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An efficient numerical approach for solving systems of high-order linear Volterra integral equations

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Abstract. In this study, a collocation method based on the Bernoulli polynomials is presented to find approximate solutions of a system of high-order linear Volterra Integral Equations (VIEs) with variable coefficients. In fact, the approximate solution of the problem in the truncated Bernoulli series form is obtained by this method. In addition, the method is presented with error and stability analysis. To show the accuracy and efficiency of the method, numerical examples are implemented and the comparisons are given by the other methods.

1. Introduction

In applied sciences, each physical event may be modeled by the differential equation, an integral equation or an integro-differential equation, or a system of these equations. So, it is very crucial to obtain the solutions of the models because these solutions provide information about the character of the modeled event. Amone the mathematical models are the linear Volterra Integral Equations (VIEs) with variable coefficients, which play a dominant role in many branches of science and engineering. The analytical solutions of most of these systems cannot be found, thus, numerical methods are required.

In recent years, several new approaches have been proposed for solving VIEs, such as the Taylor expansion method [1,2] and the Taylor collocation method [3,4]. Also, one can refer to the methods that are based on orthogonal bases (e.g. [4,5]).

During the last decade, operational matrices of differentiation and integration have received considerable attention for making an ideal base in the procedure of approximation. The Bernstein, Bernouli, Bessel and Euler matrix methods have been used in works [6-12] to solve differential, Fredholm Volterra integro difference equations and their systems. Yet, so far, to the best of our knowledge, a practical matrix method, based on Bernoulli polynomials, has had few results for approximating the solution of VIEs. This partially motivated our interest in such a method.

In this article, by developing the Bernoulli collocation method studied in [13,14], we will obtain the approximate solution of a system of linear Volterra Integral Equations (VIEs) with variable coefficients in the form:

$$\sum_{j=1}^{k} p_{i,j}(x)y_j(x) = g_i(x) + \int_a^x \sum_{j=1}^{k} K_{i,j}(x,t)y_j(t)dt,$$

$$i = 1, 2, \ldots, k, \quad 0 \le a \le x \le b, \tag{1}$$

where $y_j(x)$ is an unknown function. Also, $p_{i,j}(x)$, $g_i(x)$ and $K_{i,j}(x,t)$ are continuous functions defined on the interval $a \le x$ and $t \le b$. Moreover, functions $K_{i,j}(x,t)$ for $i,j = 1,2,\ldots,k$ can be an expanded Maclaurin series.

Our aim is to find an approximate solution of Eq. (1) expressed in the truncated Bernoulli series

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form:

$$y_i(x)=\sum_{n=0}^{N} a_{i,n}B_n(x),$$
$$i=1,2,\dots,k, \qquad 0\le a\le x\le b, \tag{2}$$

so that, $a_{i,n}$ for $n=0,1,\cdots,N$ are the unknown Bernoulli coefficients, and $B_n(x)$ for $n=0,1,\dots,N$ are the Bernoulli polynomials of the first kind, which are constructed from the following relation:

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n. \tag{3}$$

The advantages of Bernoulli polynomials for approximating an arbitrary unknown function over some classical orthogonal polynomials are expressed in [15-17].

The next sections of the paper are organized as follows: In Section 2, by introducing some mathematical preliminaries, we recall the basic concepts of Bernoulli polynomials and their relevant properties needed hereafter. Section 3 summarizes the application of this method to the systems of high-order VIEs. In Section 4, we discuss the error and stability analysis of the proposed method. In Section 5, we report our numerical findings and demonstrate the accuracy of the proposed method. The paper is closed in Section 6, by concluding.

2. Preliminaries

2.1. Polynomial interpolation

Let us consider $n+1$ pairs (x_i,y_i). The problem is to find a polynomial, p_k, called the interpolating polynomial, such that:

$$p_k(x_i)=c_0+c_1x_i+\cdots+c_kx_i^k, \qquad i=0,1,\dots,n.$$

Points x_i are called interpolation nodes. If $n\neq k$, the problem is over or under-determined. Let P_n be the $(n+1)-$dimensional subspace of $C[a,b]$ spanned by the functions $1,x,\cdots,x^n$. That is, P_n consists of all polynomials of degree, at most, n.

Theorem 1 [18]. Given $n+1$ distinct nodes, $x_0,x_1,\cdots,x_n$, and $n+1$ corresponding values, $y_0,y_1,\cdots,y_n$, there exists a unique polynomial, $p_n\in P_n$, such that $p_n(x_i)=y_i$ for $i=0,1,\dots,n$. If we define:

$$l_i\in P_n, \quad l_i=\prod_{\substack{i=0,\\ j\neq i}}^{n}\frac{(x-x_i)}{(x_i-x_j)}, \qquad i=0,1,\dots,n,$$

then, $l_i(x_j)=\delta_{ij}$. Polynomials $l_i(x)$ are called Lagrange characteristic polynomials. If $f(x_i)=y_i$ for $i=0,1,\dots,n$, f being a given function, the interpolating polynomial, $p_n(x)$, will be denoted by $p_nf(x)$. Let us introduce a lower triangular matrix, X, of infinite size, called the interpolation matrix on $[a,b]$ whose entries x_{ij} for $i,j=0,1,\dots,n$ represent the points of $[a,b]$, with the assumption that on each row, the entries are all distinct. Thus, for any $n\geqslant 0$, the $(n+1)$-th row of X contains $n+1$ distinct values that can be identified as nodes, so that, for a given function f, we can uniquely define an interpolating polynomial, p_nf, of degree n at those nodes.

Theorem 2 [19]. Let x and the abscissas, x_0, $x_1,\cdots,x_n$, be contained in an interval, $[a,b]$, on which f and its first n derivatives are continuous, and let $f^{(n+1)}$ exist in the open interval, (a,b). Then, there exists $\xi_x\in(a,b)$, which depends on x, such that:

$$f(x)-p_n(x)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{N}(x-x_i).$$

Having the fixed function, f, and the interpolation matrix, X, let us define the interpolation error by:

$$G_n(X)=\|f-p_nf\|_\infty, \qquad n=0,1,\dots.$$

Definition 1. Let $f(x)$ be defined on $[a,b]$, the modulus of continuity of $f(x)$ on $[a,b]$, $\omega(\delta)$, is defined for $\delta>0$ by:

$$\omega(\delta)=\sup_{|x-y|<\delta}|f(x)-f(y)|.$$

Corollary 1 [19]. Let $f\in C[a,b]$ and X be an interpolation matrix on $[a,b]$. Then:

$$G_n(X)=\|f-p_nf\|_\infty\le 6\Big(1+\Gamma_n(X)\Big)\omega\left(\frac{b-a}{2n}\right),$$

where $\Gamma_n(X)$ denotes the Lebesgue constant of X, defined as:

$$\Gamma_n(X)=\left\|\sum_{i=0}^{n}|l_i^{(n)}(x)|\right\|_\infty, \tag{4}$$

and where $l_i^{(n)}(x)\in P_n$ is the i-th characteristic polynomial associated with the $(n+1)$-th row of X.

