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# Intuitionistic fuzzy Choquet aggregation operator based on Einstein operation laws

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**Abstract.** In this paper, we study the intuitionistic fuzzy information aggregation operators based on Einstein operation laws under the condition in which the aggregated arguments are independent. The Einstein-based Intuitionistic Fuzzy Choquet Averaging (EIFCA) operator is proposed. Furthermore, the relationship between the EIFCA operator and the IFCA operator is investigated. The desirable properties of the EIFCA operator, such as boundeness, monotonicity, shift-invariance and homogeneity are discussed. A multi-criteria decision making approach, based on the EIFCA operator is proposed under intuitionistic fuzzy environment. A comparative example is given for demonstrating the applicability of the proposed decision procedure and for finding links with other operators-based decision approach.



## 1. Introduction

Intuitionistic Fuzzy Sets (IFS) [1-2], introduced by Atanassov, is a generalization of the concept of fuzzy set. IFS is characterized by three functions, expressing degrees of membership, non-membership and indeterminacy, so it is more powerful to deal with uncertainty and vagueness in real applications than type-2 fuzzy set [3-4], type-fuzzy set [3], fuzzy multiset [5-6] and hesitant fuzzy set [7-8] which only consider the membership degree. For example, if a boy wants to find a girlfriend and evaluates the girl from ten aspects, there are six aspects which satisfies the boy, three aspects do not satisfy, and he is uncertain with one aspect of the girl. In such a case, other types of fuzzy set can only reflect the satisfied aspects which lose some uncertain information, while intuitionistic fuzzy set can describe all the satisfied, unsatisfied and uncertain information. Because of its appearance, IFS has attracted much attention [9-34].

Research on the intuitionistic fuzzy information aggregation method is one of the hot topics of the IFS theory. Based on the famous OWA operator [35-36] and the GOWA operator [37] proposed by American scholar Yager, many extended operators have been appeared. For example, Xu [15] proposed the Intuitionistic Fuzzy Weighted Averaging (IFWA) operator, ordered IFWA (IFOWA) operator and the intuitionistic hybrid aggregation (IFHA) operator. From the geometric point of view, Xu and Yager [38] introduced the Intuitionistic Fuzzy Weighted Geometric (IFWG) operator, ordered IFWG (IFOWG) operator and the Intuitionistic Fuzzy Hybrid Geometric (IFHG) operator. Zhao et al. [39] proposed the generalized forms of the IFWA, IFOWA and IFHA operators, and they proofed that the operators proposed by Xu [15] are special cases of the operators. The above aggregation operators for intuitionistic fuzzy information are under the condition in which the aggregated arguments are independent. However, it cannot meet the requirement of real situations. Choquet integral [40,41] is a powerful tool to deal with this situation, based on which Xu [42], Tan and Chen [10] and Tan [43] developed some intuitionistic fuzzy Choquet operators such as

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the Intuitionistic Fuzzy Choquet Average (IFCA) operator and the Intuitionistic Fuzzy Choquet Geometric (IFCG) operator. Tan and Chen [11] proposed the induced intuitionistic fuzzy Choquet integral operator and applied it to decision making. The prominent characteristic of those operators is that they cannot only adapt to the intuitionistic fuzzy environment, but also reflect the interrelationship of the individual criteria.

It needs to be pointed out that the basic operational laws of IFS for the above aggregation operators are the algebraic operational laws. Einstein product and Einstein sum are good alternatives for the algebraic product and algebraic sum, respectively [44]. The purpose of this paper is to investigate intuitionistic fuzzy information aggregation methods based on the Einstein product and Einstein sum under the assumption that the aggregated arguments are correlative. To do this, the remainder of this paper is constructed as follows: Section 2 briefly reviews some basic concepts. In Section 3, we propose the Einstein-based Intuitionistic Fuzzy Choquet Averaging (EIFCA) operator; the desirable properties of the EIFCA are also studied in this section. An approach to multi-criteria decision making based on the proposed operator is proposed in Section 4; a comparative example is also illustrated in this section. S ection 5 gives some conclusion remarks.

## 2. Some basic concepts

As a generalization of fuzzy set, Intuitionistic Fuzzy Set (IFS) assigns to each element a membership degree and a non-membership degree. Atanassov [1] gave the definition of Intuitionistic Fuzzy Set (IFS) as follows.

