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A new fractional analytical approach for treatment of a system of physical models using Laplace transform

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Abstract. In this study, the Homotopy Perturbation Transform Method (HPTM) is performed to give approximate and analytical solutions of the first order linear and nonlinear system of a time fractional partial differential equation. The HPTM is a combined form of the Laplace transform, the homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The proposed scheme finds the solutions without any discretization or restrictive assumptions, and is free of round-off errors, which, therefore, reduces the numerical computations to a great extent. The speed of convergence of the method is based on a rapidly convergent series with easily computable components. The fractional derivatives are described here in the Caputo sense. Numerical results show that the HPTM is easy to implement and accurate when applied to a time-fractional system of partial differential equations.

1. Introduction

Fractional order ordinary differential equations [1-4], as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics and engineering, and other applications. Fractional derivatives provide an excellent instrument for the description of memory and the hereditary properties of various materials and processes. Half-order derivatives and integrals have proven to be more useful for the formulation of certain electrochemical problems than classical models [4-6].

The homotopy perturbation method was introduced and applied by He [7-11]. Recently, many researchers [12-17] have obtained the series solution of the fractional differential equation using HPM. The proposed method is a coupling of the Laplace transformation, the homotopy perturbation method and He's polynomials, mainly due to Ghorbani [18-19]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations using various methods and a combination of the Laplace transform. Among these are Laplace decomposition methods [20-21] and the homotopy perturbation transform method [22-25]. The system of partial differential equations has attracted much attention in a variety of applied sciences. The general ideas and essential features of this system are of wide applicability. These systems were formally derived to describe wave propagation, control shallow water waves and examine the chemical reaction-diffusion model of Brusselator [13]. Recently, Baitainah et al. [26] applied HAM to obtain the solutions of linear and nonlinear systems of first and second order PDEs, and compared their results with the results of Wazwaz [27] and Ray [28] who used VIM and ADM, respectively. Recently, Yildirim et al. [29-30], Younesian et al. [31] and Khan et al. [32]

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solved many physical models using different methods.

The main aim of this article presents an approximate analytical solution of linear and nonlinear homogenous and non-homogenous time fractional partial differential equations using the homotopy perturbation transform method. We discuss how to solve fractional homogenous and nonhomogeneous equations using HPTM.

Definition 1. The Mittag-Leffler function, $E_\alpha(z)$, with $\alpha > 0$, is defined by the following series representation, valid in the whole complex plane [33]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1}$$

Definition 2. The Laplace transform, $L[f(t)]$, of the Riemann-Louisville fractional integral is defined as follows [2]:

$$L[I_t^\alpha f(t)] = s^{-\alpha} F(s). \tag{2}$$

Definition 3. The Laplace transform, $L[f(t)]$, of the Caputo fractional derivative is defined as follows [2]:

$$L[D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad n-1 < \alpha \le n. \tag{3}$$

2. Basic idea of the new homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method, we consider the following nonlinear system of fractional differential equations:

$$\begin{cases} D_t^{n\alpha} u(x,t) + R_1(u,v) + N_1(u,v) = q_1(x,t), \\ D_t^{n\alpha} v(x,t) + R_2(u,v) + N_2(u,v) = q_2(x,t), \end{cases}$$

$$n-1 < n\alpha \le n, \quad t > 0, \tag{4}$$

with the initial conditions:

$$u(x,0) = f(x),$$

$$v(x,0) = g(x), \tag{5}$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$ is the first order linear operator in x, R_1 and R_2. N_1 and N_2 are linear and nonlinear operators, and $q_1(x,t)$ and $q_2(x,t)$ are source terms. Now, the methodology consists of applying the Laplace transform first on both sides of Eq. (4). Thus, we get:

$$\begin{cases} L[D_t^{n\alpha} u(x,t)] = L[q_1(x,t)] - L[R_1(u,v) + N_1(u,v)], \\ L[D_t^{n\alpha} v(x,t)] = L[q_2(x,t)] - L[R_2(u,v) + N_2(u,v)]. \end{cases} \tag{6}$$

Now, using the differentiation property of the Laplace transform, we have:

$$\begin{cases} L[u(x,t)] = s^{-1} f(x) + s^{-n\alpha} L[q_1(x,t)] \\ \qquad - s^{-n\alpha} L[R_1(u,v) + N_1(u,v)], \\ L[v(x,t)] = s^{-1} g(x) + s^{-n\alpha} L[q_2(x,t)] \\ \qquad - s^{-n\alpha} L[R_2(u,v) + N_2(u,v)]. \end{cases} \tag{7}$$