2.2. Bernoulli polynomials

The n-th degree truncated Bernoulli polynomials of the first kind are defined by:

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k(0)x^{n-k}, \qquad n=1,2,\dots. \tag{5}$$

We can write $B_n(x)$ in the matrix form as in Eqs. (6) and (7) shown in Box I. Let us assume that $f=$

$$\mathbf{B}^T(x) = D\mathbf{X}^T(x) \Longleftrightarrow \mathbf{B}(x) = \mathbf{X}(x)D^T, \tag{6}$$

where:

$$\mathbf{B}(x) = [B_0(x), B_1(x), \ldots, B_N(x)],$$

$$\mathbf{X}(x) = [1, x, x^2, \ldots, x^N],$$

and:

$$D = \begin{bmatrix} \binom{0}{0}B_0(0) & 0 & 0 & \ldots & 0 \\ \binom{1}{1}B_1(0) & \binom{1}{0}B_0(0) & 0 & \ldots & 0 \\ \binom{2}{2}B_2(0) & \binom{2}{1}B_1(0) & \binom{2}{0}B_0(0) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{N}{N}B_N(0) & \binom{N}{N-1}B_{N-1}(0) & \binom{N}{N-2}B_{N-2}(0) & \ldots & \binom{N}{0}B_0(0) \end{bmatrix}. \tag{7}$$

Box I

$[f_1, f_2, \ldots, f_k]^T$ is a solution of Eq. (1). We would like to interpolate f_i by:

$$p_{i,N}(x) = \sum_{n=0}^{N} a_{i,n} B_n(x),$$

$$i = 1, 2, \ldots, k, \qquad 0 \le a \le x \le b, \tag{8}$$

such that $p_{i,N}$ and f_i are equal on nodes $a \le x_0 < x_1 < \ldots < x_n \le b$. Since all $f(x_j)$ values for $j = 0, 1, \ldots, N$ are unknown, we can use Eq. (1) to find the interpolation polynomial at nodes $x_0, x_1, \ldots, x_N$ without knowing $f_i(x_j)$ values. To do this, we put the interpolation polynomials, $p_{i,N}(x) = c_{i,0} + c_{i,1}x + \cdots + c_{i,N}x^N$, for $i = 1, 2, \ldots, k$ into Eq. (1). If $p_{i,N}$ equals f_i for $i = 1, 2, \ldots, k$ on the nodes, then, $p_{i,N}$ for $i = 1, 2, \ldots, k$ satisfies Eq. (1) on the nodes. Hence, we can obtain a system of linear equations depending on $c_{1,0}, c_{1,1}, \ldots, c_{1,N}, c_{2,0}, c_{2,1} \ldots, c_{2,N}, \ldots, c_{k,0}, c_{k,1}, \ldots, c_{k,N}$. Therefore, we can find the solution of Eq. (1) with some errors, which are interpolation and computational errors.

2.3. Main matrix relations

Firstly, let us write the approximate solutions (Eq. (8)) in the matrix form:

$$[y_j(x)] = \mathbf{X}(x)\mathbf{D}^T\mathbf{A}_j, \qquad j = 1, 2, \ldots, k, \tag{9}$$

where:

$$\mathbf{A}_j = [a_{j,0}, a_{j,1}, \ldots, a_{j,N}]^T.$$

Hence, the matrix, $\mathbf{y}(x)$, can be expressed as follows:

$$\mathbf{y}(x) = \tilde{\mathbf{X}}(x)\tilde{\mathbf{D}}\mathbf{A}, \tag{10}$$

where:

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_k(x) \end{bmatrix},$$

$$\tilde{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \ldots & 0 \\ 0 & \mathbf{X}(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{X}(x) \end{bmatrix}_{k \times k},$$

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \ldots & 0 \\ 0 & \mathbf{D}^T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{D}^T \end{bmatrix}_{k \times k},$$

and $\mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k]^T$. We now write system (1) in the matrix form as:

$$\mathbf{P}(x)\mathbf{y}(x) = \mathbf{g}(x) + \mathbf{V}(x), \tag{11}$$

where:

$$\mathbf{P}(x) = \begin{bmatrix} p_{1,1}(x) & p_{1,2}(x) & \ldots & p_{1,k}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \ldots & p_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}(x) & p_{k,2}(x) & \ldots & p_{k,k}(x) \end{bmatrix},$$

$$\mathbf{K}(x,t)=\begin{bmatrix} K_{1,1}(x,t) & K_{1,2}(x,t) & \dots & K_{1,k}(x,t) \\ K_{2,1}(x,t) & K_{2,2}(x,t) & \dots & K_{2,k}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{k,1}(x,t) & K_{k,2}(x,t) & \dots & K_{k,k}(x,t) \end{bmatrix},$$

$$\mathbf{g}(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}, \qquad \mathbf{V}(x)=\begin{bmatrix} V_1(x) \\ V_2(x) \\ \vdots \\ V_k(x) \end{bmatrix},$$

$$V_i(x)=\int_a^x \sum_{j=1}^k K_{i,j}(x,t)y_j(t)dt. \tag{12}$$

3. Method for solution

The kernel functions, $K_{i,j}(x,t)$ for $i,j=1,2,\dots,k$, can be approximated by Bernoulli polynomials in the form:

$$K_{i,j}(x,t)=\sum_{m=0}^N \sum_{n=0}^N k_{mn}^{ij} B_m(x)B_n(t)=\mathbf{B}(x)\mathbf{K}^{ij}\mathbf{B}^T(t), \tag{13}$$

where:

$$\mathbf{B}(t)=[B_0(t),B_1(t),\dots,B_N(t)],$$

$$\mathbf{K}^{ij}=[k_{mn}^{ij}]_{(N+1)\times(N+1)}, \quad m,n=0,1,\dots,N,$$

and coefficients k_{mn}^{ij} can be evaluated as follows:

$$k_{mn}^{ij}=\int_0^1\int_0^1 \frac{\partial^{m+n}K_{i,j}(x,t)}{\partial x^m \partial t^n}dxdt. \tag{14}$$

By substituting the matrix forms (Eqs. (9) and (13)) into Eq. (12), we obtain the matrix relation:

$$\begin{aligned}[V_i(x)] &= \int_a^x \sum_{j=0}^k \mathbf{B}(x)\mathbf{K}^{ij}\mathbf{B}^T(t)\mathbf{B}(t)\mathbf{A}_j dt \\ &= \sum_{j=0}^k \mathbf{B}(x)\mathbf{K}^{ij}\Big(\int_a^x \mathbf{B}^T(t)\mathbf{B}(t)dt\Big)\mathbf{A}_j \\ &= \sum_{j=0}^k \mathbf{B}(x)\mathbf{K}^{ij}\mathbf{Q}(x)\mathbf{A}_j, \end{aligned} \tag{15}$$

where:

$$\mathbf{Q}(x)=[q_{rs}(x)], \quad [q_{rs}(x)]=\int_a^x B_{r-1}(t)B_{s-1}(t)dt,$$

$$r,s=1,2,\dots,N+1.$$

Note that, the elements of $\mathbf{Q}(x)$ can be computed easily by using the matrix, $\mathbf{H}(x)$, which has the following form:

$$\mathbf{H}(x)=[h_{rs}(x)]; \qquad h_{rs}(x)=\frac{x^{r+s+1}-a^{r+s+1}}{r+s+1},$$

$$r,s=0,1,\dots,N.$$

Since, from Eq. (6):

$$\begin{aligned}\mathbf{Q}(x) &= \int_a^x \mathbf{B}^T(t)\mathbf{B}(t)dt=\int_a^x (\mathbf{D}\mathbf{X}^T(t))(\mathbf{X}(t)\mathbf{D}^T)dt \\ &= \mathbf{D}\Big(\int_a^x \mathbf{X}^T(t)\mathbf{X}(t)dt\Big)\mathbf{D}^T=\mathbf{D}\mathbf{H}(x)\mathbf{D}^T.\end{aligned}$$

By substituting the matrix form (Eq. (6)) into Eq. (15), we have the matrix relation:

$$[V_i(x)]=\sum_{j=0}^k \mathbf{X}(x)\mathbf{D}^T\mathbf{K}^{ij}\mathbf{Q}(x)\mathbf{A}_j, \quad i=1,2,\dots,k. \tag{16}$$

In Eq. (11), we place the collocation points defined by:

$$x_s=a+\frac{b-a}{N}s, \qquad s=0,1,2,\dots,N, \tag{17}$$

and, thus, we have the system of the matrix equations:

$$\mathbf{P}(x_s)\mathbf{y}(x_s)=\mathbf{g}(x_s)+\mathbf{V}(x_s),$$

or, briefly, the fundamental matrix equation is:

$$\mathbf{PY}=\mathbf{G}+\mathbf{V}, \tag{18}$$

where:

$$\mathbf{P}=\begin{bmatrix} \mathbf{P}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{P}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{P}(x_N) \end{bmatrix},$$

$$\mathbf{Y}=\begin{bmatrix} \mathbf{y}(x_0) \\ \mathbf{y}(x_1) \\ \vdots \\ \mathbf{y}(x_N) \end{bmatrix}, \quad \mathbf{G}=\begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}, \quad \mathbf{V}=\begin{bmatrix} \mathbf{V}(x_0) \\ \mathbf{V}(x_1) \\ \vdots \\ \mathbf{V}(x_N) \end{bmatrix}.$$