**Definition 1.** If a set $X$ be fixed, the concept of Intuitionistic Fuzzy Set (IFS) $A$ on $X$ is defined as follows:

$$A = \{< x, \mu_A(x), v_A(x) > | x \in X\}, \tag{1}$$

where the functions $\mu_A(x)$ and $v_A(x)$ denote the degrees of membership and non-membership of the element $x \in X$ to the set $A$, respectively, with the condition that $0 \le \mu_A(x) \le 1$, $0 \le v_A(x) \le 1$ and $0 \le \mu_A(x)+v_A(x) \le 1$. For convenience, Xu [15] named $\alpha = (\mu_\alpha, v_\alpha)$ an Intuitionistic Fuzzy Value (IFV). In this paper, we let $V$ be the set of all IFVs.

For three IFVs $\alpha$, $\alpha_1$, $\alpha_2 \in V$, some Einstein operational laws were given as follows [44]:

1. $\alpha_1 \oplus_\varepsilon \alpha_2 = \left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{1+\mu_{\alpha_1}\mu_{\alpha_2}}, \frac{v_{\alpha_1}v_{\alpha_2}}{1+(1-v_{\alpha_1})(1-v_{\alpha_2})}\right)$,
2. $\alpha_1 \otimes_\varepsilon \alpha_2 = \left(\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1+(1-\mu_{\alpha_1})(1-\mu_{\alpha_2})}, \frac{v_{\alpha_1}+v_{\alpha_2}}{1+v_{\alpha_1}v_{\alpha_2}}\right)$,
3. $\alpha^\lambda = \left(\frac{2\mu_\alpha^\lambda}{(2-\mu_\alpha)^\lambda+\mu_\alpha^\lambda}, \frac{(1+v_\alpha)^\lambda-(1-v_\alpha)^\lambda}{(1+v_\alpha)^\lambda+(1-v_\alpha)^\lambda}\right)$,
4. $\lambda\alpha = \left(\frac{(1+\mu_\alpha)^\lambda-(1-\mu_\alpha)^\lambda}{(1+\mu_\alpha)^\lambda+(1-\mu_\alpha)^\lambda}, \frac{2v_\alpha^\lambda}{(2-v_\alpha)^\lambda+v_\alpha^\lambda}\right)$.

Chen and Tan [45] introduced the score function $s(\alpha) = \mu_\alpha - v_\alpha$ to get the score of $\alpha$, then Hong and Choi [46] defined the accuracy function $h(\alpha) = \mu_\alpha + v_\alpha$ to evaluate the accuracy degree of $\alpha$. Based on the score function $s$ and the accuracy function $h$, Xu and Yager [38] gave an order relation between two IFVs $\alpha_1$ and $\alpha_2$:

1. If $s(\alpha_1) < s(\alpha_2)$, then $\alpha_1 < \alpha_2$;
2. If $s(\alpha_1) = s(\alpha_2)$, then:
   i) If $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$;
   ii) If $h(\alpha_1) < h(\alpha_2)$, then $\alpha_1 < \alpha_2$.

Let $\rho(\{x_i\})$ $(i = 1, 2, \cdots, n)$ be the weights of the elements $x_i \in X$ $(i = 1, 2, \cdots, n)$ where $\rho$ is a fuzzy measure; Sugeno [47], Wang and Klir [48] and Denneberg [49] defined a fuzzy measure as follows.

**Definition 2.** A fuzzy measure $\rho$ on the set $X$ is a function $\rho : \theta(X) \to [0, 1]$ satisfying the following axioms:

1. $\rho(\emptyset) = 0$, $\rho(X) = 1$;
2. $B \subseteq C$ implies $\rho(B) \le \rho(C)$, for all $B, C \subseteq X$;
3. $\rho(B \cup C) = \rho(B)+\rho(C)+\tau\rho(B)\rho(C)$, for all $B, C \subseteq X$ and $B \cap C = \emptyset$, where $\tau > -1$.

Based on Definition 2, Xu [42], Tan and Chen [10] and Tan [43] defined Intuitionistic Fuzzy Choquet Averaging (IFCA) and Intuitionistic Fuzzy Choquet Geometric (IFCG) operators as follows.

