



Research Note

Effects of Joule heating and thermophoresis on stretched flow with convective boundary conditions

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 Stretching surface.

Abstract. The effects of heat and mass transfer on two-dimensional magnetohydrodynamic (MHD) flow of Maxwell fluid over a stretching surface are discussed. The stretching surface satisfies the convective boundary conditions. In addition, the analysis has been carried out in the presence of Joule heating, thermal radiation and thermophoresis. Governing partial differential equations are first reduced into ordinary differential equations and then computed for series solutions. Numerical values of local Nusselt and Sherwood numbers are presented and examined.

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1. Introduction

Fluid flow in non-Newtonian fluids has been extensively investigated. Wang and Tan [1] examined stability criteria for the flow of Maxwell fluid passed through a porous medium. The unsteady flow of a fractional Maxwell model between two infinite cylinders was considered by Fetecau et al. [2]. The motion in the fluid is due to the inner cylinder being subjected to time dependent shear stress. Exact solutions are developed via Hankel and Laplace transforms. Jian et al. [3] considered the eletroosmotic flow of generalized Maxwell fluid in a two-dimensional microchannel and developed series solutions. They also employed a numerical technique to analyze the volumetric flow rate. Mukhopadhyay [4] investigated the effects of suction/injection in a flow of Maxwell fluid in a porous medium. The MHD Falkner-Skan flow of Maxwell fluid was studied by Abbasbandy et al. [5]. They used the rational Chebyshev collocation method for the presentation of solutions. The boundary layer flow of Maxwell fluid with power law heat flux was investigated

by Shehzad et al. [6]. They discussed the velocity and temperature of the fluid by employing a homotopy analysis method. Moreover, flow, using a stretching surface, has applications in metallurgy and chemical engineering. Numerous studies in the huge collection of available literature on the topic deal with flow without convective boundary conditions. A few researchers, for example, Aziz [7], initiated the concept of convective boundary conditions. He discussed the boundary layer flow of viscous fluid over a flat plate subject to convective surface conditions. Makinde and Aziz [8] presented the behaviour of the boundary layer flow of a nanofluid over a stretching sheet with convective boundary conditions. Yao et al. [9] investigated the effects of convective heat transfer of Newtonian fluid over the generalized stretching/shrinking sheet. This work has been further extended by Hayat et al. [10]. They examined the characteristics of the boundary layer flow of Maxwell fluid over a stretching sheet with convective heat transfer. Three dimensional flow of Jeffery fluid with convective heat transfer over the stretching sheet is examined by Shehzad et al. [11]. Similar solutions for flow and heat transfer over a permeable surface with convective boundary conditions are investigated by Ishak [12]. Makinde and Olanrewaju [13] examined the

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effects of buoyancy forces and convective heat transfer in the thermal boundary layer.

Another important issue until now neglected is the mechanism of thermophoresis. Such a mechanism is useful for the migration of small particles in the direction of the decreasing thermal gradient [14] and for particle collection [15]. The velocity acquired by the particle is known as thermophoretic velocity, and the force experienced by the suspended particle is called thermophoretic force [16]. Specifically, it has applications in aerosol technology, deposition of silicon thin films, and radioactive particle deposition in nuclear reactor safety simulations [17–20]. Also, convective free mixed convection and forced convection flows are significant in petroleum extraction, in soil, in the storage of agricultural products, as a porous material heat exchanger, etc. Bazdid-Tehrani and Nazarpour [21] considered buoyancy-assisted flow in symmetrically heated plates combined with mixed convective-radiative heat transfer. They analyzed the radiation effects by choosing two radiative parameters. MHD mixed convection flow over a heated wall in a lid-driven cavity was investigated by Kefayati et al. [22] using the Lattice Boltzmann simulation scheme. Mozayyeni and Rahimi [23] examined the effect of constant magnetic field on the mixed convection flow in cylindrical annuli. They assumed that the magnetic field is applied to the radial direction and the flow is generated due to rotation in the outer cylinder. The mixed convection flow of Casson fluid with a convective surface condition was analytically discussed by Hayat et al. [24]. In another study, Hayat et al. [25] provided the series solution for the mixed convection flow of Maxwell fluid in the presence of thermal stratification effects.

This paper studies the mixed convection flow of MHD non-Newtonian fluid over a stretching surface. Thermal radiation, thermophoresis and Joule heating effects are included in the mathematical modelling. This is the first attempt to investigate all the above effects in the presence of convective boundary conditions. Such flow analysis is not available yet, even for the case of viscous fluid. Our main emphasis is to analyze the thermophoresis and Joule heating effects with convective type surface conditions. All previous investigations available in the literature on thermophoresis and Joule heating effects were made considering constant surface temperature conditions. We hope that this study will lead to further investigation by different researchers into various flow geometries and different fluid models. Constitutive equations for Maxwell fluid [10] are employed in the problem formulation. This paper is organized in the following fashion. Section 2 consists of problem formulation. Section 3 deals with the development of series solutions [26–30]. Convergence analysis related to the series solutions is given in

Section 4. Section 5 contains interpretation regarding various involved parameters. Section 6 includes the conclusions.

2. Mathematical model

We investigate the heat and mass transfer effects in the steady MHD flow of Maxwell fluid past a vertical stretching sheet. The surface is stretched in its own plane with a velocity proportional to its distance from the fixed origin, $x = 0$. A uniform magnetic field of strength B_0 acts parallel to the y direction. The induced magnetic field is neglected for small magnetic Reynolds number. Further, a convective boundary condition is assumed for heat transfer analysis and constant concentration, C_w . The ambient temperature and concentration are taken as T_∞ and C_∞ , respectively. We consider here $T_f > T_\infty$ and $C_w > C_\infty$. For mass deposition on the surface, the effects of thermophoresis are considered. The equations which can govern the present flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda_1 \left(u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right) \\ = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma^* B_0^2}{\rho} \left(u + \lambda_1 v \frac{\partial u}{\partial y} \right) \\ + g (\beta_T (T - T_\infty) + \beta_c (C - C_\infty)), \end{aligned} \quad (2)$$

$$\begin{aligned} u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3k^* \rho c_p} \frac{\partial^2 T}{\partial y^2} \\ + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\sigma^* B_0^2}{\rho c_p} u^2, \end{aligned} \quad (3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - \frac{\partial}{\partial y} (V_T (C - C_\infty)), \quad (4)$$

where u and v denote the velocity components in the x and y directions, respectively, λ_1 is the relaxation time, T is the fluid temperature, C is the concentration field, g is the gravitational acceleration, $\nu = (\frac{\mu}{\rho})$ is the kinematic viscosity, ρ is the density of the fluid, β_T and β_c are the thermal expansion coefficients of temperature and concentration, respectively, c_p is the specific heat, σ^* is the Stefan-Boltzmann constant, k^* is the mean absorption, D is the diffusion coefficient, and V_T is the thermophoretic velocity.