For the collocation points (Eq. (17)), the matrix relation (Eq. (12)) becomes:

$$\mathbf{V}(x_s)=\begin{bmatrix} V_1(x_s) \\ V_2(x_s) \\ \vdots \\ V_k(x_s) \end{bmatrix}=\tilde{\mathbf{X}}(x_s)\tilde{\mathbf{D}}\mathbf{K}_f\tilde{\mathbf{Q}}(x_s)\mathbf{A}, \tag{19}$$

where:

$$[V_i(x_s)] = \sum_{j=0}^{k} \mathbf{X}(x_s) D^T \mathbf{K}^{ij} \mathbf{Q}(x_s) \mathbf{A}_j,$$

$$\tilde{\mathbf{Q}}(x_s) = \begin{bmatrix} \mathbf{Q}(x_s) & 0 & \dots & 0 \\ 0 & \mathbf{Q}(x_s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{Q}(x_s) \end{bmatrix}_{k\times k},$$

$$\mathbf{K}_f = \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \dots & \mathbf{K}^{1k} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \dots & \mathbf{K}^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}^{k1} & \mathbf{K}^{k2} & \dots & \mathbf{K}^{kk} \end{bmatrix}.$$

Thus, matrix V in Eq. (18) can be expressed as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}(x_0) \\ \mathbf{V}(x_1) \\ \vdots \\ \mathbf{V}(x_N) \end{bmatrix} = \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}}\mathbf{A}, \tag{20}$$

where:

$$\tilde{\tilde{\mathbf{X}}} = \begin{bmatrix} \tilde{\mathbf{X}}(x_0) & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{X}}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\mathbf{X}}(x_N) \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{D}}} = \begin{bmatrix} \tilde{\mathbf{D}} & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{D}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\mathbf{D}} \end{bmatrix},$$

$$\tilde{\mathbf{K}}_f = \begin{bmatrix} \mathbf{K}_f & 0 & \dots & 0 \\ 0 & \mathbf{K}_f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{K}_f \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{Q}}} = \begin{bmatrix} \tilde{\mathbf{Q}}(x_0) \\ \tilde{\mathbf{Q}}(x_1) \\ \vdots \\ \tilde{\mathbf{Q}}(x_N) \end{bmatrix}.$$

On the other hand, by substituting the collocation points, $x = x_s$, defined by Eq. (17) into Eq. (10), we have:

$$\mathbf{Y} = \mathbf{X}\tilde{\mathbf{D}}\mathbf{A}, \tag{21}$$

where:

$$\mathbf{X} = \begin{bmatrix} \tilde{\mathbf{X}}(x_0) \\ \tilde{\mathbf{X}}(x_1) \\ \vdots \\ \tilde{\mathbf{X}}(x_N) \end{bmatrix} \quad \text{and} \quad \mathbf{y}(x_s) = \tilde{\mathbf{X}}(x_s)\tilde{\mathbf{D}}\mathbf{A},$$

$$s = 0, 1, \dots, N.$$

By putting Expressions (20) and (21) into Eq. (18), we obtain the matrix equation:

$$\left(\mathbf{P}\mathbf{X}\tilde{\mathbf{D}} - \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}}\right)\mathbf{A} = \mathbf{G}. \tag{22}$$

Briefly, Eq. (22) can be written as:

$$\mathbf{W}\mathbf{A} = \mathbf{G}, \quad \text{or } [\mathbf{W}; \mathbf{G}], \tag{23}$$

where:

$$\mathbf{W} = [w_{p,q}]_{k(N+1)\times k(N+1)} = \left(\mathbf{P}\mathbf{X}\tilde{\mathbf{D}} - \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}}\right),$$

$$p, q = 1, 2, \dots, k(N+1). \tag{24}$$

Hence, the unknown Bernoulli coefficients are computed by:

$$\mathbf{A} = \mathbf{W}^{-1}\mathbf{G}, \tag{25}$$

and by substituting the determined coefficients into Eq. (8):

$$y_i(x) = \sum_{n=0}^{N} a_{i,n} B_n(x), \qquad i = 1, 2, \dots, k.$$

4. Estimation of error and stability analysis

This section is devoted to providing an error bound for the approximated solution, which may be obtained by the Bernoulli polynomials.

Theorem 3 [20]. Let $\|.\| : C^{n\times n} \to \mathbf{R}$ be a consistent matrix norm. For any matrix, $\mathbf{H}$, of order n, if $\|\mathbf{H}\| \leq 1$, then, $\mathbf{I} - \mathbf{H}$ is nonsingular. Moreover:

$$\|\mathbf{X}(\mathbf{I} - \mathbf{H})^{-1}\| \leq \frac{\|\mathbf{X}\|}{1 - \|\mathbf{H}\|},$$

$$\|\mathbf{I} - (\mathbf{I} - \mathbf{H})^{-1}\| \leq \frac{\|\mathbf{H}\|}{1 - \|\mathbf{H}\|}.$$

Theorem 4 [21]. If $\mathbf{H}$ is a nonsingular matrix and $\|\delta\mathbf{H}\| \leq \frac{1}{\|\mathbf{H}^{-1}\|}$, then $\mathbf{H} + \delta\mathbf{H}$ is nonsingular. Moreover, let $h \neq 0$ and let x and $\tilde{x} = x + \delta x$ be solutions of $\mathbf{H}x = h$ and $(\mathbf{H} + \delta\mathbf{H})\tilde{x} = h$, respectively. Then:

$$\|\delta x\| \leq \|\mathbf{H}^{-1}\| \, \|\delta\mathbf{H}\| \, \|\tilde{x}\|.$$

In what follows, we formulate the assumptions under which Eq. (1) will be investigated. Namely, we assume the following hypotheses.

(H_1) $k=1$ and $p(x)=p_{i,j}(x)$, for clarity of presentation.

(H_2) $y=f(x)$ and $p_N(x)$ are the exact and approximate Bernoulli series solutions with the mentioned assumptions.

(H_3) $\overline{\mathbf{W}}=\mathbf{W}+\delta\mathbf{W}$ is the coefficient matrix of Eq. (24), where $\delta\mathbf{W}$ represents the computational error.

(H_4) There exists positive constant r, such that $\|\delta\mathbf{W}\|\leq r$.

Theorem 5. Under the tacit assumptions, (H_1)-(H_4), above, if $\|\overline{\mathbf{W}}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\leq 1$, then, the absolute error, $\|p_N-f\|_\infty$, of Eq. (1) has the following upper bound:

$$\|p_N-f\|_\infty\leq\frac{r\|\tilde{\mathbf{A}}\|_1\|\overline{\mathbf{W}}^{-1}\|_\infty}{1-r\|\overline{\mathbf{W}}^{-1}\|_\infty}\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$+6\left(1+\Gamma_n(X)\right)\omega\left(\frac{b-a}{2N}\right),$$

where $\tilde{A}$ is the solution of Eq. (25).

Proof. According to the assumptions, the basic Eq. (1) will be changed into the following equation:

$$p(x)y(x)=g(x)+\int_a^x K(x,t)y(t)dt,\quad 0\leq a\leq x\leq b.$$

Since the exact solution $y=f(x)$ is continuous on $[a,b]$, by using Corollary 1 and Theorems 3 and 4, we get:

$$\|p_N-y\|_\infty=\|p_N-p_Nf+p_Nf-y\|_\infty$$

$$\leq\|p_N-p_Nf\|_\infty+\|p_Nf-y\|_\infty$$

$$\leq\|p_N-p_Nf\|_\infty+6\left(1+\Gamma_n(X)\right)\omega\left(\frac{b-a}{2N}\right),$$

and:

$$\|p_N-p_Nf\|_\infty=\left\|\sum_{n=0}^N\tilde{a}_nB_n-\sum_{n=0}^N a_nB_n\right\|_\infty$$

$$=\left\|\sum_{n=0}^N(\tilde{a}_n-a_n)B_n\right\|_\infty$$

$$=\left\|[\tilde{a}_0-a_0\quad\tilde{a}_1-a_1...\tilde{a}_N-a_N]\mathbf{B}^T\right\|_\infty$$

