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Optimal strategies for price, warranty length, and production rate of a new product with learning production cost

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Abstract. This study investigates optimal strategies for price, warranty length, and production rate of a new product in which both static markets for non-durable and dynamic markets for durable products are involved. The mathematical model incorporates both the demand and the cost functions including production, warranty length, and inventory costs. Using the maximum principle approach, the optimal strategies and interactions among price, warranty length, and production rate in both markets are analyzed using some propositions. The analysis shows that to maximize profits in all cases, the price, the warranty length, and the production rate all must go up simultaneously, or one of them must increase and the other two must decrease concurrently.

1. Introduction and literature review

Since the conditions change rapidly in the market and the goal of many firms is to maximize profits, pricing of products has become an important decision [1]. On one hand, high price for a product puts it out of reach for many consumers, resulting in low revenue. On the other hand, a low price leads to a low profit margin. Furthermore, in many situations, the price is treated as a sign of product quality, where standard economic approaches to product quality show that there is direct and upward sloping relationship between product quality and its price [2].

As products of several firms are similar to each other based on different aspects, it is difficult for consumers to compare them just through their price and superficial characteristics [3]. In these situations, warranty length of a product is another important factor for consumers purchasing the product, since extended warranty helps them to be more confident toward its quality. Buyer's willingness for a warranty strategy that protects them against high costs of product failure, makes warranty length an important means in marketing and product development.

Abiding by warranty commitment and replacing a failed product with a new one need a good warranty service management in which product inventory is a deciding factor. Providing a new product for a consumer in the time of request has a good effect on his perspective. Adversely, lacking enough product inventories admittedly affects the consumer's perspective in an opposite way. Nonetheless, inventory management in warranty policies, in contrast to its importance, has been paid very little attention in literature [4,5].

The learning production cost refers to the effect of production size of a good on its unit price. Arrow [6] and Rosen [7] concluded that if the cumulative production doubles, then the unit production cost could be decreased by a factor of 10 to 50. As a result, the production size affects the pricing strategy.

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In 1960s, Fourt & Woodlock [8], Mansfield [9] and Bass [10] proposed information diffusion models in attempts to make the lifecycle phenomenon for consumer durable. These research works stimulated later researches, and since then, diffusion theory of new product has been paid attention to in marketing knowledge, as well as consumer behavioral analysis.

Economists always criticized the Bass [10] model to be incomplete, because of not taking into account the effects of economic variables. Later, Bass et al. [11] tried to enter marketing variables in the original model and proposed their generalized Bass model. Other extensions tried to generalize this model and consider important marketing variables such as price, advertising, distribution channels, and the like.

In some cases, only the price variable has been considered in the Bass model. Robinson & Lakhani [12] showed that the classic marginal pricing could not be effective in dynamic business environment. Since many firms will be in the market for long-term period, they suggested that the use of dynamic pricing gets more profits during this period. By combining learning curve and diffusion theory of new product, Bayus [13] tried to maximize the profit of a durable product, and showed the costs and prices decrease with an increase in cumulative production. However, the product demand is the one that increases first and then decreases in its lifecycle. Kalish [14] studied the pricing of a new product in a monopolistic market to maximize the profit function. In his research, production cost is a declining function of cumulative production and demand is a function of both cumulative sales and price.

While many researchers considered more than one variable in the diffusion model, here we review the literature that deals with price and warranty length. To maximize the profit of a product sold under warranty, Glickman and Berger [15] proposed a model in which the demand function depends exponentially on price and warranty length. Mesak [16] offered some diffusion models that incorporate price and warranty length to determine the optimal price and warranty-length policies during the lifecycle of a product. Teng and Thompson [17] considered a model to maximize the profit of a monopolist by determining the optimal price and quality strategies. Lin and Shue [18] tried to reach the optimal price and warranty-length policies in a dynamic market. They maximized the profit function based on some basic lifetime distributions of the product. Wu et al. [19] proposed a model for a manufacturer to maximize his profit by determining the optimal price and warranty-length strategies for a product with normal lifetime distribution for which Faridimehr & Niaki [20] wrote an erratum. In addition, Wu et al. [21] developed a decision model to maximize the profit of a producer in a static demand market by determining the optimal price, warranty length and production rate policies based on the Weibull lifetime distribution. While the assumptions of positive production rate and negative second derivatives of the demand function with respect to price and warranty-length were taken in their research, Faridimehr and Niaki [22] extended their model to have no restrictions on both.

In this paper, we develop a model to derive the optimal price, warranty length and production rate strategies for a new product to maximize profits in both static and dynamic markets in which the discount rate is assumed zero and the learning production cost is present. Although the non-zero discount rate assumption might seem unreasonable, it can be an approximation in markets with a very low discount rate. This means that there is no difference between present and future profits in long-term horizon. The optimal strategies provide guidelines for manufacturers to increase or decrease the price, the warranty length and the production rate that maximize their profit during product lifecycle.