**Definition 3.** Let $\rho$ be a fuzzy measure on $X$, and $\alpha(x_j) = (\mu_\alpha(x_j), v_\alpha(x_j))$ $(j = 1, 2, \cdots, n)$ be a collection of intuitionistic fuzzy sets, then:

$$\begin{aligned}
&\mathrm{IFCA}(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)) \\
&\quad= \overset{n}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right) \\
&\quad= \Big(1 - \prod_{j=1}^{n}(1 - \mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \\
&\qquad \prod_{j=1}^{n}(v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}\Big),
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\mathrm{IFCG}(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)) \\
&\quad= \overset{n}{\underset{j=1}{\otimes}} \left(\alpha(x_{\sigma(j)})\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \\
&\quad= \Big(\prod_{j=1}^{n}(\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \\
&\qquad 1 - \prod_{j=1}^{n}(1 - v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}\Big),
\end{aligned} \tag{3}$$

are called the Intuitionistic Fuzzy Choquet Averaging (IFCA) and Intuitionistic Fuzzy Choquet Geometric (IFCG) operators, respectively, where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha(x_{\sigma(1)}) \geq \alpha(x_{\sigma(2)}) \geq \cdots \alpha(x_{\sigma(n)})$, $B_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$, for $k \geq 1$, and $B_{\sigma(0)} = \emptyset$.

## 3. Intuitionistic fuzzy Choquet operator based on Einstein operation laws

In this section, we shall investigate the intuitionistic fuzzy information aggregation operator combined with Choquet integral. Based on the Einstein operational laws described by Definition 2, we give the definition of the Einstein-based Intuitionistic Fuzzy Choquet Averaging (EIFCA) operator as follows:

**Definition 4.** Let $\rho$ be a fuzzy measure on $X$ and $\alpha_j (j = 1, 2, \cdots, n)$ be a collection of IFVs, an Einstein-based Intuitionistic Fuzzy Choquet Averaging (EIFCA) operator is a mapping $V^n \to V$, and:

$$E(C) \int \alpha d\rho = \mathrm{EIFCA}(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n))$$

$$= \overset{n}{\underset{j=1}{\oplus}} \left( \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \alpha(x_{\sigma(j)}) \right), \tag{4}$$

where $(C) \int \alpha d\rho$ denotes the Choquet integral and $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha(x_{\sigma(1)}) \geq \alpha(x_{\sigma(2)}) \geq \cdots \alpha(x_{\sigma(n)})$, $B_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$, for $k \geq 1$, and $B_{\sigma(0)} = \emptyset$.

Based on the operational laws of the IFVs described in Section 2, we can derive Theorem 1 easily.

**Theorem 1.** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs and $\alpha_{\sigma(j)}$ be the $j$th largest of them, then their aggregated value by using the EIFCA operator is also an IFVs, and is defined in Eq. (5) which is shown in Box I.

**Proof.** The proof including Eqs. (6) to (13) is shown in Box II. It should be noted that the proof of Theorem 1 was done by many references such as [15,39-44].

In order to analyze the relationship between the EIFCA and IFCA operators proposed by Tan and Chen [10] and Xu [42], we introduce the following lemma [50].

**Lemma 1.** Let $x_j > 0$, $\omega_j > 0$, $j = 1, 2, \cdots, n$ and $\sum_{j=1}^{n} \omega_j = 1$, then:

$$\prod_{j=1}^{n} x_j^{\omega_j} \leq \sum_{j=1}^{n} \omega_j x_j, \tag{14}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

**Theorem 2.** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs and $\alpha_{\sigma(j)}$ be the $j$th largest of them, then:

$$\mathrm{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \mathrm{IFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n). \tag{15}$$

**Proof.** On one hand, since $\rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \geq 0$ for all $j$ and $\sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) = 1$, then based on Lemma 1, we have:

$$\begin{aligned}
&\prod_{j=1}^{n} \left(1 + \mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})} \\
&\quad + \prod_{j=1}^{n} \left(1 - \mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})} \\
&\quad \leq \sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \left(1 + \mu_{\alpha_{\sigma(j)}}\right) \\
&\quad + \sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \left(1 - \mu_{\alpha_{\sigma(j)}}\right) \\
&\quad = \sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \left(1 + \mu_{\alpha_{\delta(j)}}\right) \\
&\quad + \sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \left(1 - \mu_{\alpha_{\delta(j)}}\right) \\
&\quad = \sum_{j=1}^{n} \rho(B_{\sigma(j)} - B_{\sigma(j-1)})
\end{aligned}$$