Operating the inverse Laplace transform on both sides in Eq. (7), we get:

$$\begin{cases} u(x,t) = G_1(x,t) - L^{-1}\left(s^{-n\alpha} L[R_1(u,v) + N_1(u,v)]\right), \\ v(x,t) = G_2(x,t) - L^{-1}\left(s^{-n\alpha} L[R_2(u,v) + N_2(u,v)]\right), \end{cases} \tag{8}$$

where $G_1(x,t)$ and $G_2(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique [7-11], we can assume that the solutions can be expressed as a power series in p, as given below:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t),$$

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t), \tag{9}$$

where the homotopy parameter, p, is considered a small parameter ($p \in [0,1]$). The nonlinear terms, N_1 and N_2, can be decomposed as:

$$N_1(u,v) = \sum_{n=0}^{\infty} p^n H_{1,n}(u,v),$$

$$N_2(u,v) = \sum_{n=0}^{\infty} p^n H_{2,n}(u,v), \tag{10}$$

where $H_{1,n}(u,v)$ and $H_{2,n}(u,v)$ are He's polynomials, which can be calculated by the following formula:

$$H_{1,n}(u,v) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0},$$

$$H_{2,n}(u,v) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0},$$

$$n = 0, 1, 2, 3, \ldots$$

Substituting Eqs. (9) and (10) in Eq. (8) and using

HPM [7-11], we get:

$$\begin{cases}\sum_{n=0}^{\infty} p^n u_n(x,t) = G_1(x,t) - p\Big\{L^{-1}(s^{-n\alpha}L \\ \qquad (R_1(u,v) + \sum_{n=0}^{\infty} p^n H_{1,n}(u,v))\Big\}, \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = G_2(x,t) - p\Big\{L^{-1}(s^{-n\alpha}L \\ \qquad (R_2(u,v) + \sum_{n=0}^{\infty} p^n H_{2,n}(u,v))\Big\}.\end{cases} \tag{11}$$

This is a coupling of the Laplace transform and homotopy perturbation method using He's polynomials. Now, equating the coefficient of the corresponding power of p on both sides, the following approximations are obtained:

$$p^0 : \begin{cases} u_0(x,t) = G_1(x,t), \\ v_0(x,t) = G_2(x,t), \end{cases}$$

$$p^1 : \begin{cases} u_1(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_1(u_0,v_0) + H_{1,0}(u,v)]\right\}, \\ v_1(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_2(u_0,v_0) + H_{2,0}(u,v)]\right\}, \end{cases}$$

$$p^2 : \begin{cases} u_2(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_1(u_1,v_1) + H_{1,1}(u,v)]\right\}, \\ v_2(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_2(u_1,v_1) + H_{2,1}(u,v)]\right\}, \end{cases}$$

$$p^3 : \begin{cases} u_3(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_1(u_2,v_2) + H_{1,2}(u,v)]\right\}, \\ v_3(x,t) = -L^{-1}\left\{s^{-n\alpha}L[R_2(u_2,v_2) + H_{2,2}(u,v)]\right\}. \end{cases}$$

Proceeding in the same manner, the rest of the components, $u_n(x,t)$ and $v_n(x,t)$, $n \geq 4$ can be completely obtained and the series solutions are, thus, entirely determined.

Finally, we approximate the analytical solution, $u(x,t)$ and $v(x,t)$, by the truncated series:

$$u(x,t) = \lim_{N\to\infty} \sum_{n=1}^{N} u_n(x,t),$$

$$v(x,t) = \lim_{N\to\infty} \sum_{n=1}^{N} v_n(x,t). \tag{12}$$

The above series solutions generally converge very rapidly.

3. Numerical examples

In this section, three examples of nonlinear fractional order homogeneous and non-homogeneous time-fractional equations are solved to demonstrate the performance and efficiency of the HPM using a coupling of the Laplace transform and He's polynomials.

Example 1. We consider the following homogeneous linear system of time-fractional PDEs [34] as follows:

$$\begin{cases} D_t^{\alpha} u + D_x v - (u+v) = 0, \\ D_t^{\alpha} v + D_x u - (u+v) = 0, \end{cases} \quad 0 < \alpha \leq 1, \tag{13}$$

with initial conditions, $u(x,0) = \sinh x$ and $v(x,0) = \cosh x$. The system of Eq. (13) has the exact solution, $u(x,t) = \sinh(x+t)$ and $v(x,t) = \cosh(x+t)$ for the value of $\alpha = 1$.

