundary value problem, Eqs. (1) and (2). The proof is completed.

Theorem 3. *Suppose there exists $r_2 > r_1 > 0$, such that:*

$$\min_{k(\delta)r_1 \leqslant u \leqslant r_1} f(u) \geqslant \frac{\phi_p(r_1)}{\lambda\phi_p(k(\delta))\phi_p(A_3)},$$

$$\max_{0 \leqslant u \leqslant r_2} f(u) \leqslant \frac{\phi_p(r_2)}{\lambda\phi_p(A_1)}.$$

Then, the fractional differential equation boundary value problem, Eqs. (1) and (2), has a positive solution, $u \in P$ with $r_1 \leqslant ||u|| \leqslant r_2$.

Proof. On one hand, choose $\Omega_1 = \{u \in E \mid ||u|| < r_1\}$, then, for $u \in P \cap \partial\Omega_1$, we have:

$$||T_\lambda u(t)|| \geqslant T_\lambda u(\delta)$$

$$= \int_0^1 G(\delta,s)\phi_q\big(\lambda\int_0^1 H(s,\tau)f(u(\tau))d\tau\big)ds$$

$$\geqslant \int_0^1 k(\delta)G(1,s)\phi_q\big(\lambda\int_0^1 s^{\beta-1}H(1,\tau)f(u(\tau))d\tau\big)ds$$

$$\geqslant \int_0^1 k(\delta)G(1,s)\phi_q$$

$$\big(\lambda\int_0^1 s^{\beta-1}H(1,\tau)\min_{k(\delta)r_1 \leqslant u \leqslant r_1} f(u(\tau))d\tau\big)ds$$

$$\geqslant \phi_q(\lambda)k(\delta)A_3\frac{r_1}{\phi_q(\lambda)k(\delta)A_3} = r_1 = ||u||.$$

On the other hand, choose $\Omega_2 = \{u \in E \mid ||u|| < r_2\}$, then, for $u \in P \cap \partial\Omega_2$, we have:

$$||T_\lambda u(t)|| \leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)f(u(\tau))d\tau\big)ds$$

$$\leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)\max_{0 \leq u \leqslant r_2} f(u(\tau))d\tau\big)ds$$

$$\leqslant \phi_q(\lambda)A_1\frac{r_2}{\phi_q(\lambda)A_1} = r_2 = ||u||.$$

Thus, by Lemma 6, the fractional differential equation boundary value problem, Eqs. (1) and (2), has a positive solution, $u \in P$ with $r_1 \leqslant ||u|| \leqslant r_2$. The proof is completed.

Theorem 4. *Let $\lambda_1 = \sup_{r>0}\frac{\phi_p(r)}{\phi_p(A_1)\max_{0 \leqslant u \leqslant r} f(u)}$. If $f_0 = +\infty$ and $f_\infty = +\infty$, then, the fractional differential equation boundary value problem, Eqs. (1) and (2), has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof. Define:

$$x(r) = \frac{\phi_p(r)}{\phi_p(A_1)\max_{0 \leqslant u \leqslant r} f(u)}.$$

In view of the continuity of $f(u)$, $f_0 = +\infty$ and $f_\infty = +\infty$, we understand that $x(r) : (0, +\infty) \to (0, +\infty)$ is continuous and:

$$\lim_{r \to 0} x(r) = \lim_{r \to +\infty} x(r) = 0.$$

So, there exists $r_0 \in (0, +\infty)$, such that:

$$x(r_0) = \sup_{r>0} x(r) = \lambda_1,$$

then, for $\lambda \in (0, \lambda_1)$, there exist constants $a_1,\ a_2$ $(0 < a_1 < r_0 < a_2 < +\infty)$ with:

$$x(a_1) = x(a_2) = \lambda.$$

Thus:

$$f(u) \leqslant \frac{\phi_p(a_1)}{\lambda\phi_p(A_1)}, \quad \text{for } u \in [0, a_1], \tag{20}$$

$$f(u) \leqslant \frac{\phi_p(a_2)}{\lambda\phi_p(A_1)}, \quad \text{for } u \in [0, a_2]. \tag{21}$$