It is seen that the influence of thermophoresis is usually prescribed by means of the average velocity, which a particle will acquire when exposed to a temperature gradient. The temperature in the y direction for

boundary layer flow is larger when compared with the x direction and, thus, only the thermophoretic velocity (V_T) in the y direction is taken, i.e.:

$$V_T = -k_1 \frac{v}{T_r} \frac{\partial T}{\partial y}, \quad (5)$$

in which k_1 indicates the themophoretic coefficient and T_r denotes the reference temperature. Thermophoretic parameter, τ , satisfies the following expression:

$$\tau = \frac{-k_1(T_f - T_\infty)}{T_r}. \quad (6)$$

The appropriate boundary conditions are prescribed as follows:

$$\begin{aligned} u &= u_w(x) = ax, & v &= 0, \\ -k \frac{\partial T}{\partial y} &= h(T_f - T), & C &= C_w, \\ \text{at } y &= 0, \\ u &\rightarrow 0, & \frac{\partial u}{\partial y} &\rightarrow 0, & T &\rightarrow T_\infty, \\ C &\rightarrow C_\infty & \text{as } y &\rightarrow \infty. \end{aligned} \quad (7)$$

Here, a , b and c are the positive constants.

Defining the transformations:

$$\begin{aligned} \eta &= y \sqrt{\frac{a}{\nu}}, & u &= ax f'(\eta), & v &= -\sqrt{a\nu} f(\eta), \\ \theta(\eta) &= \frac{T - T_\infty}{T_f - T_\infty}, & \phi(\eta) &= \frac{C - C_\infty}{C_w - C_\infty}, \end{aligned} \quad (8)$$

continuity Eq. (1) is clearly satisfied and the resulting problems in f , θ and ϕ are:

$$\begin{aligned} f''' + f f'' - f'^2 + \beta (2 f f' f'' - f^2 f''') - M^2 f' \\ + M^2 \beta f f'' + \gamma^* (\theta + N \phi) = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} \left(1 + \frac{4}{3} R_d\right) \theta'' + \text{Pr } f \theta' + \text{Pr Ec } f'^2 \\ + M^2 \text{Pr Ec } f'^2 = 0, \end{aligned} \quad (10)$$

$$\phi'' + \text{Sc } f \phi' - \text{Sc } \tau (\phi' \theta' - \phi \theta'') = 0, \quad (11)$$

$$\begin{aligned} f(0) &= 0, & f'(0) &= 1, \\ f'(\infty) &\rightarrow 0, & f''(\infty) &\rightarrow 0, \\ \theta'(0) &= -\gamma(1 - \theta(0)), \\ \theta(\infty) &\rightarrow 0, & \phi(0) &= 1, & \phi(\infty) &\rightarrow 0, \end{aligned} \quad (12)$$

where the prime denotes the derivative, with respect to η . Here:

$$\begin{aligned} \beta &= \lambda_1 a, & M^2 &= \frac{\sigma^* B_0^2}{\rho a}, & \gamma^* &= \frac{\text{Gr}_x}{\text{Re}_x^2}, \\ \text{Gr}_x &= \frac{g \beta (T_w - T_\infty) x^3 / \nu^2}{u_w^2 x / \nu^2}, & \text{Re}_x &= \frac{u_w(x)}{\nu}, \\ N &= \frac{\beta_c}{\beta_T} \frac{C_w - C_\infty}{T_w - T_\infty}, & \text{Pr} &= \frac{\mu c_p}{k}, \\ \text{Ec} &= \frac{u_w^2}{c_p (T_w - T_\infty)}, & \text{Sc} &= \frac{\nu}{D}, \\ \gamma &= \frac{h}{k} \sqrt{\frac{\nu}{\alpha}}, & R_d &= \frac{4 \sigma^* T_\infty^3}{k^* k}. \end{aligned} \quad (13)$$

In the above expression, β is the Deborah number, M is the Hartman number, γ^* is the local buoyancy parameter, Gr_x is the local Grashof number, Re_x is the local Reynolds number, N is the constant dimensionless concentration buoyancy parameter, Pr is the Prandtl number, Ec is the Eckert number, Sc is the Schmidt number, γ is the Biot number and R_d is the radiation parameter.

The local Nusselt, Nu_x , and Sherwood number, Sh , can be defined as:

$$\text{Nu}_x = \frac{x q_w}{k(T_f - T_\infty)}, \quad \text{Sh} = \frac{x j_w}{D(C_w - C_\infty)}, \quad (14)$$

in which q_w and j_w denote the wall heat flux and the mass flux from the plate. These are defined as:

$$q_w = -\left(\frac{\partial T}{\partial y}\right)_{y=0}, \quad j_w = -\left(\frac{\partial C}{\partial y}\right)_{y=0}. \quad (15)$$

The above expressions in dimensionless variables are reduced as follows:

$$\text{Nu}_x \text{Re}_x^{-1/2} = -\theta'(0), \quad \text{Sh}/\text{Re}_x^{1/2} = -\phi'(0). \quad (16)$$