$$\leq\left\|[\tilde{a}_0-a_0\quad\tilde{a}_1-a_1...\tilde{a}_N-a_N]\right\|_\infty\left\|\mathbf{B}^T\right\|_\infty$$

$$\leq\|\mathbf{W}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\|\tilde{\mathbf{A}}\|_1\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$=\|(\overline{\mathbf{W}}-\delta\mathbf{W})^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\|\tilde{\mathbf{A}}\|_1\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$\leq\|\overline{\mathbf{W}}^{-1}\|_\infty\|(1-\overline{\mathbf{W}}^{-1}\delta\mathbf{W})^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\|\tilde{\mathbf{A}}\|_1$$

$$\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$\leq\frac{\|\overline{\mathbf{W}}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\|\tilde{\mathbf{A}}\|_1}{1-\|\overline{\mathbf{W}}^{-1}\delta\mathbf{W}\|_\infty}\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$\leq\frac{\|\overline{\mathbf{W}}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\|\tilde{\mathbf{A}}\|_1}{1-\|\overline{\mathbf{W}}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty}\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty$$

$$\leq\frac{r\|\tilde{\mathbf{A}}\|_1\|\overline{\mathbf{W}}^{-1}\|_\infty}{1-r\|\overline{\mathbf{W}}^{-1}\|_\infty}\|\mathbf{D}\|_\infty\|\mathbf{X}^{\mathbf{T}}(b-a)\|_\infty,$$

$$\|\overline{\mathbf{W}}^{-1}\|_\infty\|\delta\mathbf{W}\|_\infty\leq 1.\tag{26}$$

We note that a similar procedure can be applied for the case of other values of k.

Theorem 6. Suppose the tacit assumptions, (H_1)-(H_4), above. If $\|\overline{\mathbf{W}}^{-1}\|_F\|\delta\mathbf{W}\|_F\leq 1$ and $f\in C^\infty[a,b]$, then:

$$|p_N(x)-f(x)|\leq\frac{r\|\tilde{\mathbf{A}}\|_F\|\overline{\mathbf{W}}^{-1}\|_F}{1-r\|\overline{\mathbf{W}}^{-1}\|_F}\|\mathbf{D}\|_F\|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$+\left|\frac{f^{(N+1)}(\xi_x)}{(N+1)!}\prod_{i=0}^N(x-x_i)\right|.$$

Proof. By considering $\|\overline{\mathbf{W}}^{-1}\|_F\|\delta\mathbf{W}\|_F\leq 1$ and $f\in C^\infty[a,b]$, from Cauchy-Schwarz inequalities and Theorems 2 to 4, we obtain that:

$$|p_N(x)-y|=|p_N(x)-p_Nf(x)+p_Nf(x)-y|$$

$$\leq|p_N(x)-p_Nf(x)|+|p_Nf(x)-y|$$

$$\leq|p_N(x)-p_Nf(x)|$$

$$+\left|\frac{f^{(N+1)}(\xi_x)}{(N+1)!}\prod_{i=0}^N(x-x_i)\right|,$$

and:

$$|p_N(x)-p_Nf(x)|=\left|\sum_{n=0}^N\tilde{a}_nB_n-\sum_{n=0}^N a_nB_n\right|$$

$$=\left|\sum_{n=0}^N(\tilde{a}_n-a_n)B_n\right|$$

$$\leq\left\|[\tilde{a}_0-a_0\tilde{a}_1-a_1\ldots\tilde{a}_N-a_N]\right\|_F\left\|\mathbf{B}^T\right\|_F$$

$$\leq\|\mathbf{W}^{-1}\|_F\|\delta\mathbf{W}\|_F\|\tilde{\mathbf{A}}\|_F\|\mathbf{D}\|_F\|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$= \|(\overline{\mathbf{W}} - \delta\mathbf{W})^{-1}\|_F \|\delta\mathbf{W}\|_F \|\tilde{\mathbf{A}}\|_F$$

$$\|\mathbf{D}\|_F \|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$\leq \|\overline{\mathbf{W}}^{-1}\|_F \|(1 - \overline{\mathbf{W}}^{-1}\delta\mathbf{W})^{-1}\|_F \|\delta\mathbf{W}\|_F$$

$$\|\tilde{\mathbf{A}}\|_F \|\mathbf{D}\|_F \|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$\leq \frac{\|\overline{\mathbf{W}}^{-1}\|_F \|\delta\mathbf{W}\|_F \|\tilde{\mathbf{A}}\|_F}{1 - \|\overline{\mathbf{W}}^{-1}\delta\mathbf{W}\|_F} \|\mathbf{D}\|_F \|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$\leq \frac{\|\overline{\mathbf{W}}^{-1}\|_F \|\delta\mathbf{W}\|_F \|\tilde{\mathbf{A}}\|_F}{1 - \|\overline{\mathbf{W}}^{-1}\|_F \|\delta\mathbf{W}\|_F} \|\mathbf{D}\|_F \|\mathbf{X}^{\mathbf{T}}(b-a)\|_F$$

$$\leq \frac{r\|\tilde{\mathbf{A}}\|_F \|\overline{\mathbf{W}}^{-1}\|_F}{1 - r\|\overline{\mathbf{W}}^{-1}\|_F} \|\mathbf{D}\|_F \|\mathbf{X}^{\mathbf{T}}(b-a)\|_F,$$

$$\|\overline{\mathbf{W}}^{-1}\|_F \|\delta\mathbf{W}\|_F \leq 1.$$

To complete the proof of Theorems 5 and 6, we need to find r, the highest value of $\|\delta\mathbf{W}\|$. To do this, we will use the following lemma.

Lemma 1 [22]. If $\mathbf{X}_j + \Delta\mathbf{X}_j \in \mathbf{R}^{n\times n}$ satisfies $\|\Delta\mathbf{X}_j\| \leq \sigma_j \|\mathbf{X}_j\|$ for all j for a consistent norm, then:

$$\left\| \prod_{j=0}^{m} (\mathbf{X}_j + \Delta\mathbf{X}_j) - \prod_{j=0}^{m} \mathbf{X}_j \right\| \leq \left(\prod_{j=0}^{m} (1+\sigma_j) - 1 \right) \prod_{j=0}^{m} \|\mathbf{X}_j\|.$$

Using the above Lemma, we find the upper bound of $\|\delta\mathbf{W}\|$ as:

$$\|\delta\mathbf{W}\| = \left\| \mathbf{PX\tilde{D}} - \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}} - fl\left(\mathbf{PX\tilde{D}} - \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}} \right) \right\|$$

$$\leq \left(\prod_{j=0}^{2} (1+\sigma_j) - 1 \right) \|\mathbf{P}\|\|\mathbf{X}\|\|\tilde{\mathbf{D}}\|$$

$$+ \left(\prod_{j=0}^{3} (1+\sigma_j) - 1 \right) \|\tilde{\tilde{\mathbf{X}}}\|\|\tilde{\tilde{\mathbf{D}}}\|\|\tilde{\mathbf{K}}_f\|\|\tilde{\tilde{\mathbf{Q}}}\| + s\theta\mathbf{W},$$

where $\theta \leq u$, u is the unit round off, and s is the number of terms summed in $\mathbf{W}$.