The remainder of the paper is organized as follows: In Section 2, the mathematical formulation of the problem at hand is given. The detailed information about the model comes in Section 3. In this section, a solution approach based on maximum principle is presented and optimal strategies are analyzed in both static and dynamic markets. Finally, conclusions are made in Section 4.

2. Problem formulation

Assuming the remaining unsold product at the end of the period has no value, the developed model of this section includes the demand and the cost functions. This means that the manufacturer only pays attention to his profit during the period rather than to his profit after it [16]. In what comes next, the demand function is first derived. Then, the cost function is developed based on production, warranty length and inventory costs.

2.1. Demand function

The demand function (f) considered in this paper is analyzed in both static and dynamic markets. While in both markets, this function depends on price and warranty length, in the dynamic market it depends on cumulative sales as well. We assume the demand function is twice as much differentiable and decreases with an increase in price and increases with an increase in warranty length (as the case usually happens in practice.) Moreover, the rate of increase in demand subject to increase in warranty length decreases by increasing price. In other words:

$$f_p = \frac{\partial f}{\partial p} < 0,$$

$$f_w = \frac{\partial f}{\partial w} > 0,$$

$$f_{pw} = \frac{\partial^2 f}{\partial p \partial w} < 0,$$

where f_p is the partial derivatives of the demand function with respect to the price, f_w is the partial derivatives of the demand function with respect to the warranty length, and f_{pw} represents the second derivatives of the demand function with respect to price and warranty-length. The optimal strategies based on both demand functions are discussed in subsequent sections.

2.2. Cost function

The cost function that is used in the mathematical model involves two components: (1) the production and warranty-length cost and (2) the inventory cost. The following two subsections are devoted to these components.

2.2.1. Production and warranty length cost function

Here, we assume that the unit cost, C, is a function of the warranty length at time t, $w(t)$, the cumulative production volume at time t, $X(t)$, and the production rate at time t, $q(t)$. Similar to Lin [23], C is assumed twice as much differentiable, satisfying the following conditions:

$$C_w = \frac{\partial C}{\partial w} > 0,$$

$$C_{ww} = \frac{\partial^2 C}{\partial w^2} > 0,$$

$$C_q = \frac{\partial C}{\partial q} > 0,$$

$$C_{qq} = \frac{\partial^2 C}{\partial q^2} > 0.$$

In other words, the unit cost function is a strictly increasing convex function with respect to both warranty length and production rate. However, it decreases with an increase in the cumulative production, where the rate of decrease increases. Mathematically we have:

$$C_X = \frac{\partial C}{\partial X} < 0,$$

$$C_{XX} = \frac{\partial^2}{\partial X^2} > 0.$$

Although the unit cost function increases with an increase in the warranty length, the rate of increase increases with an increase in the production rate and decreases with an increase in the cumulative production. In addition, although the unit cost function increases with an increase in the production rate, the rate of this increase decreases by an increase in the cumulative production. These characteristics can be shown as:

$$C_{wq} = \frac{\partial^2 C}{\partial w \partial q} > 0,$$

$$C_{wX} = \frac{\partial^2 C}{\partial w \partial X} < 0,$$

$$C_{qX} = \frac{\partial^2 C}{\partial q \partial X} < 0.$$

2.2.2. Inventory cost

On one hand, having extra product in the warehouse causes extra holding cost for the manufacturer. On the other hand, insufficient inventory results in delays in offering warranty service to the customers. Thus, inventory management is a key component of managing service warranty.

Let $Q(t)$ be the cumulative sales at time t, $f(t)$ be the sales rate at time t, and $I(t)$ be the inventory volume at time t. According to Feichtinger and Hartl [24], Jorgensen et al. [25] and Warburton [26], the equations of inventory balance are:

$$I(t) = X(t) - Q(t),$$

$$\frac{dI}{dt} = q(t) - f(t). \qquad (1)$$

Moreover, similar to Jorgensen et al. [25], we propose a dynamic system with Q, X and I as state variables in which the possibility of shortage occurrence during the period is considered. Besides, in contrast with Wu et al. [21], we assume the holding and the shortage cost per unit of the product per period, δ_1, are equal. Then, the holding or the shortage cost function, $h(I)$, can be modeled as:

$$h(I) = \begin{cases} \delta_1 I(t); & \text{if } I(t) > 0 \\ 0; & \text{if } I(t) = 0 \\ -\delta_1 I(t); & \text{if } I(t) < 0 \end{cases}. \qquad (2)$$

2.3. Mathematical model

Aiming to maximize the profit in period T, the mathematical formulation of the problem becomes:

$$Max\, J = \int_0^T \{[p(t) - C(w(t), q(t), X(t))] \times f(t) - h(I)\}\, dt. \qquad (3)$$

Subject to:

$$\frac{dQ}{dt} = f(t),$$

$$\frac{dX}{dt} = q(t), \tag{4}$$

where $p(t)$ is the price function of the product at time t.