Now, applying Laplace transform on both sides in Eq. (13), we get:

$$\begin{cases} L[u(x,t)] = s^{-1}\sinh x + s^{-\alpha}L[D_x v - (u+v)], \\ L[v(x,t)] = s^{-1}\cosh x + s^{-\alpha}L[D_x v - (u+v)]. \end{cases} \tag{14}$$

The inverse Laplace transform on both sides implies that:

$$\begin{cases} u(x,t) = \sinh x + L^{-1}\left(s^{-\alpha}L[D_x v - (u+v)]\right), \\ v(x,t) = \cosh x + L^{-1}\left(s^{-\alpha}L[D_x v - (u+v)]\right). \end{cases} \tag{15}$$

Now, we apply the homotopy perturbation method:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = \sinh x \\ \qquad + p\left(L^{-1}\left(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_{1,n}(u,v)]\right)\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = \cosh x \\ \qquad + p\left(L^{-1}\left(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_{2,n}(u,v)]\right)\right), \end{cases} \tag{16}$$

where $H_{1,n}(u,v)$ and $H_{2,n}(u,v)$ are He's polynomials, and all components of He's polynomials can be obtained by $H_{1,n}(u,v) = v_{nx} - u_n - v_n$ and $H_{2,n}(u,v) = u_{nx} - u_n - v_n$, $\forall n \in N$.

Now, equating the coefficient of the corresponding power of p on both sides of Eq. (16), we get:

$$p^0 : \begin{cases} u_0(x,t) = \sinh x, \\ v_0(x,t) = \cosh x, \end{cases}$$

$$p^1 : \begin{cases} u_1(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{1,0}(u,v)]\right) = \cosh x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ v_1(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,0}(u,v)]\right) = \sinh x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{cases}$$

$$p^2 : \begin{cases} u_2(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{1,1}(u,v)]\right) = \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,1}(u,v)]\right) = \cosh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases}$$

$$p^3 : \begin{cases} u_3(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{1,2}(u,v)]\right) = \cosh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,2}(u,v)]\right) = \sinh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases}$$

$$p^4 : \begin{cases} u_4(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{1,3}(u,v)]\right) = \sinh x \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \cdots \\ v_4(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,3}(u,v)]\right) = \cosh x \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \cdots \end{cases}$$

Using the above terms, solution $u(x,t)$ is given as:

$$u(x,t) = \sinh x\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\ldots\right) + \cosh x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right),$$

$$v(x,t) = \cosh x\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\ldots\right) + \sinh x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right).$$

Hence:

$$u(x,t) = \sinh x\left(\frac{E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)}{2}\right) + \cosh x\left(\frac{E_\alpha(t^\alpha)-E_\alpha(-t^\alpha)}{2}\right), \quad (17)$$

$$v(x,t) = \cosh x\left(\frac{E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)}{2}\right) + \sinh x\left(\frac{E_\alpha(t^\alpha)-E_\alpha(-t^\alpha)}{2}\right). \quad (18)$$

As $\alpha = 1$, this series has the closed form $u(x,t) = \sinh(x+t)$ and $v(x,t) = \cosh(x+t)$ which is an exact solution of the given homogenous system of the partial differential equation (Eq. (13)) for $\alpha = 1$.

Example 2. In this example, we consider the following non-homogeneous linear system of time-fractional PDEs [34] as follows:

$$\begin{cases} D_t^\alpha u - D_x v - (u-v) = -2, \\ D_t^\alpha v - D_x u - (u-v) = -2, \end{cases} \quad 0<\alpha\le 1, \quad (19)$$

with initial conditions $u(x,0) = 1+e^x$ and $v(x,0) = -1+e^x$. This system of the non-homogenous equation (Eq. (14)) has the exact solution, $u(x,t) = 1+e^{x+t}$ and $v(x,t) = -1+e^{x-t}$, for the value of $\alpha = 1$.