$$\mathrm{EIFCA}\left(\alpha_1, \alpha_2, \cdots, \alpha_n\right) = \left( \frac{\prod_{j=1}^{n} (1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})} - \prod_{j=1}^{n} (1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})}}{\prod_{j=1}^{n} (1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})} + \prod_{j=1}^{n} (1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})}}, \frac{2 \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})}}{\prod_{j=1}^{n} (2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})} + \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)} - B_{\sigma(j-1)})}} \right). \tag{5}$$

**Box I.**

$$\mathrm{EIFCA}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=\left(\frac{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}-\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}},\right.$$
$$\left.\frac{2\prod_{j=1}^{n}v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n}v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}\right), \tag{6}$$

by using mathematical induction on $n$:
For $n=2$: since:

$$\rho(B_{\sigma(1)}-B_{\sigma(1-1)})\alpha(x_{\sigma(1)})=\left(\frac{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}-(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}},\frac{2v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(2-v_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}\right), \tag{7}$$

$$\rho(B_{\sigma(2)}-B_{\sigma(2-1)})\alpha(x_{\sigma(2)})=\left(\frac{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}-(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}},\frac{2v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(2-v_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}\right). \tag{8}$$

Then:

$$\rho(B_{\sigma(1)}-B_{\sigma(1-1)})\alpha(x_{\sigma(1)})\oplus\rho(B_{\sigma(2)}-B_{\sigma(2-1)})\alpha(x_{\sigma(2)})$$
$$=\left(\frac{\frac{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}-(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}+\frac{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}-(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}}{1+\frac{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}-(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(1+\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+(1-\mu_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}\frac{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}-(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(1+\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+(1-\mu_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}}\right.,$$
$$\left.\frac{\frac{2v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(2-v_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}\frac{2v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(2-v_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}}{1+\left(1-\frac{2v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}{(2-v_{\sigma(1)})^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}+v_{\sigma(1)}^{\rho(B_{\sigma(1)}-B_{\sigma(1-1)})}}\right)\left(1-\frac{2v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}{(2-v_{\sigma(2)})^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}+v_{\sigma(2)}^{\rho(B_{\sigma(2)}-B_{\sigma(2-1)})}}\right)}\right) \tag{9}$$

$$=\left(\frac{\prod_{j=1}^{2}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}-\prod_{j=1}^{2}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{2}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{2}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}},\right.$$
$$\left.\frac{2\prod_{j=1}^{2}v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{2}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{2}v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}\right). \tag{10}$$

**Box II.** Continued on the next page.

$$+\sum_{j=1}^{n}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})$$
$$+\sum_{j=1}^{n}\mu_{\alpha_{\sigma(j)}}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})$$
$$-\sum_{j=1}^{n}\mu_{\alpha_{\sigma(j)}}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})=2. \tag{16}$$

Therefore, Eq. (17) is given in Box III.

On the other hand, since:

$$\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}$$
$$+\prod_{j=1}^{n}v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}$$
$$\leq\sum_{j=1}^{n}\rho(B_{\sigma(j)-B_{\sigma(j-1)}})(2-v_{\alpha_{\sigma(j)}})$$
$$+\sum_{j=1}^{n}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})v_{\alpha_{\sigma(j)}}=2, \tag{18}$$

then Eq. (19) is given in Box IV. Let EIFCA$(\alpha_1,\alpha_2,$

If Eq. (6) holds for $n = k$, that is:

$$\overset{k}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right) = \left( \frac{\prod_{j=1}^{k}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} - \prod_{j=1}^{k}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{k}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{k}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}, \frac{2\prod_{j=1}^{k} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{k}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{k} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \right), \tag{11}$$

then, when $n = k+1$, by the operational laws of IFVs, we have:

$$\overset{k+1}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right) = \overset{k}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right) \oplus \left(\rho(B_{\sigma(k+1)} - B_{\sigma(k)})\alpha(x_{\sigma(k+1)})\right)$$

$$= \left( \frac{\prod_{j=1}^{k+1}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} - \prod_{j=1}^{k+1}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{k+1}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{k+1}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}, \frac{2\prod_{j=1}^{k+1} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{k+1}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{k+1} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \right), \tag{12}$$