3. A solution method

The mathematical model in Eqs. (3) and (4) can be solved using the maximum principle method [27]. The present value of Hamiltonian, H, is formulated as:

$$H = (p - C) f - h(I) + \lambda_1 \frac{dQ}{dt} + \lambda_2 \frac{dX}{dt},$$

$$H = (p - C + \lambda_1) f - h(I) + \lambda_2 q, \tag{5}$$

where λ_1 and λ_2 are the current values of adjoint variables (shadow price of f and q respectively) that must satisfy the following differential equations:

$$\frac{d\lambda_1}{dt} = -H_Q = -(p - C + \lambda_1) f_Q + h_Q(I),$$

$$\lambda_1(T) = 0, \tag{6}$$

$$\frac{d\lambda_2}{dt} = -H_X = C_X f + h_X(I),$$

$$\lambda_2(T) = 0. \tag{7}$$

In the above formulae $H_Q = \frac{\partial H}{\partial Q}$, $H_X = \frac{\partial H}{\partial X}$, and $f_Q = \frac{\partial f}{\partial Q}$.

To have an optimal solution, the partial derivatives of H with respect to the main variables, p, w and q must be zero. In other words:

$$H_p = \frac{\partial H}{\partial p} = 0,$$

$$H_w = \frac{\partial H}{\partial w} = 0,$$

$$H_q = \frac{\partial H}{\partial q} = 0. \tag{8}$$

In addition, the non-singular Hessian matrix of the second-order partial derivatives of H with respect to price, warranty length and production rate must be negative definite. This matrix is:

$$HM = \begin{bmatrix} H_{pp} & H_{pw} & H_{pq} \\ H_{wp} & H_{ww} & H_{wq} \\ H_{qp} & H_{qw} & H_{qq} \end{bmatrix}.$$

Thus, the following conditions must hold:

$$H_{pp} < 0,$$

$$\begin{vmatrix} H_{pp} & H_{pw} \\ H_{wp} & H_{ww} \end{vmatrix} > 0,$$

$$\begin{vmatrix} H_{pp} & H_{pw} & H_{pq} \\ H_{wp} & H_{ww} & H_{wq} \\ H_{qp} & H_{qw} & H_{qq} \end{vmatrix} < 0.$$

Hence, the optimal decisions can be obtained using Cramer's rule shown in Box 1 (see Appendix A for the proof).

To discuss the optimal strategies, we first consider the sign of H_{pwq} to determine the sign of the derivatives of price, warranty length and production rate, with respect to time. For the considered demand function, we have:

$$H_{pwq} = -C_{wq} f_p - C_q f_{pw}. \tag{10}$$

Then, based on the assumptions on f and C given in Subsections 2.1 and 2.2.1, H_{pwq} is always positive. Therefore, in order to increase the current value of

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dw}{dt} \\ \frac{dq}{dt} \end{bmatrix} = -HM^{-1}$$

$$\times \begin{bmatrix} f\left(f_Q + (p - C + \lambda_1) f_{pQ}\right) - C_X q f_p + f_p\left(-(p - C + \lambda_1) f_Q + h_Q(I)\right) \\ f\left(-C_w f_Q + (p - C + \lambda_1) f_{wQ}\right) - q\left(C_{wX} f + C_X f_w\right) + f_w\left(-(p - C + \lambda_1) f_Q + h_Q(I)\right) \\ -C_q f f_Q - C_{qX} q f + C_X f + h_X(I) \end{bmatrix}. \tag{9}$$

where HM^{-1} is the inverse of HM.

Box I.

Table 1. Optimal policies of price, warranty length and production rate in static market.

Conditions	Results
$(-C_X q h' k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix} > 0$ and $(-C_X q h' k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix} > 0$	$p\uparrow, w\uparrow, q\uparrow$
$(-C_X q h' k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix} < 0$ and $(-C_X q h' k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix} > 0$	$p\downarrow, w\uparrow, q\downarrow$
$(-C_X q h' k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix} > 0$ and $(-C_X q h' k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix} < 0$	$p\uparrow, w\downarrow, q\downarrow$
$(-C_X q h' k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix} < 0$ and $(-C_X q h' k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(C_X - C_{qX} q\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ q\left(C_X k' + C_{wX} k\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix} < 0$	$p\downarrow, w\downarrow, q\uparrow$

Hamiltonian, either all of the time derivatives of price, warranty length and production rate must be positive or two of them must be negative and one positive.

In the subsequent sections, the optimal strategy on price, warranty length and production rate are discussed in both static and dynamic markets.

3.1. Static market

The static market belongs to non-durable products in which the word-of-mouth is not important in the development of the product and that almost the whole lifecycle of the product lies in the maturity phase. In this type of market, there is almost no saturation effect; therefore, demand is just a function of price and warranty length. Moreover, $f_Q = 0$, $f_{pQ=0}$ and $f_{wQ=0}$.