3. Homotopy analysis solutions

Taking the base functions in the form:

$$\{\eta^k \exp(-n\eta), k \geq 0, n \geq 0\}, \quad (17)$$

we express that:

$$f_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \quad (18)$$

$$\theta_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta), \quad (19)$$

$$\phi_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{m,n}^k \eta^k \exp(-n\eta), \quad (20)$$

where $a_{m,n}^k$, $b_{m,n}^k$ and $c_{m,n}^k$ are the coefficients. The initial guesses f_0 , θ_0 and ϕ_0 , of $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$, are:

$$f_0(\eta) = (1 - \exp(-\eta)),$$

$$\theta_0(\eta) = \exp(-\eta),$$

$$\phi_0(\eta) = \exp(-\eta), \quad (21)$$

in which the auxiliary linear operators are chosen through the equations:

$$L_f = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta},$$

$$L_\theta = \frac{d^2 \theta}{d\eta^2} - \theta,$$

$$L_\phi = \frac{d^2 \phi}{d\eta^2} - \theta, \quad (22)$$

$$L_f [C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta)] = 0,$$

$$L_\theta [C_4 \exp(\eta) + C_5 \exp(-\eta)] = 0,$$

$$L_\phi [C_6 \exp(\eta) + C_7 \exp(-\eta)] = 0, \quad (23)$$

in which C_i ($i = 1 - 7$) are the arbitrary constants.

3.1. Zeroth and m th order deformation problems

Defining the non-linear operators $\mathbf{N}_f$, $\mathbf{N}_\theta$ and $\mathbf{N}_\phi$ in the forms:

$$\begin{aligned} \mathbf{N}_f [\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)] &= \frac{\partial^3 \hat{f}(\eta; p)}{\partial \eta^3} \\ &+ (M^2 \beta + 1) \hat{f}(\eta; p) \frac{\partial^2 \hat{f}(\eta; p)}{\partial \eta^2} \\ &- \left(\frac{\partial \hat{f}(\eta; p)}{\partial \eta} \right)^2 - M^2 \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \\ &+ \beta \left(2 \hat{f}(\eta; p) \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \frac{\partial^2 \hat{f}(\eta; p)}{\partial \eta^2} \right. \\ &\left. - \hat{f}(\eta; p)^2 \frac{\partial^3 \hat{f}(\eta; p)}{\partial \eta^3} \right) \\ &+ \gamma^* \left(\hat{\theta}(\eta; p) + N \hat{\phi}(\eta; p) \right), \end{aligned} \quad (24)$$

$$\begin{aligned} \mathbf{N}_\theta [\hat{f}(\eta; p), \hat{\theta}(\eta; p)] &= \left(1 + \frac{4}{3} R_d \right) \frac{\partial^2 \hat{\theta}(\eta; p)}{\partial \eta^2} \\ &+ \text{Pr} \left(\hat{f}(\eta; p) \frac{\partial \hat{\theta}(\eta; p)}{\partial \eta} - \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \hat{\theta}(\eta; p) \right) \\ &+ \text{Pr Ec} \left(\frac{\partial^2 \hat{f}(\eta; p)}{\partial \eta^2} \right)^2 + M^2 \text{Pr Ec} \left(\frac{\partial \hat{f}(\eta; p)}{\partial \eta^2} \right)^2, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathbf{N}_\phi [\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)] &= \frac{\partial^2 \hat{\phi}(\eta; p)}{\partial \eta^2} \\ &+ \text{Sc} \left(\hat{f}(\eta; p) \frac{\partial \hat{\phi}(\eta; p)}{\partial \eta^2} - \frac{\partial \hat{f}(\eta; p)}{\partial \eta^2} \hat{\phi}(\eta; p) \right) \\ &- \text{Sc} \tau \left(\frac{\partial \hat{\theta}(\eta; p)}{\partial \eta} \frac{\partial \hat{\phi}(\eta; p)}{\partial \eta} - \hat{\phi}(\eta; p) \frac{\partial^2 \hat{\theta}(\eta; p)}{\partial \eta^2} \right), \end{aligned} \quad (26)$$

the corresponding problems at the zeroth and m th orders can be written as:

$$\begin{aligned} (1-p)L_f [\hat{f}(\eta; p) - f_0(\eta)] &= p \hbar_f \mathbf{N}_f [\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)], \end{aligned} \quad (27)$$

$$\begin{aligned} (1-p)L_\theta [\hat{\theta}(\eta; p) - \theta_0(\eta)] &= p \hbar_\theta \mathbf{N}_\theta [\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)], \end{aligned} \quad (28)$$

$$\begin{aligned} (1-p)L_\phi [\hat{\phi}(\eta; p) - \phi_0(\eta)] &= p \hbar_\phi \mathbf{N}_\phi [\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)], \end{aligned} \quad (29)$$

$$\begin{aligned} \hat{f}(\eta; p) \Big|_{\eta=0} &= 0, \\ \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \Big|_{\eta=0} &= 1, \\ \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \Big|_{\eta=\infty} &= 0, \\ \frac{\partial \hat{\theta}(\eta; p)}{\partial \eta} \Big|_{\eta=0} &= -\gamma (1 - \theta(0)), \end{aligned} \quad (30)$$

$$\hat{\theta}(\eta; p) \Big|_{\eta=\infty} = 0, \quad (31)$$

$$\hat{\phi}(\eta; p) \Big|_{\eta=0} = 1, \quad \hat{\phi}(\eta; p) \Big|_{\eta=\infty} = 0, \quad (32)$$

$$L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathbf{R}_m^f(\eta), \quad (33)$$

$$L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathbf{R}_m^\theta(\eta), \quad (34)$$

$$L_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = \hbar_\phi \mathbf{R}_m^\phi(\eta), \quad (35)$$

$$\mathbf{R}_m^f(\eta) = f_{m-1}'''(\eta) + \gamma(\theta_{m-1-k} + N \phi_{m-1-k})$$

$$\begin{aligned} & \sum_{k=0}^{m-1} \left[(M^2 \beta + 1) f_{m-1-k} f_k'' - f_{m-1-k}' f_k' \right. \\ & + \beta \left(2 f_{m-1-k} \sum_{l=0}^k f_{k-l}' f_l'' \right. \\ & \left. \left. - f_{m-1-k} \sum_{l=0}^k f_{k-l} f_l''' \right) \right], \end{aligned} \quad (36)$$