Let us consider a set of function values $\{\tilde{f}_i(x_j) : i = 1,2,...,k, j = 0,1,...,N\}$ which is a perturbation of the data $\{f_i(x_j) : i = 1,2,...,k, j = 0,1,...,N\}$ relative to the nodes $\{x_j : j = 0,1,...,N\} \subset [a,b]$. The perturbation may be due, for instance, to the effect of rounding errors, or may be caused by an error in the experimental measure of the data. Denoting, by $\tilde{p}_N f(x)$, the vector of interpolating polynomials on the set of values $\tilde{f}_i(x_j)$, we have:

$$\|p_N(x) - \tilde{p}_N(x)\|_\infty$$

$$= \left\| \begin{bmatrix} p_{1,N}f(x) - \delta p_{1,N}(x) - \tilde{p}_{1,N}f(x) + \delta\tilde{p}_{1,N}(x) \\ p_{2,N}f(x) - \delta p_{2,N}(x) - \tilde{p}_{2,N}f(x) + \delta\tilde{p}_{2,N}(x) \\ \vdots \\ p_{k,N}f(x) - \delta p_{k,N}(x) - \tilde{p}_{k,N}f(x) + \delta\tilde{p}_{k,N}(x) \end{bmatrix} \right\|_\infty$$

$$\leq \left\| \begin{bmatrix} p_{1,N}f(x) - \tilde{p}_{1,N}f(x) \\ p_{2,N}f(x) - \tilde{p}_{2,N}f(x) \\ \vdots \\ p_{k,N}f(x) - \tilde{p}_{k,N}f(x) \end{bmatrix} \right\|_\infty$$

$$+ \left\| \begin{bmatrix} \delta p_{1,N}(x) - \delta\tilde{p}_{1,N}(x) \\ \delta p_{2,N}(x) - \delta\tilde{p}_{2,N}(x) \\ \vdots \\ \delta p_{k,N}(x) - \delta\tilde{p}_{k,N}(x) \end{bmatrix} \right\|_\infty$$

$$= \max_{\substack{0\leq x\leq 1 \\ i}} \left| \sum_{j=0}^{n} (f_i(x_j) - \tilde{f}_i(x_j)) l_j(x) \right|$$

$$+ \max_{\substack{0\leq x\leq 1 \\ i}} \left\| (\delta p_{i,N}(x) - \delta\tilde{p}_{i,N}(x)) \right\|_\infty$$

$$\leq \Gamma_n(X) \max_{\substack{0\leq x\leq 1 \\ i}} \left| (f_i(x_j) - \tilde{f}_i(x_j)) \right|$$

$$+ \max_{\substack{0\leq x\leq 1 \\ i}} \left\| (\delta p_{i,N}(x) - \delta\tilde{p}_{i,N}(x)) \right\|_\infty, \qquad (27)$$

where $\delta p_{i,N}(x)$ and $\delta\tilde{p}_{i,N}(x))$ for $i = 1,2,\ldots,k$ represent the differences between the interpolating polynomial and the Bernoulli series solution. If the Lebesgue constant, $\Gamma_n(X)$, given in Eq. (4) and the difference between the interpolating polynomial and the Bernoulli series solution are small, the small changes on the data give rise to small changes on the Bernoulli series solution. On equally spaced nodes, as proved in [23], $\Gamma_n(X)$ grows exponentially, that is, as $N \to \infty$:

$$\Gamma_n(X) \approx \frac{2^{N+1}}{e^N \log(N)}.$$

This shows that, for large N and equally spaced nodes, the polynomial interpolation method can become unstable. On the other hand, using Eq. (26), the difference between the interpolating polynomial and the Bernoulli series solution grows whenever N increases. As a result, the Bernoulli series solution can become unstable for large N.

5. Numerical experiments

In this section, some numerical examples are considered to demonstrate the efficiency and accuracy of the proposed method. The errors have been calculated using an absolute error function defined by:

$$e_{i,N}(x) = \left|y_i(x) - y_{i,N}(x)\right|, \qquad i = 1,2,\ldots,k,$$

and maximum absolute error defined by:

$$e_{i,N} = \max\left\{|y_{i,N}(x_j) - y_i(x_j)|; \quad x_j = \frac{j}{1000}, \quad j = 1(1)1000,\right\}, \quad i = 1,2,\ldots,k.$$

All computations are modeled using the mathematical software package, Matlab.

Example 1. At first we consider the following system of linear Volterra integral equations:

$$\begin{cases} y_1(x) + xy_2(x) = \sin(x) + x\cos(x) \\ \quad + \int_0^x x^2\cos(t)y_1(t)dt - \int_0^x x^2\sin(t)y_2(t)dt, \\ \\ y_2(x) - 2xy_1(x) = \cos(x) - 2x\sin(x) \\ \quad + \int_0^x \sin(x)\cos(t)y_1(t)dt \\ \quad - \int_0^x \sin(x)\sin(t)y_2(t)dt, \qquad 0 \le x \le 1 \end{cases} \tag{28}$$

with the exact solutions, $y_1(x) = \sin(x)$ and $y_2(x) = \cos(x)$.

By using the method in Section 3, we get the approximate solutions of the problem for $N = 3$. The fundamental matrix equation of the problem is:

$$(\mathbf{P}\mathbf{X}\tilde{\mathbf{D}} - \tilde{\tilde{\mathbf{X}}}\tilde{\tilde{\mathbf{D}}}\tilde{\mathbf{K}}_f\tilde{\tilde{\mathbf{Q}}})\mathbf{A} = \mathbf{G},$$

where:

$$\mathbf{P}(x) = \begin{bmatrix} 1 & x \\ -2x & 1 \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}(0) & 0 & 0 & 0 \\ 0 & \mathbf{P}(1/3) & 0 & 0 \\ 0 & 0 & \mathbf{P}(2/3) & 0 \\ 0 & 0 & 0 & \mathbf{P}(1) \end{bmatrix},$$

$$\mathbf{g}(x) = \begin{bmatrix} \sin(x) + x\cos(x) \\ \cos(x) - 2x\sin(x) \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(0) \\ \mathbf{g}(1/3) \\ \mathbf{g}(2/3) \\ \mathbf{g}(1) \end{bmatrix},$$

$$\mathbf{X}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{X}(1/3) = \begin{bmatrix} 1 & 1/3 & 1/9 & 1/27 \end{bmatrix},$$

$$\mathbf{X}(2/3) = \begin{bmatrix} 1 & 2/3 & 4/9 & 8/27 \end{bmatrix},$$

$$\mathbf{X}(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{X}}(0) = \begin{bmatrix} \mathbf{X}(0) & 0 \\ 0 & \mathbf{X}(0) \end{bmatrix},$$

$$\tilde{\mathbf{X}}(1/3) = \begin{bmatrix} \mathbf{X}(1/3) & 0 \\ 0 & \mathbf{X}(1/3) \end{bmatrix},$$

$$\tilde{\mathbf{X}}(2/3) = \begin{bmatrix} \mathbf{X}(2/3) & 0 \\ 0 & \mathbf{X}(2/3) \end{bmatrix},$$

$$\tilde{\mathbf{X}}(1) = \begin{bmatrix} \mathbf{X}(1) & 0 \\ 0 & \mathbf{X}(1) \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} \tilde{\mathbf{X}}(0) \\ \tilde{\mathbf{X}}(1/3) \\ \tilde{\mathbf{X}}(2/3) \\ \tilde{\mathbf{X}}(1) \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{X}}} = \begin{bmatrix} \tilde{\mathbf{X}}(0) & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{X}}(1/3) & 0 & 0 \\ 0 & 0 & \tilde{\mathbf{X}}(2/3) & 0 \\ 0 & 0 & 0 & \tilde{\mathbf{X}}(1) \end{bmatrix},$$

$$\mathbf{Q}(x) = \begin{bmatrix} \mathbf{q}_1, & \mathbf{q}_2, & \mathbf{q}_3, & \mathbf{q}_4 \end{bmatrix},$$

$$\mathbf{q}_1 = \begin{bmatrix} x \\ x^2/2 - x/2 \\ x^3/3 - x^2/2 + x/6 \\ x^4/4 - x^3/2 + x^2/4 \end{bmatrix},$$

$$\mathbf{q}_2 = \begin{bmatrix} x^2/2 - x/2 \\ x^3/3 - x^2/2 + x/4 \\ x^4/4 - x^3/2 + x^2/3 - x/12 \\ x^5/5 - x^4/2 + 5x^3/12 - x^2/8 \end{bmatrix},$$

$$\mathbf{q}_3 = \begin{bmatrix} x^3/3 - x^2/2 + x/6 \\ x^4/4 - x^3/2 + x^2/3 - x/12 \\ x^5/5 - x^4/2 + 4x^3/9 - x^2/6 + x/36 \\ x^6/6 - x^5/2 + 13x^4/24 - x^3/4 + x^2/24 \end{bmatrix},$$

$$\mathbf{q}_4 = \begin{bmatrix} x^4/4 - x^3/2 + x^2/4 \\ x^5/5 - x^4/2 + 5x^3/12 - x^2/8 \\ x^6/6 - x^5/2 + 13x^4/24 - x^3/4 + x^2/24 \\ x^7/7 - x^6/2 + 13x^5/20 - 3x^4/8 + x^3/12 \end{bmatrix},$$