Assuming the inventory cost of an item is relatively low and negligible; the optimal strategies in this type of market can be characterized through the following proposition.

Proposition 1: *For a given $f = h(p)k(w)$, the optimal policies for price, warranty length, and production rate are characterized in Table 1.*

Proof: See Appendix B.

If the interaction between any two of the warranty length, the cumulative production, and the production rate can be relatively low, then more clear results are obtained in Table 2.

Proof: See Appendix C.

3.2. Dynamic market

In the dynamic market that is for durable products, demand exhibits diffusion and saturation effects, and the word-of-mouth is an important factor in diffusion of product in a market. Moreover, current demand affects the quantity of future demand; therefore, the demand is a function of price, warranty length and cumulative sales.

In addition to the assumptions considered for zero value of unsold products, the second assumption rising

Table 2. Optimal policies of price, warranty length and production rate in static market (low interactions between any two of warranty length, cumulative production and production rate).

Conditions	Results
$(-qh'k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ qk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix}>0$ and $(-qh'k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ qk'\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix}>0$	$p\downarrow, w\downarrow, q\uparrow$
$(-qh'k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ qk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix}<0$ and $(-qh'k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ qk'\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix}>0$	$p\uparrow, w\downarrow, q\downarrow$
$(-qh'k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ qk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix}>0$ and $(-qh'k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ qk'\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix}<0$	$p\downarrow, w\uparrow, q\downarrow$
$(-qh'k)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\begin{pmatrix} k\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)- \\ qk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{pmatrix}<0$ and $(-qh'k)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\begin{pmatrix} k\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)- \\ qk'\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{pmatrix}<0$	$p\uparrow, w\uparrow, q\uparrow$

here is that each person purchases the product only once in the period. This assumption has a higher impact in short planning horizons [16]. Furthermore, similar to static demand markets, to reach results that are more tractable, we assume that the unit inventory cost of the product is relatively low and negligible. As a result, the following proposition is made to obtain optimal policies.

Proposition 2: *For a given* $f = h(p)k(w)g(Q)$, *the optimal policies for price, warranty length and production rate are characterized in Table 3.*

Proof: See Appendix D.

To reach more tractable results, one can consider the case in which the production and warranty length cost functions are independent of the cumulative production volume. It means that C can be defined as $C(w(t), q(t))$. Besides, if we assume that the interaction between the warranty length and the production rate is relatively low, as in the static market, the optimal policies can be characterized as given in Table 4.

Proof: See Appendix E.

The above proposition shows that the optimal price, warranty length and production rate strategies are dependent on each other. Note that $g'(Q)$ is positive in the diffusion phase and negative during the saturation phase.

4. Conclusion and recommendation for future works

In this paper, a mathematical decision model for a new product (both non-durable and durable) was proposed to determine the optimal price, warranty length and production rate strategies, where the discount rate was assumed zero. Two cost functions, (1) production and warranty length and (2) inventory were consid-

Table 3. Optimal policies of price, warranty length and production rate in dynamic market.

Conditions	Diffusion effect	Saturation effect
$h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)>0$ and $h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)>0$	$p\downarrow, w\downarrow, q\uparrow$	$p\uparrow, w\uparrow, q\uparrow$
$h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)<0$ and $h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)>0$	$p\uparrow, w\downarrow, q\downarrow$	$p\downarrow, w\uparrow, q\downarrow$
$h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)>0$ and $h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)<0$	$p\downarrow, w\uparrow, q\downarrow$	$p\uparrow, w\downarrow, q\downarrow$
$h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)<0$ and $h'k\left(h^2k-\frac{C_X q h'}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\left(h^2kk'-\frac{qh'\left(C_X k'+C_{wX}k\right)}{g'}\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)+hh'k\left(-C_q hk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)<0$	$p\uparrow, w\uparrow, q\uparrow$	$p\downarrow, w\downarrow, q\uparrow$

Table 4. Optimal policies of price, warranty length and production rate in dynamic market ($C(w(t), q(t))$ and low interaction between warranty length and production rate).