i.e. Eq. (6) holds for $n = k+1$. Thus, Eq. (6) holds for all $n$. Then:

$$\text{EIFCA}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left( \frac{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} - \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}, \frac{2\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \right), \tag{13}$$

which completes the proof of Theorem 1.□

**Box II.** Continued.

$$\frac{\prod_{j=1}^{n}\left(1+\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} - \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}\left(1+\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}$$

$$= 1 - \frac{2\prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}\left(1+\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} + \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \leq 1 - \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{17}$$

where the equality holds if and only if $\mu_{\alpha_{\sigma(j)}}$ $(j = 1, 2, \cdots, n)$ are equal.

**Box III.**

$\cdots, \alpha_n) = (\mu_\alpha, v_\alpha) = \alpha$ and $\text{IFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\mu'_\alpha, v'_\alpha) = \alpha'$, then Eq. (19) can be transformed to:

$$\mu_\alpha \leq \mu'_\alpha, \qquad \text{and} \qquad v_\alpha \geq v'_\alpha. \tag{20}$$

Based on Eq. (20), we have:

$$s(\alpha) = \mu_\alpha - v_\alpha \leq \mu'_\alpha - v'_\alpha = s(\alpha'). \tag{21}$$

If $s(\alpha) < s(\alpha^+)$, then:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) < \text{IFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n). \tag{22}$$

If $s(\alpha) = \mu_\alpha - v_\alpha = \mu'_\alpha - v'_\alpha = s(\alpha')$, i.e. Eqs. (23) to (25) are given in Box V. Therefore:

$$h(\alpha) = \mu_\alpha + v_\alpha = h'(\alpha) = \mu'_\alpha + v'_\alpha. \tag{26}$$

Thus, is follows that:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{IFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n). \tag{27}$$

From Eqs. (22) and (27), we know that Eq. (15) always holds.

$$\frac{2\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \geq \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}. \quad (19)$$

**Box IV.**

$$\frac{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}-\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} - \frac{2\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}$$

$$= \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}\right) - \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \qquad j=1,2,\cdots,n \quad (23)$$

Then we have:

$$\frac{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}-\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} = \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}\right) \quad (24)$$

$$\frac{2\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^{n}(2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} = \prod_{j=1}^{n} v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \quad (25)$$

**Box V.**

**Theorem 3 (Idempotency).** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j=(\mu_{\alpha_j},v_{\alpha_j})$ $(j=1,2,\cdots,n)$ be a collection of IFVs and $\alpha_{\sigma(j)}$ be the $j$th largest of them. If all $\alpha_j$ $(j=1,2,\cdots,n)$ are equal, i.e. $\alpha_j=\alpha$, for all $j$, then:

$$\mathrm{EIFCA}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha. \quad (28)$$

**Proof.** By Definition 4, we have:

$$\begin{aligned}
&\mathrm{EIFCA}(\alpha_1,\alpha_2,\cdots,\alpha_n)\\
&= \mathop{\oplus}_{j=1}^{n}\left(\rho(B_{\sigma(j)}-B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right)\\
&= \left(\rho(B_{\sigma(1)}-B_{\sigma(1-1)})\alpha(x_{\sigma(1)})\right)\\
&\oplus \left(\rho(B_{\sigma(2)}-B_{\sigma(2-1)})\alpha(x_{\sigma(2)})\right)\\
&\oplus \left(\rho(B_{\sigma(n)}-B_{\sigma(n-1)})\alpha(x_{\sigma(n)})\right)\\
&= \left(\rho(B_{\sigma(1)}-B_{\sigma(1-1)})\alpha\right)\\
&\oplus \left(\rho(B_{\sigma(2)}-B_{\sigma(2-1)})\alpha\right)\cdots\\
&\oplus \left(\rho(B_{\sigma(n)}-B_{\sigma(n-1)})\alpha\right)\\
&= \left(\sum_{j=1}^{n}\left(\rho(B_{\sigma(j)}-B_{\sigma(j-1)})\right)\alpha\right)\\
&= \left(\rho(B_{\sigma(n)}-B_{\sigma(1-1)})\alpha\right)=\alpha. \quad (29)
\end{aligned}$$