$$\begin{aligned} \mathbf{R}_m^\theta(\eta) = & \left(1 + \frac{4}{3} R_d \right) \theta_{m-1}''(\eta) \\ & + \text{Pr} \sum_{k=0}^{m-1} \left[f_{m-1-k} \theta_k' - \theta_{m-1-k} f_k' \right. \\ & \left. + \text{Ec} f_{m-1-k}'' f_k'' + M^2 \text{Ec} f_{m-1-k}' f_k' \right], \end{aligned} \quad (37)$$

$$\begin{aligned} \mathbf{R}_m^\phi(\eta) = & \phi_{m-1}''(\eta) + \text{Sc} \sum_{k=0}^{m-1} [f_{m-1-k} \phi_k' - \phi_{m-1-k} f_k'] \\ & + \text{Sc} \tau (\phi' \theta' - \phi \theta_{m-1}''), \end{aligned} \quad (38)$$

$$\begin{aligned} f_m(0) = 0, \quad f_m'(\infty) = 0, \quad f_m'(\infty) = 0, \\ \theta_m'(0) - \gamma \theta_m(0) = \theta_m(\infty) = 0, \quad \theta_m(0) = 0, \\ \phi_m(0) = 0, \quad \phi_m(\infty) = 0, \end{aligned} \quad (39)$$

where $p \in [0, 1]$ is an embedding parameter and $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are the auxiliary parameters.

According to Taylor series, we obtain:

$$\begin{aligned} \hat{f}(\eta; p) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \\ \hat{\theta}(\eta; p) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \end{aligned} \quad (40)$$

$$\hat{\phi}(\eta; p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) p^m, \quad (41)$$

$$f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \hat{f}(\eta; p)}{\partial p^m} \right|_{p=0},$$

$$\theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \hat{\theta}(\eta; p)}{\partial p^m} \right|_{p=0}, \quad (42)$$

$$\phi_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \hat{\phi}(\eta; p)}{\partial p^m} \right|_{p=0}. \quad (43)$$

The auxiliary parameters are selected in such a manner that:

$$\begin{aligned} f(\eta) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \\ \theta(\eta) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \end{aligned} \quad (44)$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \quad (45)$$

The general solutions of Eqs. (34)-(36) are:

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta), \quad (46)$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 \exp(\eta) + C_5 \exp(-\eta), \quad (47)$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_6 \exp(\eta) + C_7 \exp(-\eta), \quad (48)$$

in which $f_m^*(\eta)$, $\theta_m^*(\eta)$ and $\phi_m^*(\eta)$ are the special solutions.

4. Convergence of the homotopy solutions

Clearly, the homotopy solutions consist of auxiliary parameters, $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$. For such interest, the $\hbar$ -curves for the 19th order of approximations are displayed. It is found that the admissible values of $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are $-1.53 \leq \hbar_f \leq -0.60$, $-1.50 \leq \hbar_\theta \leq -0.55$ and $-1.50 \leq \hbar_\phi \leq -0.5$ (see Figure 1).

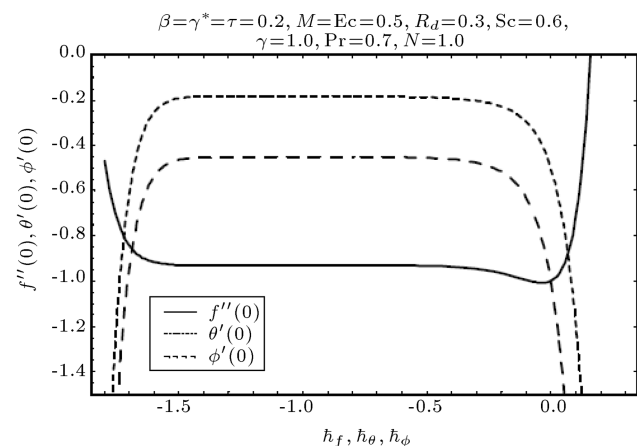


Figure 1. $\hbar$ curves for the functions $f''(0)$, $\theta'(0)$ and $\phi'(0)$ at 19th order of approximation.

5. Graphical results and discussion

This section is organized just to see the behaviour of different emerging parameters on the fluid velocity, temperature and concentration. Figures 2-9 illustrate the variations of β , γ , N , Ec , Pr , M , R_d and Sc on the velocity $f'(\eta)$. Figure 2 depicts that the velocity and momentum boundary layer thickness are decreasing functions of Deborah number. The Deborah number is dependent on the relaxation time, and relaxation time opposes the fluid flow that corresponds to the decrease in velocity and boundary layer thickness. Further, it is noticed that the effect of Deborah number on the velocity profile is similar to that of [4]. An increase in the local buoyancy parameter increases the fluid velocity and its associated boundary layer thickness (see Figure 3). Here, increase in local buoyancy parameter leads to an enhancement in the buoyancy force. An enhancement in the buoyancy force corresponds to an increase in the fluid velocity. The buoyancy force plays an important role in the petroleum industry. Figures 4 and 5 show the influences of N and Ec on $f'(\eta)$. These

figures show that both N and Ec increase the velocity. It is also observed that the variation in velocity due to N increases rapidly in comparison to the variation of $f'(\eta)$ due to Ec . Figure 6 depicts that the fluid velocity and its associated boundary layer thickness

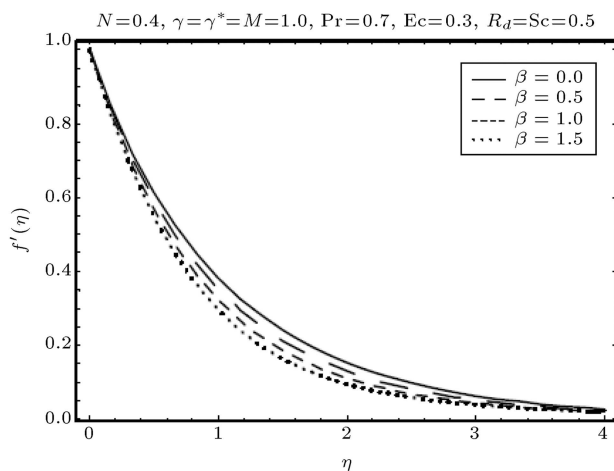


Figure 2. Influence of β on velocity f' .