Conditions	Diffusion effect	Saturation effect
$h'k\left(\left(H_{ww}H_{qq}-H_{wq}^2\right)-C_q\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\right)$ $+hk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)>0$ and $h'k\left(\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)-C_q\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\right)$ $+hk'\left(H_{pp}H_{qq}-H_{pq}^2\right)>0$	$p\downarrow, w\downarrow, q\uparrow$	$p\uparrow, w\uparrow, q\uparrow$
$h'k\left(\left(H_{ww}H_{qq}-H_{wq}^2\right)-C_q\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\right)$ $+hk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)<0$ and $h'k\left(\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)-C_q\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\right)$ $+hk'\left(H_{pp}H_{qq}-H_{pq}^2\right)>0$	$p\uparrow, w\downarrow, q\downarrow$	$p\downarrow, w\uparrow, q\downarrow$
$h'k\left(\left(H_{ww}H_{qq}-H_{wq}^2\right)-C_q\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\right)$ $+hk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)>0$ and $h'k\left(\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)-C_q\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\right)$ $+hk'\left(H_{pp}H_{qq}-H_{pq}^2\right)<0$	$p\downarrow, w\uparrow, q\downarrow$	$p\uparrow, w\downarrow, q\downarrow$
$h'k\left(\left(H_{ww}H_{qq}-H_{wq}^2\right)-C_q\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\right)$ $+hk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)<0$ and $h'k\left(\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)-C_q\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\right)$ $+hk'\left(H_{pp}H_{qq}-H_{pq}^2\right)<0$	$p\uparrow, w\uparrow, q\uparrow$	$p\downarrow, w\downarrow, q\uparrow$

ered in the proposed model. The first is a function of warranty length, production rate and cumulative production volume, and the later includes both holding and shortage costs. The maximum principle method was then utilized to obtain the optimal strategies. Next, the optimal strategies were discussed both for static market, the market of non-durable products, and dynamic market which is for durable products. Before analyzing optimal strategies in both markets, we concluded that the product of the price derivatives with respect to time, the warranty length derivatives with respect to time, and the production rate derivatives with respect to time must be positive. Moreover, we considered the optimal strategies, when the warranty length, production rate and cumulative production do not interact with each other in the production and warranty length cost function. This assumption ends in more clear and tractable results.

While the marketing variables considered in the proposed model were price and warranty length, future works on incorporating other marketing variables such as advertising and distribution are recommended. In addition, future study can focus on various warranty strategies, and compare them with each other. Analyzing the cases of duopoly and oligopoly markets will be an interesting field of research because in reality, many markets include more than one producer.

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Appendix A

In order to analyze the relationships among price, warranty length and production rate, we first take the time derivatives on both sides of Eqs. (8) to reach:

$$\frac{d}{dt}\left[\frac{\partial H}{\partial p}\right] = \frac{\partial^2 H}{\partial p^2}\frac{dp}{dt} + \frac{\partial^2 H}{\partial p \partial w}\frac{dw}{dt} + \frac{\partial^2 H}{\partial p \partial q}\frac{dq}{dt} + \frac{\partial^2 H}{\partial p \partial Q}\frac{dQ}{dt} + \frac{\partial^2 H}{\partial p \partial X}\frac{dX}{dt} + \frac{\partial^2 H}{\partial p \partial \lambda_1}\frac{d\lambda_1}{dt} + \frac{\partial^2 H}{\partial p \partial \lambda_2}\frac{d\lambda_2}{dt} = 0, \tag{A.1}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial w}\right] = \frac{\partial^2 H}{\partial w \partial p}\frac{dp}{dt} + \frac{\partial^2 H}{\partial w^2}\frac{dw}{dt} + \frac{\partial^2 H}{\partial w \partial q}\frac{dq}{dt} + \frac{\partial^2 H}{\partial w \partial Q}\frac{dQ}{dt} + \frac{\partial^2 H}{\partial w \partial X}\frac{dX}{dt} + \frac{\partial^2 H}{\partial w \partial \lambda_1}\frac{d\lambda_1}{dt} + \frac{\partial^2 H}{\partial w \partial \lambda_2}\frac{d\lambda_2}{dt} = 0, \tag{A.2}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial q}\right] = \frac{\partial^2 H}{\partial q \partial p}\frac{dp}{dt} + \frac{\partial^2 H}{\partial q \partial w}\frac{dw}{dt} + \frac{\partial^2 H}{\partial q^2}\frac{dq}{dt} + \frac{\partial^2 H}{\partial q \partial Q}\frac{dQ}{dt} + \frac{\partial^2 H}{\partial q \partial X}\frac{dX}{dt} + \frac{\partial^2 H}{\partial q \partial \lambda_1}\frac{d\lambda_1}{dt} + \frac{\partial^2 H}{\partial q \partial \lambda_2}\frac{d\lambda_2}{dt} = 0. \tag{A.3}$$