$$\tilde{\mathbf{Q}}(x) = \begin{bmatrix} \mathbf{Q}(x) & 0 \\ 0 & \mathbf{Q}(x) \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{Q}}} = \begin{bmatrix} \tilde{\mathbf{Q}}(0) \\ \tilde{\mathbf{Q}}(1/3) \\ \tilde{\mathbf{Q}}(2/3) \\ \tilde{\mathbf{Q}}(1) \end{bmatrix},$$

$$\mathbf{D}^T = \begin{bmatrix} 1 & -1/2 & 1/6 & 0 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 \\ 0 & \mathbf{D}^T \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{D}}} = \begin{bmatrix} \tilde{\mathbf{D}} & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{D}} & 0 & 0 \\ 0 & 0 & \tilde{\mathbf{D}} & 0 \\ 0 & 0 & 0 & \tilde{\mathbf{D}} \end{bmatrix},$$

$$\mathbf{K}_f = \begin{bmatrix} 5/18 & -1/6 & -1/6 & 0 & -11/72 & -5/18 & 1/12 & 1/18 \\ 5/6 & -1/2 & -1/2 & 0 & -11/24 & -5/6 & 1/4 & 1/6 \\ 5/6 & -1/2 & -1/2 & 0 & -11/24 & -5/6 & 1/4 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & -1 & -2/3 & -1/3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and:

$$\tilde{\mathbf{K}}_f = \begin{bmatrix} \mathbf{K}_f & 0 & 0 & 0 \\ 0 & \mathbf{K}_f & 0 & 0 \\ 0 & 0 & \mathbf{K}_f & 0 \\ 0 & 0 & 0 & \mathbf{K}_f \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is shown in Box II. By solving this system, the unknown Bernoulli coefficients matrix is obtained as:

$$A = [2159/4663 \quad 1538/1791 \quad -973/4821 \quad -226/2189 \quad 594/709 \quad -702/1471 \quad -1154/2517 \quad 223/4247]^T.$$

Thus, the solutions of this problem become:

$$y_{1,3}(x) = (0.185037170778e-17) + 1.00894176194x - 0.04696004873x^2 - 0.10324352180x^3,$$

and:

$$y_{2,3}(x) = 1 + (0.750969476e-02)x - 0.53724376915x^2 + 0.0525076505x^3.$$

In a similar way, we obtain the approximate solutions of the problem for $N = 7$ and $N = 10$, respectively:

$$\begin{aligned} y_{1,7}(x) = & -(0.6885873799451457e-16) \\ & + 0.999998974293757x \\ & + (0.1743120966369345e-06)x^2 \\ & - 0.166776412237619x^3 \\ & + (0.3270882229227112e-05)x^4 \\ & + 0.007874461317911x^5 \\ & + (0.1946019045888647e-05)x^6 \\ & - (0.7665666442591322e-06)x^7, \end{aligned}$$

$$\begin{aligned} y_{2,7}(x) = & 1 + (0.1253346629516529e-07)x \\ & - 0.500022898557948x^2 \\ & + (0.1614552116369807e-05)x^3 \\ & + 0.041091123213102x^4 \\ & + 0.001106215189335x^5 \\ & - 0.002458746044216x^6 \\ & - (0.3467610831506825e-05)x^7, \end{aligned}$$

and:

$$\begin{aligned} y_{1,10}(x) = & (0.5.032274756712580e-13) \\ & + 1.000000000610761x \\ & + (0.1827377657090948e-09)x^2 \\ & - 0.166666445369482x^3 \\ & - (0.1485583416080523e-07)x^4 \\ & + 0.008339503886760x^5 \\ & - (0.1669629678726148e-06)x^6 \\ & - (0.1684843630304439e-05)x^7 \\ & - (0.3496584690737268e-06)x^8 \\ & + (0.2757626134545171e-06)x^9 \end{aligned}$$

$$\mathbf{W}=\begin{bmatrix} 1 & -1/2 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & 1/6 & 0 \\ 1405/1458 & -239/1547 & -263/4619 & 133/3726 & 5939/17496 & -832/14531 & -231/12475 & 49/3887 \\ -175/243 & 252/1999 & 77/2082 & -236/8751 & 82/81 & -55/324 & -271/4860 & 158/4205 \\ 529/729 & 35/162 & -291/5647 & -79/1856 & 1666/2187 & 209/1981 & -91/2153 & -535/22358 \\ -370/243 & -761/3645 & 173/2057 & 547/11526 & 89/81 & 1/6 & -151/2430 & -98/2633 \\ 1/6 & 13/24 & 61/360 & -1/240 & 35/24 & 137/240 & 119/720 & -1/140 \\ -7/3 & -31/30 & -59/180 & 1/280 & 4/3 & 7/12 & 31/180 & -1/120 \end{bmatrix}$$

Box II

Table 1. Absolute values of the error at the selected points for Example 1.

x_i	Absolute errors of $y_1(x)$			Absolute errors of $y_2(x)$		
	$N=3$	$N=7$	$N=10$	$N=3$	$N=7$	$N=10$
0	0	$6.8860e-13$	$5.0323e-12$	0	$2.4425e-13$	$2.2352e-11$
0.2	$4.1467e-04$	$4.6562e-09$	$1.1006e-12$	$3.6567e-04$	$4.6077e-09$	$5.7333e-12$
0.4	$3.7170e-05$	$2.5457e-08$	$7.6667e-12$	$6.5563e-04$	$1.5792e-08$	$2.6564-11$
0.6	$1.5164e-03$	$1.1235e-06$	$7.2593e-11$	$2.8959e-03$	$1.8114e-06$	$1.1861-09$
0.8	$6.8822e-03$	$1.3938e-05$	$2.9694e-08$	$7.6510e-03$	$1.6112e-05$	$3.4832e-08$
1	$1.7268e-02$	$8.8503e-05$	$4.9404e-07$	$1.7529e-02$	$7.7142e-05$	$4.4235e-07$

$$-(0.8494266467093018e-07)x^{10},$$

$$\begin{aligned} y_{2,10}(x) ={}&1+(0.3004526426968851e-11)x \\ &-(0.500000008139665)x^2 \\ &+(0.1027174623903725e-08)x^3 \\ &+(0.041665993007845)x^4 \\ &+(0.2678895248177806e-07)x^5 \\ &-(0.001395675693326)x^6 \\ &+(0.1090797968583095e-06)x^7 \\ &+(0.1437084029013844e-06)x^8 \\ &+(0.4742891633234149e-07)x^9 \\ &-(0.3645566266664232e-08)x^{10}. \end{aligned}$$

Absolute values of the error function at points $x=\frac{i}{10}$, where $i=0,2,\ldots,10$, are provided in Table 1 by taking $N=3,7$ and 10. From this table, one can see the high order of accuracy of the presented method. We see that the errors decrease rapidly as N increases. The error histories for $N=3,7$ and 10 are depicted in Figures 1 and 2.

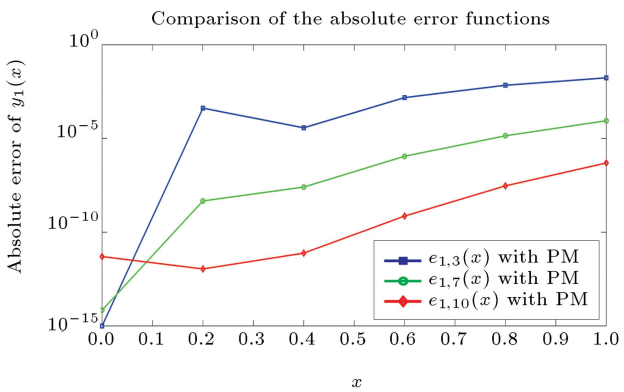

Figure 1. Comparison of the absolute error functions $e_{1,N}(x)$ of the Presented Method (PM) for Example 1.