On the other hand, applying the conditions $f_0 = +\infty$ and $f_\infty = +\infty$, there exist constants $b_1,\ b_2$ $(0 < b_1 < a_1 < r_0 < a_2 < b_2 < +\infty)$ with:

$$\frac{f(u)}{\phi_p(u)} \geqslant \frac{1}{(\phi_p(k(\delta)))^2\lambda\phi_p(A_3)},$$
$$\text{for } u \in (0, b_1) \cup (k(\delta)b_2, +\infty),$$

then:

$$\min_{k(\delta)b_1 \leqslant u \leqslant b_1} f(u) \geqslant \frac{\phi_p(b_1)}{\lambda\phi_p(k(\delta))\phi_p(A_3)}, \tag{22}$$

$$\min_{k(\delta)b_2 \leqslant u \leqslant b_2} f(u) \geqslant \frac{\phi_p(b_2)}{\lambda\phi_p(k(\delta))\phi_p(A_3)}. \tag{23}$$

By Relations (20) and (22), (21) and (23), and combining Theorem 3 and Lemma 6, the fractional differential equation boundary value problem, Eqs. (1) and (2), has at least two positive solutions for each $\lambda \in (0, \lambda_1)$. The proof is completed.

Corollary 1. Let $\lambda_1 = \sup\limits_{r>0} \frac{\phi_p(r)}{\phi_p(A_1)\max\limits_{0\leqslant u\leqslant r} f(u)}$. If $f_0 = +\infty$ or $f_\infty = +\infty$, then, the fractional differential equation boundary value problem, Eqs. (1) and (2), has at least one positive solution for each $\lambda \in (0, \lambda_1)$.

4. Nonexistence

In this section, we derive some sufficient conditions for nonexistence of a positive solution to the fractional differential equation boundary value problem, Eqs. (1) and (2).

Theorem 5. *If $F_0 < +\infty$ and $F_\infty < +\infty$, then, there exists $\lambda_0 > 0$, such that for all $0 < \lambda < \lambda_0$, the fractional differential equation boundary value problem, Eqs. (1) and (2), has no positive solution.*

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers, $M_1,\ M_2,\ r_1$ and r_2, such that $r_1 < r_2$ and:

$$f(u) \leqslant M_1\phi_p(u), \quad \text{for } u \in [0, r_1],$$

$$f(u) \leqslant M_2\phi_p(u), \quad \text{for } u \in [r_2, +\infty).$$

Let $M_0 = \max\{M_1,\ M_2,\ \max\limits_{r_1\leqslant u\leqslant r_2}\{\frac{f(u)}{\phi_p(u)}\}\}$. Then, we have:

$$f(u) \leqslant M_0\phi_p(u), \quad \text{for } u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of the fractional differential equation boundary value problem, Eqs. (1) and (2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (M_0\phi_p(A_1))^{-1}$. Since $T_\lambda v(t) = v(t)$ for $t \in [0, 1]$:

$$\begin{aligned}
||v|| &= ||T_\lambda v|| \\
&\leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)f(v(\tau))d\tau\big)ds \\
&\leqslant \int_0^1 G(1,s)\phi_q\big(\lambda\int_0^1 H(\tau,\tau)M_0\phi_p(v)d\tau\big)ds \\
&\leqslant \phi_q(\lambda M_0)||v||A_1 < ||v||,
\end{aligned}$$

which is a contradiction. Therefore, the fractional differential equation boundary value problem, Eqs. (1) and (2), has no positive solution. The proof is completed.

Theorem 6. If $f_0 > 0$ and $f_\infty > 0$, then, there exists $\lambda_0 > 0$, such that for all $\lambda > \lambda_0$, the fractional differential equation boundary value problem, Eqs. (1) and (2), has no positive solution.