Substituting $\frac{dQ}{dt} = f$, $\frac{dX}{dt} = q$ and $\frac{d\lambda_1}{dt}$ and $\frac{d\lambda_2}{dt}$ from Eqs. (6) and (7) in Eqs. (A.1), (A.2), and (A.3), we have:

$$\frac{d}{dt}\left[\frac{\partial H}{\partial p}\right] = \frac{\partial^2 H}{\partial p^2}\frac{dp}{dt} + \frac{\partial^2 H}{\partial p \partial w}\frac{dw}{dt} + \frac{\partial^2 H}{\partial p \partial q}\frac{dq}{dt} + \frac{\partial^2 H}{\partial p \partial Q}f + \frac{\partial^2 H}{\partial p \partial X}q + \frac{\partial^2 H}{\partial p \partial \lambda_1}(-H_Q) + \frac{\partial^2 H}{\partial p \partial \lambda_2}(-H_X) = 0, \tag{A.4}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial w}\right] = \frac{\partial^2 H}{\partial w \partial p}\frac{dp}{dt} + \frac{\partial^2 H}{\partial w^2}\frac{dw}{dt} + \frac{\partial^2 H}{\partial w \partial q}\frac{dq}{dt} + \frac{\partial^2 H}{\partial w \partial Q}f + \frac{\partial^2 H}{\partial w \partial X}q + \frac{\partial^2 H}{\partial w \partial \lambda_1}(-H_Q) + \frac{\partial^2 H}{\partial w \partial \lambda_2}(-H_X) = 0, \tag{A.5}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial q}\right] = \frac{\partial^2 H}{\partial q \partial p}\frac{dp}{dt} + \frac{\partial^2 H}{\partial q \partial w}\frac{dw}{dt} + \frac{\partial^2 H}{\partial q^2}\frac{dq}{dt} + \frac{\partial^2 H}{\partial q \partial Q}f + \frac{\partial^2 H}{\partial q \partial X}q + \frac{\partial^2 H}{\partial q \partial \lambda_1}(-H_Q) + \frac{\partial^2 H}{\partial q \partial \lambda_2}(-H_X) = 0. \tag{A.6}$$

Then, the following matrix can be constructed based on Eqs. (A.4), (A.5), and (A.6).

$$\begin{bmatrix} \frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial w} & \frac{\partial^2 H}{\partial p \partial q} \\ \frac{\partial^2 H}{\partial w \partial p} & \frac{\partial^2 H}{\partial w^2} & \frac{\partial^2 H}{\partial w \partial q} \\ \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q \partial w} & \frac{\partial^2 H}{\partial q^2} \end{bmatrix} \begin{bmatrix} \frac{dp}{dt} \\ \frac{dw}{dt} \\ \frac{dq}{dt} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 H}{\partial p \partial Q}f + \frac{\partial^2 H}{\partial p \partial X}q - \frac{\partial^2 H}{\partial p \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial p \partial \lambda_2}\frac{\partial H}{\partial X} \\ \frac{\partial^2 H}{\partial w \partial Q}f + \frac{\partial^2 H}{\partial w \partial X}q - \frac{\partial^2 H}{\partial w \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial w \partial \lambda_2}\frac{\partial H}{\partial X} \\ \frac{\partial^2 H}{\partial w \partial Q}f + \frac{\partial^2 H}{\partial w \partial X}q - \frac{\partial^2 H}{\partial w \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial w \partial \lambda_2}\frac{\partial H}{\partial X} \end{bmatrix}. \tag{A.7}$$

So:

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dw}{dt} \\ \frac{dq}{dt} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial w} & \frac{\partial^2 H}{\partial p \partial q} \\ \frac{\partial^2 H}{\partial w \partial p} & \frac{\partial^2 H}{\partial w^2} & \frac{\partial^2 H}{\partial w \partial q} \\ \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q \partial w} & \frac{\partial^2 H}{\partial q^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 H}{\partial p \partial Q}f + \frac{\partial^2 H}{\partial p \partial X}q - \frac{\partial^2 H}{\partial p \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial p \partial \lambda_2}\frac{\partial H}{\partial X} \\ \frac{\partial^2 H}{\partial w \partial Q}f + \frac{\partial^2 H}{\partial w \partial X}q - \frac{\partial^2 H}{\partial w \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial w \partial \lambda_2}\frac{\partial H}{\partial X} \\ \frac{\partial^2 H}{\partial w \partial Q}f + \frac{\partial^2 H}{\partial w \partial X}q - \frac{\partial^2 H}{\partial w \partial \lambda_1}\frac{\partial H}{\partial Q} - \frac{\partial^2 H}{\partial w \partial \lambda_2}\frac{\partial H}{\partial X} \end{bmatrix}. \tag{A.8}$$

Eq. (A.8) leads to Eq. (A.9) by calculating the partial derivatives of current Hamiltonian. Note that:

$$HM^{-1} = \begin{bmatrix} \frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial w} & \frac{\partial^2 H}{\partial p \partial q} \\ \frac{\partial^2 H}{\partial w \partial p} & \frac{\partial^2 H}{\partial w^2} & \frac{\partial^2 H}{\partial w \partial q} \\ \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q \partial w} & \frac{\partial^2 H}{\partial q^2} \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} H_{ww}H_{qq} - H_{wq}^2 & H_{pq}H_{wq} - H_{pw}H_{qq} & H_{pw}H_{wq} - H_{ww}H_{pq} \\ H_{pq}H_{wq} - H_{pw}H_{qq} & H_{pp}H_{qq} - H_{pq}^2 & H_{pw}H_{pq} - H_{pp}H_{wq} \\ H_{pw}H_{wq} - H_{ww}H_{pq} & H_{pw}H_{pq} - H_{pp}H_{wq} & H_{pp}H_{ww} - H_{pw}^2 \end{bmatrix}, \tag{A.9}$$

where Δ is the determinant of the HM matrix.