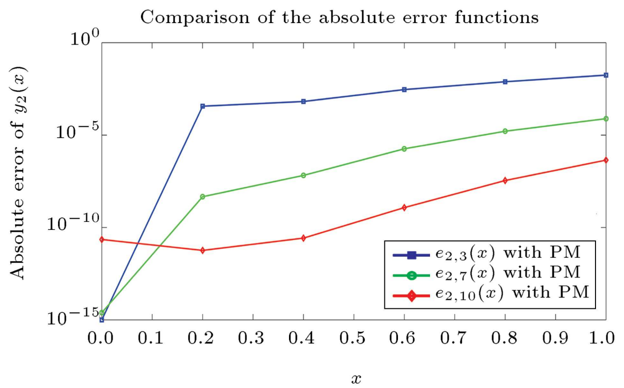

Figure 2. Comparison of the absolute error functions $e_{2,N}(x)$ of the Presented Method (PM) for Example 1.

Table 2. Comparison of the absolute error functions $e_{1,N}(x)$ for $y_{1,N}(x)$ of Example 2.

x_i	Taylor expansion method [2]	Euler matrix method [12]	Present method	
		$N=10$	$N=10$	$N=12$
0	0	0	0	0
0.2	$9.2719e-04$	$4.8098e-10$	$7.4397e-12$	$2.7127e-14$
0.4	$5.4551e-03$	$9.0010e-10$	$2.2300e-11$	$6.4010e-14$
0.6	$1.3864e-02$	$9.8703e-10$	$3.3965e-11$	$6.6891e-13$
0.8	$2.5635e-02$	$9.6784e-10$	$2.8303e-10$	$2.2369e-12$
1	$4.1981e-02$	$3.3247e-09$	$3.1772e-09$	$7.9290e-11$

Table 3. Comparison of the absolute error functions $e_{2,N}(x)$ for $y_{2,N}(x)$ of Example 2.

x_i	Taylor expansion method [2]	Euler matrix method [12]	Present method	
		$N=10$	$N=10$	$N=12$
0	0	0	0	0
0.2	$1.1471e-04$	$1.8414e-09$	$6.2610e-13$	$1.5013e-14$
0.4	$8.5720e-03$	$2.8160e-09$	$2.0128e-12$	$2.6135e-13$
0.6	$2.7824e-02$	$2.7157e-09$	$1.2683e-12$	$3.1138e-13$
0.8	$5.9121e-02$	$1.5968e-09$	$2.0646e-11$	$6.8084e-12$
1	$8.7089e-02$	$1.1052e-10$	$2.7642e-10$	$3.5298e-11$

Example 2 [2]. Consider the system of linear Volterra integral equations given by:

$$\begin{cases} y_1(x) = \cos(x)(2+\sin(x)-x\cos(x)) \\ \quad +\frac{1}{4}(\cos(x-1)-\cos(x+1)) \\ \quad -\frac{1}{2}x\sin(x-1)-1 \\ \quad +\int_0^x \sin(x-t-1)y_1(t) \\ \quad +\int_0^x (1-t\cos(x))y_2(t)dt, \\ y_2(x)=\sin(x)-x+\int_0^x y_1(t)dt+\int_0^x (x-t)y_2(t)dt, \\ \quad 0\leq x\leq 1, \end{cases} \tag{29}$$

with exact solutions $y_1(x)=\cos(x)$ and $y_2(x)=\sin(x)$.

We solve this problem by considering notations in Section 3 for $N=10$ and 12. We compare the absolute error functions computed by the present method, the Euler matrix method [12] and Taylor expansion method [2] in Tables 2 and 3 and Figures 3 and 4. According to these, one can see that not only is our method superior in results, but also, that the behaviour of the error of the presented method has a stable manner, with respect to the Taylor expansion method [2] and Euler matrix method [12].

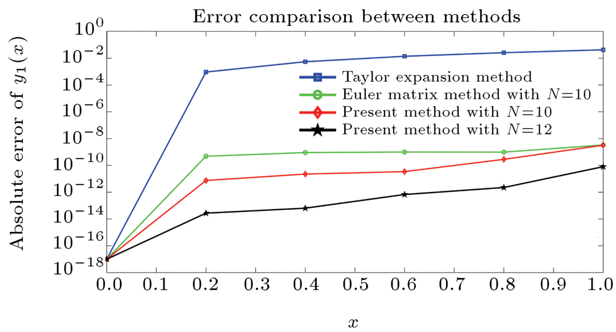

Figure 3. Comparison of the presented method, Taylor expansion and Euler matrix method for $y_{1,N}(x)$ of Example 2.

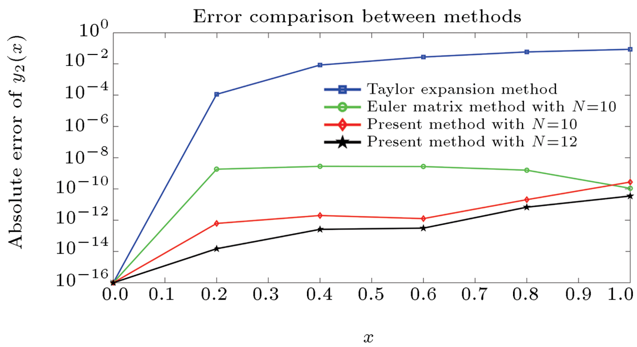

Figure 4. Comparison of the presented method, Taylor expansion and Euler matrix method for $y_{2,N}(x)$ of Example 2.

Example 3 [4]. Let us consider the system of the linear Volterra integral equation given by:

$$\begin{cases} y_1(x) = e^{2x}(-\frac{1}{2}x^2+\frac{1}{4}x+1)+e^{-2x}(x+\frac{1}{4}) \\ \quad -\frac{3}{4}x-\frac{1}{4}+\int_0^x xty_1(t)dt \\ \quad +\int_0^x (x+t)y_2(t)dt, \\ y_2(x)=e^{-2x}(2x^2+x+\frac{5}{4})-\frac{1}{4}e^{2x}-\frac{1}{2}x^2 \\ \quad +\int_0^x (x-t)y_1(t)dt+\int_0^x (x+t)^2y_2(t)dt, \\ \quad 0\leq x\leq 1, \end{cases} \tag{30}$$

with the exact solutions, $y_1(x)=e^{2x}$ and $y_2(x)=e^{-2x}$.

Similar to the previous two examples, we solve this system for different values of N. Tables 4 and 5 and Figures 5 and 6 display the numerical results of the absolute error functions obtained by the Taylor method [4], Bessel method [10] and present method for $N=5$, $N=7$ and $N=10$. It is seen from Tables 4

Table 4. Comparison of the absolute errors for $y_1(x) = e^{2x}$ of Example 3.

x_i	Taylor method [4]		Bessel method [10]		Present method		
	$e_{1,5}(x_i)$	$e_{1,7}(x_i)$	$e_{1,5}(x_i)$	$e_{1,7}(x_i)$	$e_{1,5}(x_i)$	$e_{1,7}(x_i)$	$e_{1,10}(x_i)$
0	0	0	0	0	0	0	0
0.2	$6.0310e-06$	$1.7000e-08$	$2.1190e-06$	$3.6977e-07$	$2.1189e-06$	$3.6976e-07$	$4.0034e-13$
0.4	$4.1026e-04$	$4.5620e-06$	$2.4829e-06$	$1.0646e-07$	$2.4828e-06$	$1.0645e-07$	$7.8619e-12$
0.6	$4.9809e-03$	$1.2277e-04$	$1.0519e-05$	$1.3253e-07$	$1.0519e-05$	$1.3252e-07$	$7.3454e-12$
0.8	$2.9917e-02$	$1.2899e-03$	$1.5004e-05$	$3.1732e-07$	$1.5003e-05$	$3.1732e-07$	$5.8746e-12$
1	$1.2238e-01$	$8.1037e-03$	$9.3000e-05$	$4.9275e-07$	$9.2999e-05$	$4.9274e-07$	$7.4116e-12$

Table 5. Comparison of the absolute errors for $y_2(x) = e^{-2x}$ of Example 3.

x_i	Taylor method [4]		Bessel method [10]		Present method		
	$e_{2,5}(x_i)$	$e_{2,7}(x_i)$	$e_{2,5}(x_i)$	$e_{2,7}(x_i)$	$e_{2,5}(x_i)$	$e_{2,7}(x_i)$	$e_{2,10}(x_i)$
0	0	0	0	0	0	0	0
0.2	$5.3793e-06$	$1.5500e-08$	$3.8983e-06$	$3.7623e-08$	$3.8983e-06$	$3.7622e-08$	$1.1124e-13$
0.4	$3.2623e-04$	$3.8187e-06$	$9.6593e-06$	$6.8382e-08$	$9.6593e-06$	$6.8381e-08$	$2.0695e-11$
0.6	$3.5302e-03$	$9.3961e-05$	$2.0735e-05$	$1.1996e-07$	$2.0735e-05$	$1.1996e-07$	$2.9825e-11$
0.8	$1.8878e-02$	$9.0226e-04$	$3.9343e-05$	$1.7505e-07$	$3.9343e-05$	$1.7504e-07$	$3.4090e-11$
1	$6.8667e-02$	$5.1766e-03$	$1.2025e-04$	$6.4723e-07$	$1.2025e-04$	$6.4722e-07$	$9.1456e-11$

Table 6. Comparison of the maximum absolute errors of Example 3 for various values of N.