Proof. From $f_0 > 0$ and $f_\infty > 0$, we know that there exist positive numbers, $m_1,\ m_2,\ r_3$ and r_4, such that $r_3 < r_4$ and:

$$f(u) \geqslant m_1\phi_p(u), \quad \text{for } u \in [0, r_3],$$

$$f(u) \geqslant m_2\phi_p(u), \quad \text{for } u \in [r_4, +\infty).$$

Let $m_0 = \min\{m_1,\ m_2,\ \min\limits_{r_3\leqslant u\leqslant r_4}\{\frac{f(u)}{\phi_p(u)}\}\} > 0$. Then, we get:

$$f(u) \geqslant m_0\phi_p(u), \quad \text{for } u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of the fractional differential equation boundary value problem, Eqs. (1) and (2). We will show that this leads to a contradiction for $\lambda > \lambda_0 := (\phi_p(k(\delta))m_0\phi_p(A_2))^{-1}$. Since $T_\lambda v(t) =$

$v(t)$ for $t \in [0,1]$:

$$\begin{aligned}||v|| &= ||T_\lambda v(t)||\\ &\geqslant \int_0^1 k(\delta)G(1,s)\phi_q\big(\lambda\int_0^1 s^{\beta-1}H(1,\tau)f(v(\tau))d\tau\big)ds\\ &\geqslant \int_0^1 k(\delta)G(1,s)\phi_q\big(\lambda\int_0^1 s^{\beta-1}H(1,\tau)m_0\phi_p(v)d\tau\big)ds\\ &\geqslant \int_0^1 k(\delta)\phi_q(s^{\beta-1})G(1,s)\phi_q\\ &\big(\int_0^1 H(1,\tau)\lambda m_0\phi_p(\tau^{\alpha-1}||v||)d\tau\big)ds\\ &\geqslant \phi_q(\lambda m_0)k(\delta)||v||A_2 > ||v||,\end{aligned}$$

which is a contradiction. Thus, the fractional differential equation boundary value problem, Eqs. (1) and (2), has no positive solution. The proof is completed.

5. Examples

In this section, we will present some examples to illustrate the main results.

Example 1. Consider the fractional differential equation boundary value problem:

$$D_{0+}^{\frac{3}{2}}(\phi_p(D_{0+}^{\frac{5}{2}}u(t))) = \lambda u^a,\ 0<t<1,\ a>1, \tag{24}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_{0+}^{\frac{5}{2}}u(0)) = (\phi_p(D_{0+}^{\frac{5}{2}}u(1)))' = 0. \tag{25}$$

Let $p=2$. Since $\alpha=\frac{5}{2}$ and $\beta=\frac{3}{2}$, by a simple calculation, we obtain:

$$\begin{aligned}A_1 &= \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\\ &= \int_0^1 \frac{(1-s)^{\frac{1}{2}}-(1-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\\ &\quad\Big(\int_0^1 \frac{\tau^{\frac{1}{2}}(1-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}d\tau\Big)ds = 0.3556,\end{aligned}$$

$$\begin{aligned}A_2 &= \int_0^1 \phi_q(s^{\beta-1})G(1,s)\phi_q\Big(\int_0^1 \phi_p(\tau^{\alpha-1})H(1,\tau)d\tau\Big)ds\\ &= \int_0^1 \frac{s^{\frac{1}{2}}((1-s)^{\frac{1}{2}}-(1-s)^{\frac{3}{2}})}{\Gamma(\frac{5}{2})}\\ &\quad\Big(\int_0^1 \frac{\tau^{\frac{3}{2}}((1-\tau)^{-\frac{1}{2}}-(1-\tau)^{\frac{1}{2}})}{\Gamma(\frac{3}{2})}d\tau\Big)ds = 0.1636.\end{aligned}$$

Let $f(u)=u^a$, $a>1$. Then, we have $F_0=0$, $f_\infty=+\infty$. Choose $\delta=\frac{1}{2}$. Then, $k(\frac{1}{2})=\frac{\sqrt{2}}{4}=0.3536$. So, $\phi_p(k(\delta))f_\infty\phi_p(A_2) > F_0\phi_p(A_1)$ holds. Thus, by Theorem 1, the fractional differential equation boundary value problem (Eqs. (24) and (25)), has a positive solution for each $\lambda \in (0,+\infty)$.