Taking the Laplace transform on both sides in the system of Eq. (19) and, then, using the differentiation property of the Laplace transform and given initial conditions, we get:

$$\begin{cases} L[u(x,t)] = s^{-1}(e^x+1) - 2s^{-(\alpha+1)} + s^{-\alpha}L[u-v+D_x v], \\ L[v(x,t)] = s^{-1}(e^x-1) - 2s^{-(\alpha+1)} + s^{-\alpha}L[u-v+D_x u]. \end{cases} \quad (20)$$

Applying the inverse Laplace transform on both sides in Eq. (20), we get:

$$\begin{cases} u(x,t) = (e^x+1) - \frac{2t^\alpha}{\Gamma(\alpha+1)} + L^{-1}(s^{-\alpha}L[u-v+D_x v]), \\ v(x,t) = (e^x-1) - \frac{2t^\alpha}{\Gamma(\alpha+1)} + L^{-1}(s^{-\alpha}L[u-v-D_x u]). \end{cases} \quad (21)$$

Now, we apply the homotopy perturbation method:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = (e^x+1) - \frac{2t^\alpha}{\Gamma(\alpha+1)} \\ \quad + p\left(L^{-1}\left(s^{-\alpha}L\left[\sum_{n=0}^{\infty} p^n H_{1,n}(u,v)\right]\right)\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = (e^x-1) - \frac{2t^\alpha}{\Gamma(\alpha+1)} \\ \quad + p\left(L^{-1}\left(s^{-\alpha}L\left[\sum_{n=0}^{\infty} p^n H_{2,n}(u,v)\right]\right)\right), \end{cases} \quad (22)$$

where $H_n(u,v)$ and $H_n(u,v)$ are He's polynomials, and all components of He's polynomials can be obtained by $H_{1,n}(u) = u_n - v_n + D_x v_n$ and $H_{2,n}(v) = u_n - v_n - D_x u_n$, $\forall n \in N$.

Now, equating the coefficient of the corresponding power of p on both sides in Eq. (22), we get:

$$p^0: \begin{cases} u_0(x,t) = e^x - \frac{2t^\alpha}{\Gamma(\alpha+1)} + 1, \\ v_0(x,t) = e^x - \frac{2t^\alpha}{\Gamma(\alpha+1)} - 1, \end{cases}$$

$$p^1: \begin{cases} u_1(x,t) = L^{-1}(s^{-\alpha}L[H_{1,0}(u,v)]) = (2+e^x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x,t) = L^{-1}(s^{-\alpha}L[H_{2,0}(u,v)]) = (2-e^x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \end{cases}$$

$$p^2: \begin{cases} u_2(x,t) = L^{-1}(s^{-\alpha}L[H_{1,1}(u,v)]) = e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x,t) = L^{-1}(s^{-\alpha}L[H_{2,1}(u,v)]) = e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{cases}$$

$$p^3: \begin{cases} u_3(x,t) = -L^{-1}(s^{-\alpha}L[H_{1,2}(u,v)]) = e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3(x,t) = -L^{-1}(s^{-\alpha}L[H_{2,2}(u,v)]) = -e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{cases}$$

Using the above terms, the solution $u(x,t)$ is given as:

$$u(x,t) = e^x\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right)+1 = e^x E_\alpha(t^\alpha)+1,$$

$$v(x,t) = e^x\left(1-\frac{t^\alpha}{\Gamma(alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right)-1 = e^x E_\alpha(-t^\alpha)-1.$$

Now, for a standard case, i.e. for $\alpha = 1$, this series has the closed form of the solution, $u(x,t) = e^{x+t}+1$ and $v(x,t) = e^{x-t}-1$ which is an exact solution of the given non-homogenous system of the partial differential equation (Eq. (16)) for $\alpha = 1$.

Example 3. In this example, we consider the following non-homogeneous nonlinear system of time-fractional PDEs [34] as follows:

$$\begin{cases} D_t^\alpha u - v D_x u - u = 1, \\ D_t^\alpha v + u D_x v + v = 1, \end{cases} \quad 0<\alpha\le 1, \quad (23)$$

with initial conditions $u(x,0) = e^{-x}$ and $v(x,0) = e^{x}$. This system of the nonlinear non-homogenous equation (Eq. (23)) has the exact solution $u(x,t) = e^{-x+t}$ and $v(x,t) = e^{x-t}$ for the value of $\alpha = 1$.