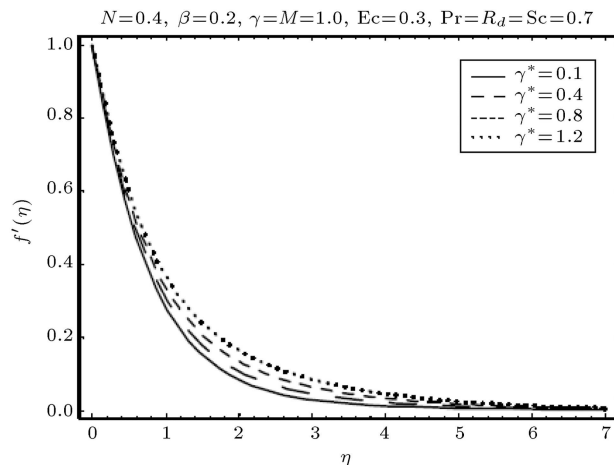


Figure 3. Influence of γ^* on velocity f' .

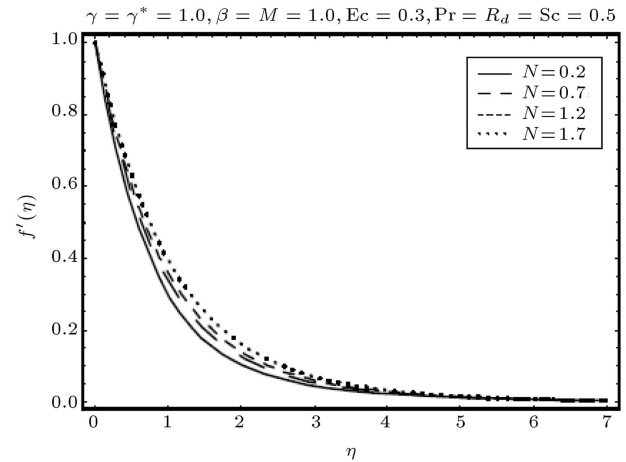


Figure 4. Influence of N on velocity f' .

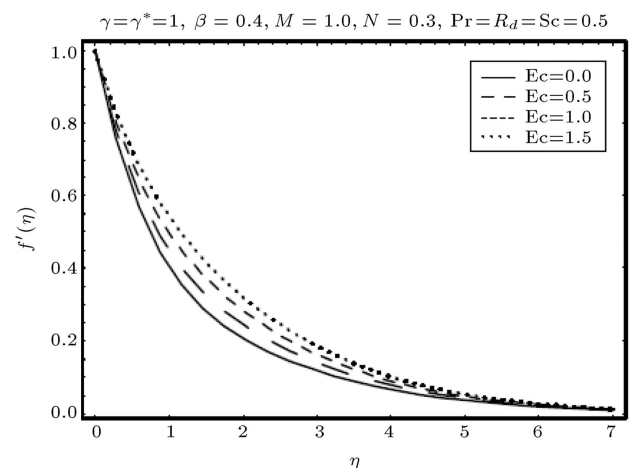


Figure 5. Influence of Ec on velocity f' .

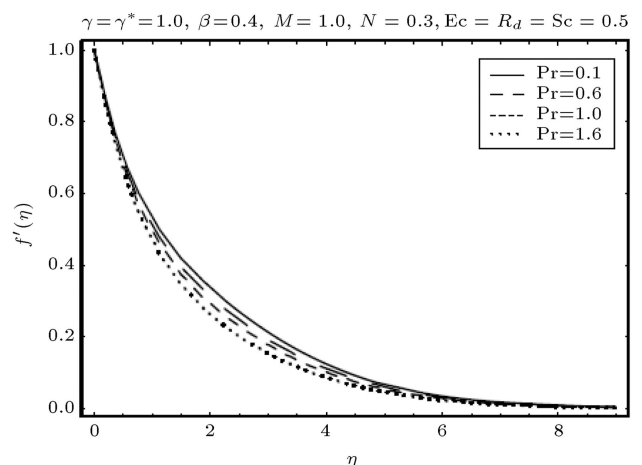


Figure 6. Influence of Pr on velocity f' .

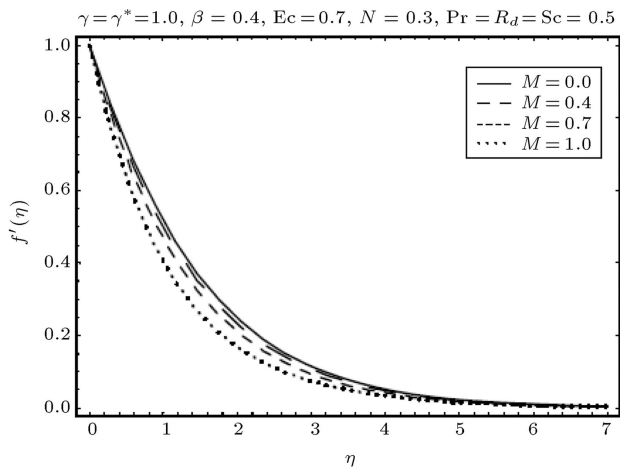


Figure 7. Influence of M on velocity f' .

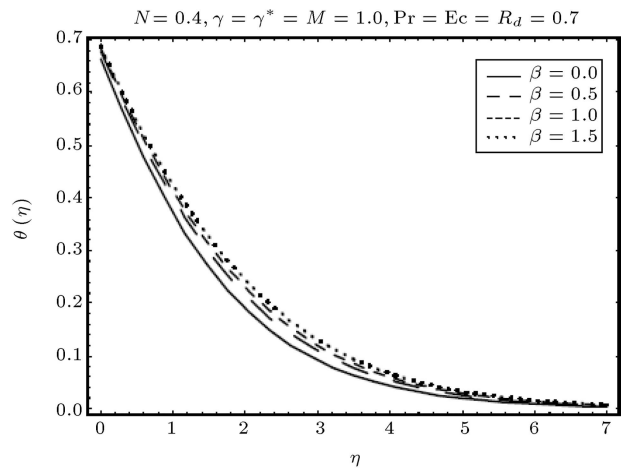


Figure 10. Influence of β on temperature θ .