N	Maximum absolute error $e_{1,N}$ of $y_1(x)$		Maximum absolute error $e_{2,N}$ of $y_2(x)$	
	Euler matrix method [12]	Present method	Euler matrix method [12]	Present method
3	$3.3934e-02$	$7.3516e-03$	$1.7421e-02$	$4.1424e-03$
5	$2.9720e-04$	$9.2999e-05$	$1.2026e-04$	$1.2025e-04$
7	$1.1469e-06$	$4.9274e-07$	$6.4723e-07$	$6.4722e-07$
10	$4.1474e-09$	$6.1789e-10$	$1.3517e-09$	$1.4248e-10$
12	$5.8207e-10$	$2.0433e-11$	$3.2316e-10$	$1.3728e-11$

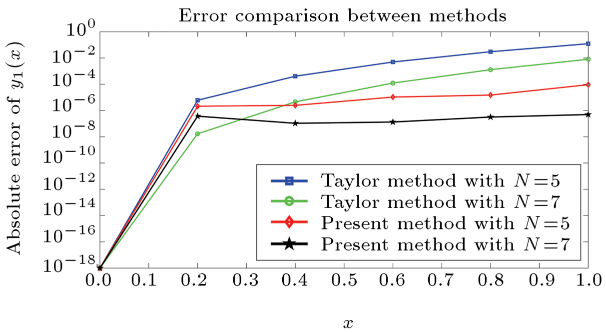

Figure 5. Comparison of the presented method and Taylor method for $y_{1,N}(x)$ of Example 3.

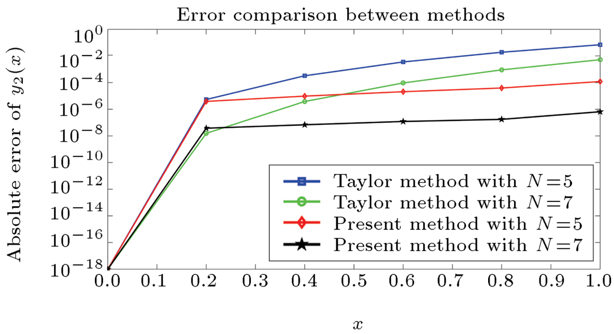

Figure 6. Comparison of the presented method and Taylor method for $y_{2,N}(x)$ of Example 3.

and 5 and Figures 5 and 6 that the results obtained by the present method is greatly superior to that obtained by the Taylor method [4]. However, the absolute errors of the Bessel method [10] and the present method are similar to each other. Also, we compare the maximum absolute error computed by the present method and Euler matrix method [12] for various values of N in Table 6. This fact is obvious from Table 6 that the results obtained by the present method are better than those obtained by the Euler matrix method.

6. Conclusion

In this article, we have studied a numerical scheme to solve a system of Volterra integral equations with variable coefficients. This method is based on the Bernoulli collocation method used for some problems of differential equations. The numerical results obtained by the proposed method are in good agreement with the exact solutions. Also, we notice that the proposed method provides us with a smaller absolute error than the absolute error obtained in [2,4,12]. Moreover, it could be applied to other integral equations to get the desired good accuracy. Here, we have examined one-dimensional problems only, but believe it would be straightforward to extend the method to further dimensions.

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References

1. Huang, Y. and Li, X-F. "Approximate solution of a class of linear integro-differential equations by Taylor expansion method", *Int. J. Comput. Math.*, **87**, pp. 1277-1288 (2010).

2. Rabbani, M., Maleknejad, K. and Aghazadeh, N. "Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method", *Appl. Math. Comput.*, **187**, pp. 1143-1146 (2007).

3. Gulsu, M. and Sezer, M. "Taylor collocation method for solution of systems of high-order linear Fredholm Volterra integro-differential equations", *Int. J. Comput. Math.*, **83**, pp. 429-448 (2006).

4. Sorkun, H.H. and Yalinbas, S. "Approximate solutions of linear Volterra integral equation systems with variable coefficients", *Appl. Math. Model.*, **34**, pp. 3451-3464 (2010).

5. Akyuz-Dacoglu, A. and Sezer, M. "Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations", *J. Franklin Inst.*, **342**, pp. 688-701 (2005).

6. Doha, E.H., Bhrawy, A.H. and Saker, M.A. "Integrals of Bernstein polynomials: An application for the solution of high even-order differential equations", *Appl. Math. Lett.*, **24**, pp. 559-565 (2011).

7. Doha, E.H., Bhrawy, A.H. and Saker, M.A. "On the derivatives of Bernstein polynomials: An application for the solution of higher even-order differential equations", *Bound. Value Prob.*, Article ID 829543 (2011).

8. Bhrawy, A.H., Tohidi, E. and Soleymani, F. "A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals", *Appl. Math. Comput.*, **219**, pp. 482-497 (2012).

9. Tohidi, E., Bhrawy, A.H. and Erfani, K. "A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation", *Appl. Math. Model.*, **37**, pp. 4283-4294 (2013).

10. Sahin, N., Yuzbasi, S. and Gulsu, M. "A collocation approach for solving systems of linear Volterra integral equations with variable coefficients", *Appl. Math. Comput.*, **62**, pp. 755-769 (2011).

11. Mirzaee, F. and Bimesl, S. "A new approach to numerical solution of second-order linear hyperbolic partial differential equations arising from physics and engineering", *Resu. Phys.*, **3**, pp. 241-247 (2013).

12. Mirzaee, F. and Bimesl, S. "A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients", *J. Egypt. Math. Soci.*, **22**, pp. 238-248 (2014).

13. Tohidi, E. and Kilicman, A. "A collocation method based on the Bernoulli operational matrix for solving nonlinear BVPs which arise from the problems in calculus of variation", *Math. Prob. Eng.*, Article ID 757206 (2013).

14. Toutounian, F., Tohidi, E. and Shateyi, S. "A collocation method based on Bernoulli operational matrix for solving high order linear complex differential equations in a rectangular domain", *Abst. Appl. Anal.*, Article ID 823098 (2013).

15. Doha, E.H., Bhrawy, A.H. and Ezz-Eldien, S.S. "A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order", *Comput. Math. Appl.*, **62**, pp. 2364-2373 (2011).

16. Doha, E.H., Bhrawy, A.H. and Ezz-Eldien, S.S. "A new Jacobi operational matrix: An application for solving fractional differential equation", *Appl. Math. Model.*, **36**, pp. 4931-4943 (2012).

17. Mashayekhi, S., Ordokhani, Y. and Razzaghi, M. "Hybrid functions approach for nonlinear constrained optimal control problems", *Commun. Nonlinear. Sci. Numer. Simulat.*, **17**, pp. 1831-1843 (2012).

18. Quarteroni, A., Sacco, R. and Saleri, F., *Numerical Mathematics*, Springer, NY (2007).

19. Rivlin, T.J., *An Introduction to the Approximation of Functions*, Dover Publications, NY (2003).

20. Stewart, G.W. "Matrix Algorithms", *Volume I: Basic Decompositions*, SIAM, Philadelphia (1998).

21. Golub, H.G. and Van Loan, C.F., *Matrix Computa-*

tions, The Johns Hopkins University Press, London (1996).

22. Higham, J.N., *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia (2002).

23. Natanson, I., *Constructive Function Theory*, **III**, Ungar, NY (1965).

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