Taking the Laplace transform on both sides in Eq. (23) and, then, using the differentiation property of the Laplace transform and given initial conditions, we get:

$$\begin{cases} L[u(x,t)] = s^{-1}e^{-x} + s^{-(\alpha+1)} + s^{-\alpha}L[u + vD_x u], \\ L[v(x,t)] = s^{-1}e^{x} + s^{-(\alpha+1)} - s^{-\alpha}L[v + vD_x v]. \end{cases} \quad (24)$$

The inverse Laplace transform on both sides implies that:

$$\begin{cases} u(x,t) = e^{-x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + L^{-1}\left(s^{-\alpha}L[u + vD_x u]\right), \\ v(x,t) = e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - L^{-1}\left(s^{-\alpha}L[v + uD_x v]\right). \end{cases} \quad (25)$$

We represent $u(x,t)$ and $v(x,t)$ by the infinite series Eq. (21). Then, by inserting these series into both sides of Eq. (25), and using HPM [5-9], we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ \qquad + p\left(L^{-1}\left(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_{1,n}(u,v)]\right)\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ \qquad - p\left(L^{-1}\left(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_{2,n}(u,v)]\right)\right), \end{cases} \quad (26)$$

where $H_{1,n}(u,v)$ and $H_{2,n}(u,v)$ are He's polynomials, which represent nonlinear terms $u + vD_x u$ and $v + uD_x v$, respectively. We have a few terms of He's polynomials for $u + vD_x u$ and $v + uD_x v$ which are given by:

$$H_{1,0} = u_0 + v_0 u_{0x},$$

$$H_{1,1} = u_1 + v_1 u_{0x} + v_0 u_{1x},$$

$$H_{1,2} = u_2 + v_0 u_{2x} + v_1 u_{1x} + v_2 u_{0x},$$

$$H_{1,3} = u_3 + v_0 u_{3x} + v_1 u_{2x} + v_2 u_{1x} + v_3 u_{0x}, \cdots$$

$$H_{2.0} = v_0 + u_0 v_{0x}$$

$$H_{2,1} = v_1 + u_1 v_{0x} + u_0 v_{1x},$$

$$H_{2,2} = v_2 + u_0 v_{2x} + u_1 v_{1x} + u_2 v_{0x},$$

$$H_{1,3} = v_3 + u_0 v_{3x} + u_1 v_{2x} + u_2 v_{1x} + u_3 v_{0x}, \cdots$$

Now, equating the coefficient of the corresponding power of p on both sides of Eq. (26), we get:

$$p^0 : \begin{cases} u_0(x,t) = e^{-x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ v_0(x,t) = e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{cases}$$

$$p^1 : \begin{cases} u_1(x,t) = L^{-1}\left(s^{-\alpha}L[H_{1,0}(u,v)]\right) \\ \qquad = (e^{-x} - 1)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right), \\ v_1(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,0}(u,v)]\right) \\ \qquad = -(e^{x} + 1)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right), \end{cases}$$

$$p^2 : \begin{cases} u_2(x,t) = L^{-1}\left(s^{-\alpha}L[H_{1,1}(u,v)]\right) = \\ \quad (2e^{-x} - 1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(3 - \frac{e^{-x}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right) \\ \quad \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{e^{-x}\Gamma(3\alpha 1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\ v_2(x,t) = -L^{-1}\left(s^{-\alpha}L[H_{2,1}(u,v)]\right) = \\ \quad (2e^{x} + 1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(3 + \frac{e^{x}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right) \\ \quad \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{e^{x}\Gamma(3\alpha 1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}. \end{cases}$$

Proceeding in this manner, the rest of the components $u_n(x,t)$ and $v_n(x,t)$, $n \geq 3$, can be completely obtained, and the series solutions are, thus, entirely determined. Finally, we approximate the analytical solution, $u(x,t)$ and $v(x,t)$, by the truncated series:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=1}^{N} u_n(x,t),$$

$$v(x,t) = \lim_{N\to\infty}\sum_{n=1}^{N} v_n(x,t). \quad (27)$$

The above series solutions generally converge very rapidly.

4. Numerical result and discussion

In this section, Figures 1-4 show approximate solutions for different fractional Brownian motions, $\alpha = 0.7$, 0.8, 0.9, and, also, for the standard motion, $\alpha = 1$. The numerical values of $u(x,t)$ and $v(x,t)$ vs. time t at $x = 1$ are shown by Figures 1-4. It is seen from Example 1, which is described by Figures 1 and 2, that $u(x,t)$ and