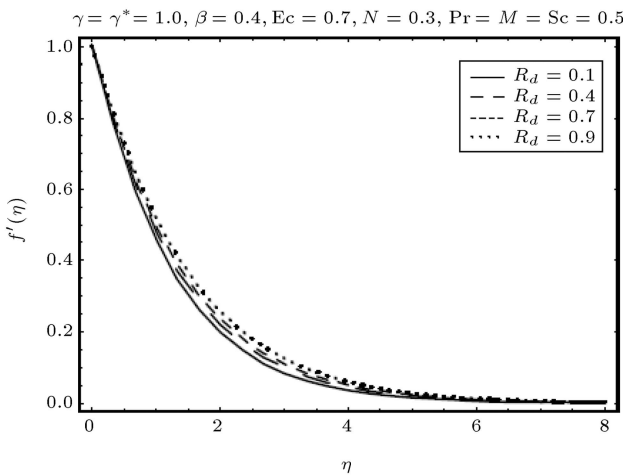


Figure 8. Influence of R_d on velocity f' .

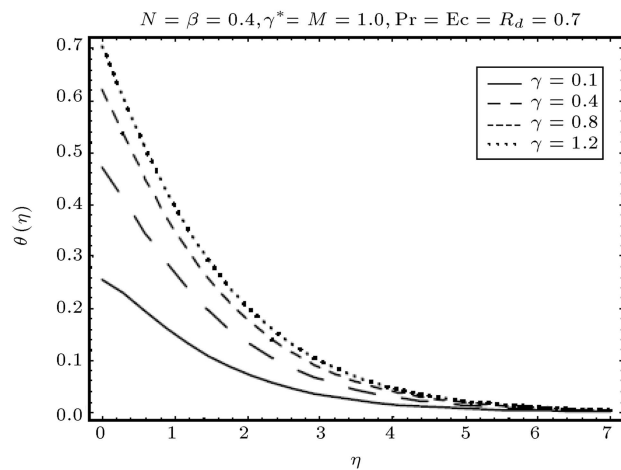


Figure 11. Influence of γ on temperature θ .

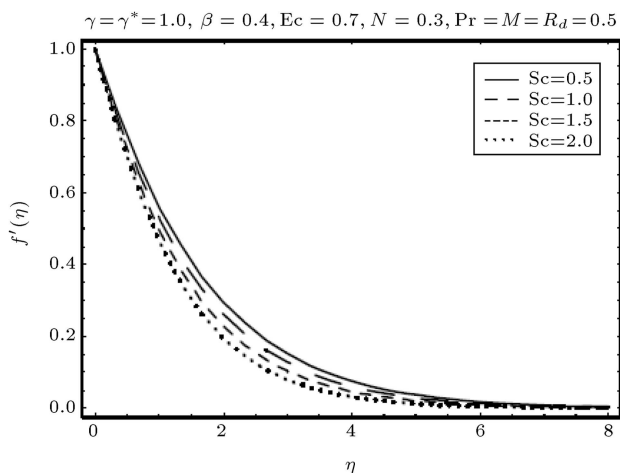


Figure 9. Influence of Sc on velocity f' .

decrease by increasing Prandtl number. The velocity, $f'(\eta)$, is reduced with an increase in Hartman number. Hartman number is through the Lorentz force, and the increase in Lorentz force resists the fluid flow, which causes reduction in the velocity, $f'(\eta)$ (see Figure 7).

To see the effects of R_d and Sc on $f'(\eta)$, Figures 8 and 9 are presented. We observed that R_d and Sc have quite opposite effects on velocity. The thermal radiation parameter increases velocity, but Schmidt number reduces it.

Figures 10-14 are plotted for the variations of β , γ , Ec , Pr and R_d on the fluid temperature, $\theta(\eta)$. Figure 10 indicates that the temperature increases with an increase in β . Comparative study of Figures 2 and 10 shows that the decrease in velocity is rapid and the increase in temperature is slow when Deborah number β increases. Figure 11 is prepared to see the influence of Biot number on temperature, $\theta(\eta)$. The temperature and thermal boundary layer thickness are increasing functions of Biot number, γ . The results for variations in Biot number, γ , are similar in a qualitative way, as obtained [11,12]. It is also noticed that $\gamma \rightarrow \infty$ leads to the results of [20]. Further, an enhancement in the temperature with an increase in Biot number is actually due to the heat transfer coefficient, h , as discussed in [12]. Biot number is dependent on the heat transfer coefficient. Higher

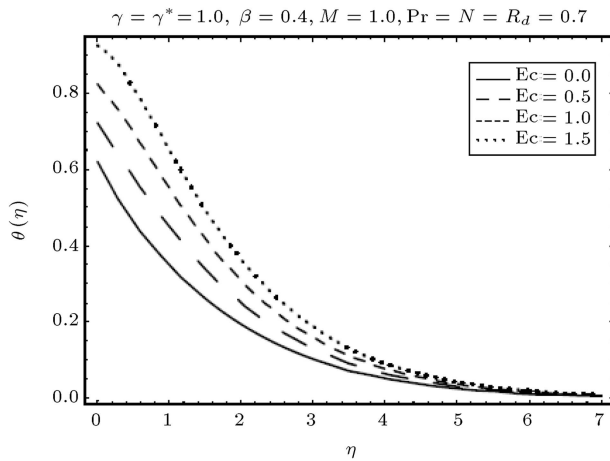


Figure 12. Influence of Ec on temperature θ .