Example 2. Consider the fractional differential equation boundary value problem:

$$D_0^{\frac{3}{2}} + (\phi_p(D_0^{\frac{5}{2}} + u(t))) = \lambda u^b,$$

$$0<t<1, \quad 0<b<1, \tag{26}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_0^{\frac{5}{2}} + u(0)) = (\phi_p(D_0^{\frac{5}{2}} + u(1)))' = 0. \tag{27}$$

Let $p=2$. Since $\alpha=\frac{5}{2}$ and $\beta=\frac{3}{2}$, we have $A_1=0.3556$ and $A_2=0.1636$. Let $f(u)=u^b$, $0<b<1$. Then, we have $F_\infty=0$ and $f_0=+\infty$. Choose $\delta=\frac{1}{2}$. Then, $k(\frac{1}{2})=\frac{\sqrt{2}}{4}=0.3536$. So, $\phi_p(k(\delta))f_0\phi_p(A_2) > F_\infty\phi_p(A_1)$ holds. Thus, by Theorem 2, the fractional differential equation boundary value problem (Eqs. (26) and (27)), has a positive solution for each $\lambda \in (0,+\infty)$.

Example 3. Consider the fractional differential equation boundary value problem:

$$D_{0+}^{\frac{3}{2}}(\phi_p(D_{0+}^{\frac{5}{2}}u(t))) = \lambda\frac{(200u^2+u)(2+\sin u)}{u+1},$$

$$0<t<1, \tag{28}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_{0+}^{\frac{5}{2}}u(0)) = (\phi_p(D_{0+}^{\frac{5}{2}}u(1)))' = 0. \tag{29}$$

Let $p=2$. Since $\alpha=\frac{5}{2}$ and $\beta=\frac{3}{2}$, we have $A_1=0.3556$ and $A_2=0.1636$. Let $f(u)=\frac{(200u^2+u)(2+\sin u)}{u+1}$. Then, we have $F_0=f_0=2$, $F_\infty=600$, $f_\infty=200$ and $2u<f(u)<600u$.

i Choose $\delta=\frac{1}{2}$. Then, $k(\frac{1}{2})=\frac{\sqrt{2}}{4}=0.3536$. So, $\phi_p(k(\delta))f_\infty\phi_p(A_2) > F_0\phi_p(A_1)$ holds. Thus, by Theorem 1, the fractional differential equation boundary value problem (Eqs. (28) and (29)) has a positive solution for each $\lambda \in (0.0864, 1.4061)$.

ii By Theorem 5, the fractional differential equation boundary value problem (Eqs. (28) and (29)), has no positive solution for all $\lambda \in (0, 0.0047)$.

iii By Theorem 6, the fractional differential equation boundary value problem (Eqs. (28) and (29)) has no positive solution for all $\lambda \in (8.6432, +\infty)$.

Example 4. Consider the fractional differential equation boundary value problem:

$$D_{0^+}^{\frac{3}{2}}(\phi_p(D_{0^+}^{\frac{5}{2}}u(t))) = \lambda\frac{(u^2+u)(2+\sin u)}{150u+1},$$

$$0 < t < 1, \tag{30}$$

$$u(0) = u'(0) = u'(1) = 0,$$

$$\phi_p(D_{0^+}^{\frac{5}{2}}u()) = (\phi_p(D_{0^+}^{\frac{5}{2}}u(1)))' = 0. \tag{31}$$

Let $p = 2$. Since $\alpha = \frac{5}{2}$ and $\beta = \frac{3}{2}$, we have $A_1 = 0.3556$ and $A_2 = 0.1636$. Let $f(u) = \frac{(u^2+u)(2+\sin u)}{150u+1}$. Then, we have $F_0 = f_0 = 2$, $F_\infty = \frac{1}{50}$, $f_\infty = \frac{1}{150}$ and $\frac{u}{150} < f(u) < 2u$.

i Choose $\delta = \frac{1}{2}$. Then, $k(\frac{1}{2}) = \frac{\sqrt{2}}{4} = 0.3536$. So, $\phi_p(k(\delta))f_0\phi_p(A_2) > F_\infty\phi_p(A_1)$ holds. Thus, by Theorem 2, the fractional differential equation boundary value problem (Eqs. (30) and (31)) has a positive solution for each $\lambda \in (8.6432, 140.6074)$.

ii By Theorem 5, the fractional differential equation boundary value problem (Eqs. (30) and (31)) has no positive solution for all $\lambda \in (0, 1.4061)$.

iii By Theorem 6, the fractional differential equation boundary value problem (Eqs. (30) and (31)) has no positive solution for all $\lambda \in (2592.9593, +\infty)$.

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