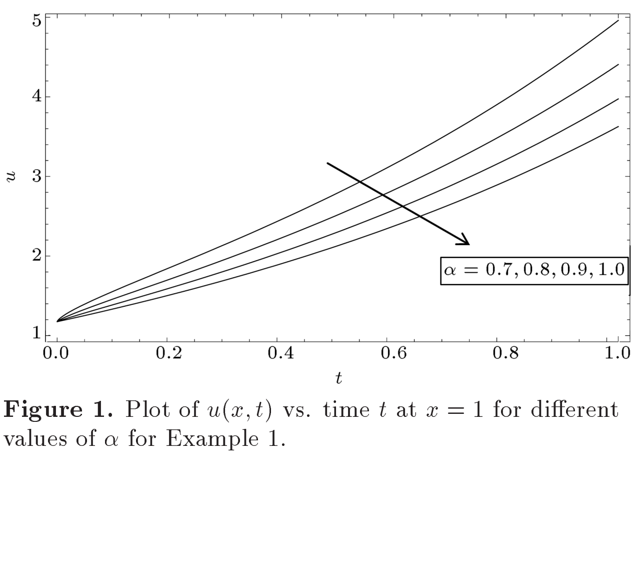

Figure 1. Plot of $u(x,t)$ vs. time t at $x = 1$ for different values of α for Example 1.

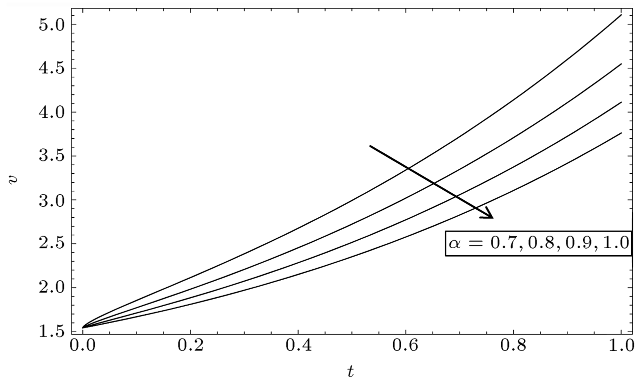

Figure 2. Plot of $v(x,t)$ vs. time t at $x = 1$ for different values of α for Example 1.

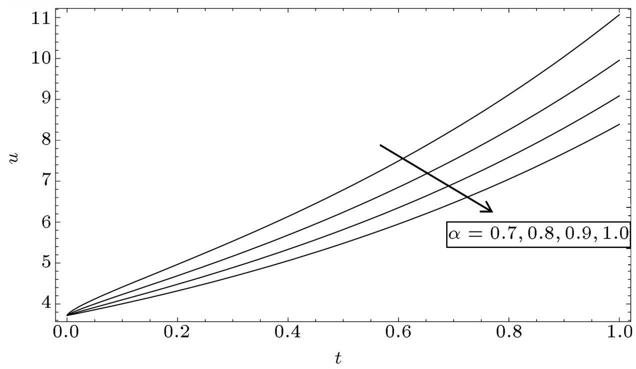

Figure 3. Plot of $u(x,t)$ vs. time t at $x = 1$ for different values of α for Example 2.

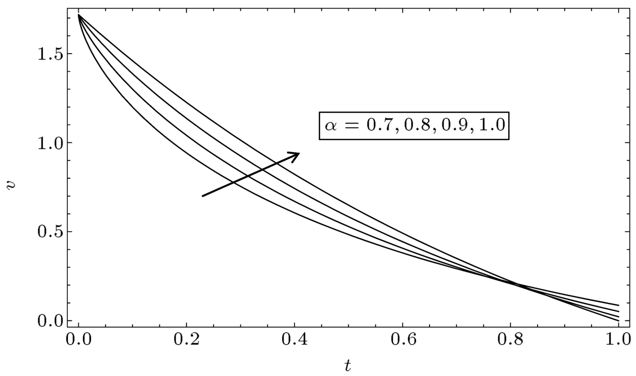

Figure 4. Plot of $v(x,t)$ vs. time t at $x = 1$ for different values of α for Example 2.

$v(x,t)$ increase with an increase in t for different values of α.

However, for Example 2, the nature of the approximate solution, $u(x,t)$, which is depicted in Figure 3, is similar to Example 1, and the behavior of the approximate solution, $v(x,t)$, which is depicted in Figure 4, is remarkably opposite. It is observed that the approximate solution, $u(x,t)$, increases with the increase in t for different values of α, and the approximate solution, $v(x,t)$, decreases with the increase in t for different values of α.

5. Concluding remarks

This paper develops an effective modification of the homotopy perturbation method, which is coupled with the Laplace transform and He's polynomials, and its validity is studied over a wide range, with three examples of linear and nonlinear time fractional homogenous and non-homogenous PDEs. It is clear that the Laplace homotopy perturbation method yields very accurate approximate solutions using only a few iterates. Thus, it can be concluded that the LHPM methodology is very powerful and efficient in finding approximate solutions, as well as numerical solutions.

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