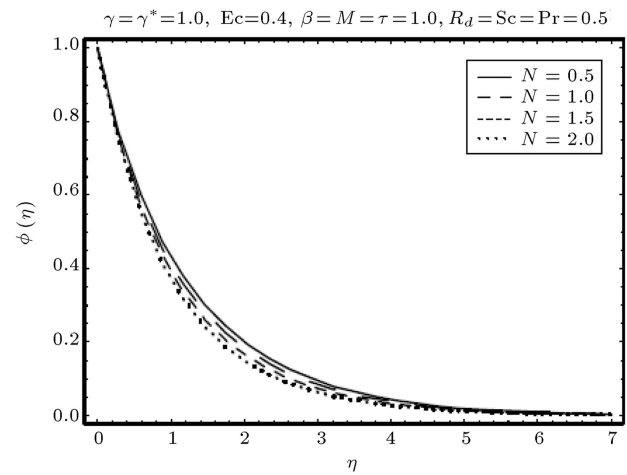


Figure 15. Influence of N on concentration ϕ .

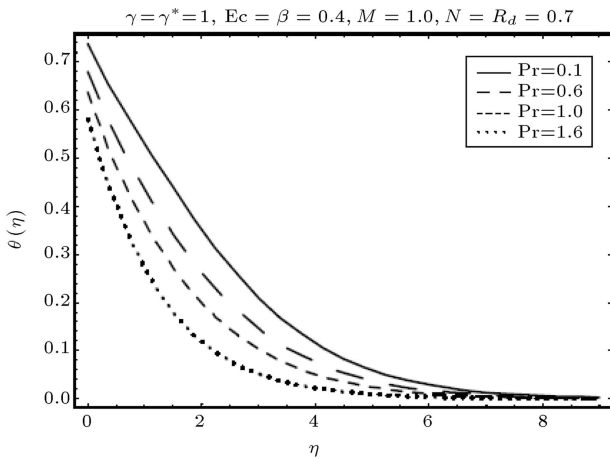


Figure 13. Influence of Pr on temperature θ .

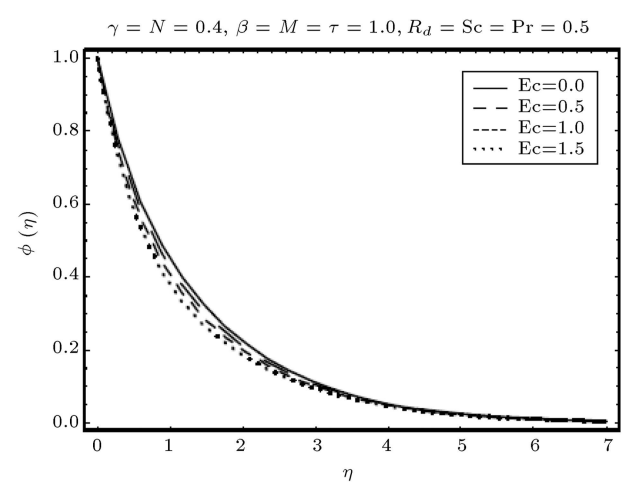


Figure 16. Influence of Ec on concentration ϕ .

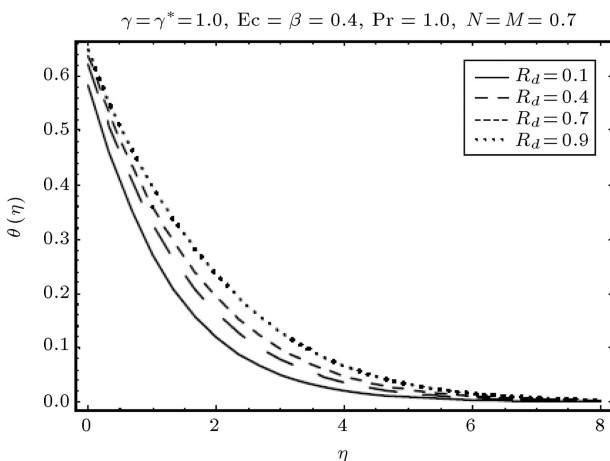


Figure 14. Influence of R_d on temperature θ .

values of Biot number correspond to an increase in heat transfer coefficient. An increase in heat transfer coefficient gives more heat to the fluid, which leads to higher fluid temperature. Due to an increase in Ec , the temperature and thermal boundary layer thickness are increased (see Figure 12). Figure 13 clearly shows

that an increase in Prandtl number causes a decrease in temperature and its associated thermal boundary layer thickness. Comparison of Figures 8 and 14 clearly illustrate that the radiation parameter, R_d , has similar effects on fluid velocity and temperature. However, it is observed that an increase in temperature is more significant than the increase in velocity.

Figures 15-18 are prepared to see the variations of some interesting parameters, N , Ec , M and Sc , on the concentration profile, $\phi(\eta)$. The concentration field and its associated boundary layer thickness show a decrease with an increase in N . From Figure 16, we have seen that the Eckert number reduces the concentration profile and concentration boundary layer thickness. We also noticed that the increase in velocity and temperature is more significant in comparison to the decrease in concentration. An increase in Hartman number leads to an increase in concentration (see Figure 17). Figure 18 shows that the concentration and associated boundary layer thickness are decreasing functions of Sc . Schmidt number is the ratio of momentum to mass diffusivities. It is used to characterize the fluid flows in

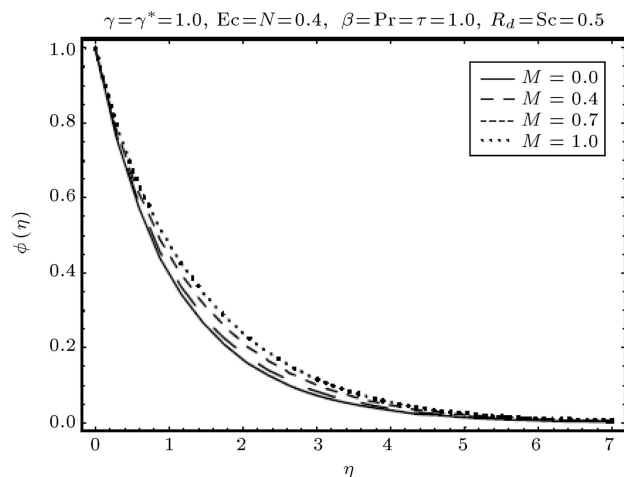


Figure 17. Influence of M on concentration ϕ .