Appendix B

Considering the fact that in the static markets, the demand function does not depend on the cumulative sales, the partial derivatives of price, warranty length, and production rate can be found as follows:

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dw}{dt} \\ \frac{dq}{dt} \end{bmatrix} = -\frac{1}{\Delta} \begin{bmatrix} H_{ww}H_{qq} - H_{wq}^2 & H_{pq}H_{wq} - H_{pw}H_{qq} & H_{pw}H_{wq} - H_{ww}H_{pq} \\ H_{pq}H_{wq} - H_{pw}H_{qq} & H_{pp}H_{qq} - H_{pq}^2 & H_{pw}H_{pq} - H_{pp}H_{wq} \\ H_{pw}H_{wq} - H_{ww}H_{pq} & H_{pw}H_{pq} - H_{pp}H_{wq} & H_{pp}H_{ww} - H_{pw}^2 \end{bmatrix} \times \begin{bmatrix} h^2k^2gg' \\ \frac{h^3kk'gg'}{h'} \\ -C_qh^2k^2gg' \end{bmatrix}. \tag{B.1}$$

Thus,

$$\frac{dp}{dt} = -\frac{1}{\Delta}\Bigg((-C_Xqh'k)\left(H_{ww}H_{qq} - H_{wq}^2\right) + h\begin{pmatrix} k\left(C_X - C_{qX}q\right)\left(H_{pw}H_{wq} - H_{ww}H_{pq}\right) - \\ q\left(C_Xk' + C_{wX}k\right)\left(H_{pq}H_{wq} - H_{pw}H_{qq}\right) \end{pmatrix}\Bigg), \tag{B.2}$$

$$\frac{dw}{dt}=-\frac{1}{\Delta}\Bigg(\left(-C_Xqh'k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)$$

$$+h\left(\begin{array}{c}k\left(C_X-C_{qX}q\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)-\\ q\left(C_Xk'+C_{wX}k\right)\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{array}\right)\Bigg), \tag{B.3}$$

$$\frac{dq}{dt}=-\frac{1}{\Delta}\Bigg(\left(-C_Xqh'k\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)$$

$$+h\left(\begin{array}{c}k\left(C_X-C_{qX}q\right)\left(H_{pp}H_{ww}-H_{pw}^2\right)-\\ q\left(C_Xk'+C_{wX}k\right)\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\end{array}\right)\Bigg). \tag{B.4}$$

Appendix C

Knowing that in the static markets the demand function does not depend on the cumulative sales and also considering the interaction terms between any two of warranty length, cumulative production and production rate, the partial time derivatives of price, warranty length and production are obtained as follows:

$$\begin{bmatrix}\frac{dp}{dt}\\ \frac{dw}{dt}\\ \frac{dq}{dt}\end{bmatrix}=-\frac{1}{\Delta}\begin{bmatrix}H_{ww}H_{qq}-H_{wq}^2 & H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pw}H_{wq}-H_{ww}H_{pq}\\ H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pp}H_{qq}-H_{pq}^2 & H_{pw}H_{pq}-H_{pp}H_{wq}\\ H_{pw}H_{wq}-H_{ww}H_{pq} & H_{pw}H_{pq}-H_{pp}H_{wq} & H_{pp}H_{ww}-H_{pw}^2\end{bmatrix}\begin{bmatrix}-C_Xqh'k\\ -C_Xqhk\\ C_Xhk\end{bmatrix}. \tag{C.1}$$

In other words,

$$\frac{dp}{dt}=-\frac{C_X}{\Delta}\Bigg(\left(-qh'k\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)+h\left(\begin{array}{c}k\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)-\\ qk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\end{array}\right)\Bigg), \tag{C.2}$$

$$\frac{dw}{dt}=-\frac{C_X}{\Delta}\Bigg(\left(-qh'k\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)+h\left(\begin{array}{c}k\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)-\\ qk'\left(H_{pp}H_{qq}-H_{pq}^2\right)\end{array}\right)\Bigg), \tag{C.3}$$

$$\frac{dq}{dt}=-\frac{C_X}{\Delta}\Bigg(\left(-qh'k\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)+h\left(\begin{array}{c}k\left(H_{pp}H_{ww}-H_{pw}^2\right)-\\ qk'\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\end{array}\right)\Bigg). \tag{C.4}$$