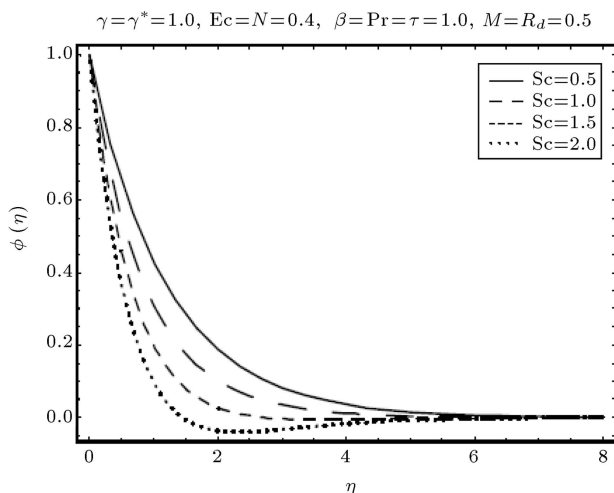


Figure 18. Influence of Sc on concentration ϕ .

petroleum reservoirs, where momentum and mass diffusion convection processes arise simultaneously. The relative thickness of the hydrodynamic layer and mass transfer layer are related, due to Schmidt number.

Table 1 is prepared to see the convergent values of $-f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ for different order of approximations. From this table, we noticed that the series solutions converge from the 30th order of approximation for velocity, temperature and concentration. Tables 2 and 3 are given for the numerical values of the local Nusselt and Sherwood number for the different values of involved parameters of interest. From Table 2, it is found that the magnitude of $-\theta'(0)$ decreases for large values of β . The magnitude of $-\phi'(0)$ increases when γ^* is increased. Table 3 is prepared for the variations of M , R_d , Sc , τ and Pr on $-\theta'(0)$ and $-\phi'(0)$. It is obvious from this table that when R_d , Sc and τ are increased, the magnitude of $-\theta'(0)$ increases, whereas the magnitude of $-\phi'(0)$ decreases. Our present analysis reduced to the analysis of Ref. [8], when $\beta = M = \gamma = N = R_d = Ec =$

Table 1. Convergence of homotopy solutions for different order of approximations when $N = 1$, $\beta = 0.2$, $\gamma = 1.0$, $Pr = 0.7$, $Ec = 0.5$, $M = 0.5$, $R_d = 0.3$, $Sc = 0.6$, $\tau = 0.2$, $\gamma^* = 0.2$ and $h_f = h_\theta = h_\phi = -0.6$.

Order of approximations	$-f''(0)$	$-\theta'(0)$	$-\phi'(0)$
1	1.0167	0.39063	0.80000
5	0.97400	0.23820	0.53157
10	0.94307	0.19903	0.47048
15	0.93367	0.18820	0.45620
20	0.93101	0.18452	0.45250
25	0.93043	0.18320	0.45180
30	0.93040	0.18273	0.45189
40	0.93040	0.18273	0.45189

Table 2. Numerical values of local Nusselt number $-\theta(0)$ and Sherwood number $-\phi'(0)$ when $M = 0.5$, $R_d = 0.3$, $Sc = 0.6$, $c = 0.2$, $Pr = 0.7$ and $\gamma = 1.0$.

N	β	γ^*	Ec	$-\theta'(0)$	$-\phi'(0)$
0.0	0.2	0.6	0.5	0.46443	0.16457
0.4				0.48040	0.17638
0.6				0.48728	0.18119
0.6	0.0			0.49850	0.18791
	0.4			0.47822	0.17490
	0.7			0.46607	0.16606
	0.4	0.5		0.45563	0.14673
		0.8		0.48426	0.19882
		1.0		0.50002	0.22513
	0.6	0.0		0.48314	0.29506
		0.4		0.49651	0.23870
		1.0		0.51574	0.15876

Table 3. Numerical values of local Nusselt number $-\theta'(0)$ and Sherwood number $-\phi'(0)$ when $N = 0.6$, $\beta = 0.7$, $\gamma = \gamma^* = 1.0$, and $Ec = 1.0$.

M	R_d	Sc	τ	Pr	$-\theta'(0)$	$-\phi'(0)$
0.0	0.3	0.6	0.2	0.7	0.51775	0.22790
0.5					0.51580	0.18283
0.7					0.51313	0.059840
0.4	0.0				0.50916	0.18818
	0.2				0.51388	0.18507
	0.4				0.51807	0.18074
	0.4	0.7			0.57162	0.17789
		1.0			0.71363	0.17243
		1.3			0.83561	0.16899
		0.6	0.0		0.67373	0.17317
			0.3		0.73457	0.17245
			0.5		0.77550	0.17198
			0.5	0.5	0.76906	0.16472
				1.0	0.78498	0.17414
				1.5	0.80404	0.18212

$Sc = \tau = 0$, and reduced to the study [9], when $M = \gamma^* = N = R_d = Ec = Sc = \tau = 0$. Our results also reduced to Ref. [20], when $\gamma \rightarrow \infty$.

6. Conclusions

This work is a theoretical investigation to analyze the thermophoresis and Joule heating effects in an electrically conductive Maxwell fluid over a stretching surface with convective conditions. Existing attempts related to thermophoresis and Joule effects have been reported with references to the prescribed surface temperature. In many metallurgical and industrial processes, the temperature at the surface is not constant. The prescribed surface temperature is not useful under such conditions. Hence, we adopt the more appropriate condition known as convective or Robin's type boundary condition. Such a condition, in fact, relates heat flux with temperature. To our knowledge, such flow analysis is examined for the first time and it provides a basis for convective conditions of heat transfer through various fluid models and geometrical configurations. The following points of the presented study are worth mentioning:

- Effects of β , Pr , M , and Sc on velocity profile, f' , are similar in a qualitative sense [20].
- Velocity profile, f' , is increased for larger N .
- Behaviours of R_d and Pr on the temperature, θ , are opposite.
- Both temperature and thermal boundary layer are decreasing functions of Prandtl number.
- An enhancement in the temperature is noticed when we increase the values of Biot number. A similar behaviour was observed in [7].
- Concentration field decreases by increasing Schmidt number, Sc .

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Biographies

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