Appendix D

Since the demand function in dynamic markets depends not only on price and warranty length, but also on cumulative sales, the partial time derivatives of price, warranty length, and production rate are:

$$\begin{bmatrix}\frac{dp}{dt}\\ \frac{dw}{dt}\\ \frac{dq}{dt}\end{bmatrix}=-\frac{1}{\Delta}\begin{bmatrix}H_{ww}H_{qq}-H_{wq}^2 & H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pw}H_{wq}-H_{ww}H_{pq}\\ H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pp}H_{qq}-H_{pq}^2 & H_{pw}H_{pq}-H_{pp}H_{wq}\\ H_{pw}H_{wq}-H_{ww}H_{pq} & H_{pw}H_{pq}-H_{pp}H_{wq} & H_{pp}H_{ww}-H_{pw}^2\end{bmatrix}$$

$$\times\begin{bmatrix}kgg'\left(h^2k-\frac{C_Xqh'}{g'}\right)\\ \frac{hgg'}{h'}\left(h^2kk'-\frac{qh'\left(C_Xk'+C_{wX}k\right)}{g'}\right)\\ hkgg'\left(-C_qhk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\end{bmatrix}. \tag{D.1}$$

In other words,

$$\frac{dp}{dt}=-\frac{gg'}{\Delta h'}\begin{pmatrix}h'k\left(h^2k-\frac{C_Xqh'}{g'}\right)\left(H_{ww}H_{qq}-H_{wq}^2\right)\\ +h\left(h^2kk'-\frac{qh'\left(C_Xk'+C_{wX}k\right)}{g'}\right)\\ \left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\\ +hh'k\left(-C_qhk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\\ \left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\end{pmatrix}, \tag{D.2}$$

$$\frac{dw}{dt}=-\frac{gg'}{\Delta h'}\begin{pmatrix}h'k\left(h^2k-\frac{C_Xqh'}{g'}\right)\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right)\\ +h\left(h^2kk'-\frac{qh'\left(C_Xk'+C_{wX}k\right)}{g'}\right)\\ \left(H_{pp}H_{qq}-H_{pq}^2\right)\\ +hh'k\left(-C_qhk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\\ \left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\end{pmatrix}, \tag{D.3}$$

$$\frac{dq}{dt}=-\frac{gg'}{\Delta h'}\begin{pmatrix}h'k\left(h^2k-\frac{C_Xqh'}{g'}\right)\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\\ +h\left(h^2kk'-\frac{qh'\left(C_Xk'+C_{wX}k\right)}{g'}\right)\\ \left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\\ +hh'k\left(-C_qhk+\frac{\left(-C_{qX}q+C_X\right)}{g'}\right)\\ \left(H_{pp}H_{ww}-H_{pw}^2\right)\end{pmatrix}. \tag{D.4}$$

Appendix E

Since the demand function in dynamic markets depends on price, warranty length and cumulative sales, assuming the production and warranty-length cost a function of warranty length and production rate (not a function of the cumulative production volume) and that the interaction between the warranty length and production rate in the cost function is negligible, the partial time derivatives of price, warranty length and production rate are:

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dw}{dt} \\ \frac{dq}{dt} \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} H_{ww}H_{qq}-H_{wq}^2 & H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pw}H_{wq}-H_{ww}H_{pq} \\ H_{pq}H_{wq}-H_{pw}H_{qq} & H_{pp}H_{qq}-H_{pq}^2 & H_{pw}H_{pq}-H_{pp}H_{wq} \\ H_{pw}H_{wq}-H_{ww}H_{pq} & H_{pw}H_{pq}-H_{pp}H_{wq} & H_{pp}H_{ww}-H_{pw}^2 \end{bmatrix} \times \begin{bmatrix} h^2k^2gg' \\ \frac{h^3kk'gg'}{h'} \\ -C_q h^2k^2gg' \end{bmatrix}. \quad \text{(E.1)}$$

In other words,

$$\frac{dp}{dt} = -\frac{h^2kgg'}{\Delta h'} \begin{pmatrix} h'k\Big(\left(H_{ww}H_{qq}-H_{wq}^2\right) \\ -C_q\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right)\Big) \\ +hk'\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right) \end{pmatrix}, \quad \text{(E.2)}$$

$$\frac{dw}{dt} = -\frac{h^2kgg'}{\Delta h'} \begin{pmatrix} h'k\Big(\left(H_{pq}H_{wq}-H_{pw}H_{qq}\right) \\ -C_q\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right)\Big) \\ +hk'\left(H_{pp}H_{qq}-H_{pq}^2\right) \end{pmatrix}, \quad \text{(E.3)}$$

$$\frac{dq}{dt} = -\frac{h^2kgg'}{\Delta h'} \begin{pmatrix} h'k\Big(\left(H_{pw}H_{wq}-H_{ww}H_{pq}\right) \\ -C_q\left(H_{pp}H_{ww}-H_{pw}^2\right)\Big) \\ +hk'\left(H_{pw}H_{pq}-H_{pp}H_{wq}\right) \end{pmatrix}. \quad \text{(E.4)}$$

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