**Theorem 4 (Boundeness).** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j=(\mu_{\alpha_j},v_{\alpha_j})$ $(j=1,2,\cdots,n)$ be a collection of IFVs and $\alpha_{\sigma(j)}$ be the $j$th largest of them, and also let:

$$\alpha^- = \left(\min_j(\mu_{\alpha_{\sigma(j)}}),\max_j(v_{\alpha_{\sigma(j)}})\right),$$

$$\alpha^+ = \left(\max_j(\mu_{\alpha_{\sigma(j)}}),\min_j(v_{\alpha_{\sigma(j)}})\right).$$

Then:

$$\alpha^- \leq \mathrm{EIFCA}\ (\alpha_1,\alpha_2,\cdots,\alpha_n) \leq \alpha^+. \quad (30)$$

**Proof.** Let $f(x)=\frac{1-x}{1+x}$, $x\in[0,1]$, then $f'(x)=\frac{-2}{(1+x)^2}<0$, i.e. $f(x)$ is a decreasing function. Since, $\min_j(\mu_{\alpha_{\sigma(j)}})\leq\mu_{\alpha_j}\leq\max_j(\mu_{\alpha_{\sigma(j)}})$ for all $j$, then:

$$f(\max_j(\mu_{\alpha_{\sigma(j)}}))\leq f(\mu_{\alpha_{\sigma(j)}})\leq f(\min_j(\mu_{\alpha_{\sigma(j)}})), \quad (31)$$

i.e.:

$$\frac{1-\max_j(\mu_{\alpha_{\sigma(j)}})}{1+\max_j(\mu_{\alpha_{\sigma(j)}})}\leq\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\leq\frac{1-\min_j(\mu_{\alpha_{\sigma(j)}})}{1+\min_j(\mu_{\alpha_{\sigma(j)}})},$$

$$j=1,2,\cdots,n. \quad (32)$$

Since $\rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \geq 0$ for all $j$, then we have:

$$\left(\frac{1-\max_j(\mu_{\alpha_{\sigma(j)}})}{1+\max_j(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)}} \leq \left(\frac{1-\min_j(\mu_{\alpha_{\sigma(j)}})}{1+\min_j(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \quad j=1,2,\cdots,n. \tag{33}$$

Thus:

$$\prod_{j=1}^{n}\left(\frac{1-\max_j(\mu_{\alpha_{\delta(j)}})}{1+\max_j(\mu_{\alpha_{\delta(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\delta(j)}})}{1+(\mu_{\alpha_{\delta(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \prod_{j=1}^{n}\left(\frac{1-\min_j(\mu_{\alpha_{\delta(j)}})}{1+\min_j(\mu_{\alpha_{\delta(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{34}$$

$$\Rightarrow \left(\frac{1-\max_j(\mu_{\alpha_{\sigma(j)}})}{1+\max_j(\mu_{\alpha_{\sigma(j)}})}\right)^{\sum_{j=1}^{n}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \left(\frac{1-\min_j(\mu_{\alpha_{\sigma(j)}})}{1+\min_j(\mu_{\alpha_{\sigma(j)}})}\right)^{\sum_{j=1}^{n}\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{35}$$

$$\Rightarrow \frac{1-\max_j(\mu_{\alpha_{\sigma(j)}})}{1+\max_j(\mu_{\alpha_{\sigma(j)}})} \leq \prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \frac{1-\min_j(\mu_{\alpha_{\sigma(j)}})}{1+\min_j(\mu_{\alpha_{\sigma(j)}})}, \tag{36}$$

$$\Rightarrow \frac{2}{1+\max_j(\mu_{\alpha_{\sigma(j)}})} \leq 1+\prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \leq \frac{2}{1+\min_j(\mu_{\alpha_{\sigma(j)}})}, \tag{37}$$

$$\Rightarrow \frac{1+\min_j(\mu_{\alpha_{\sigma(j)}})}{2} \leq \frac{1}{1+\prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \leq \frac{1+\max_j(\mu_{\alpha_{\sigma(j)}})}{2}, \tag{38}$$

$$\Rightarrow 1+\min_j(\mu_{\alpha_{\sigma(j)}}) \leq \frac{2}{1+\prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \leq 1+\max_j(\mu_{\alpha_{\sigma(j)}}), \tag{39}$$

$$\Rightarrow \min_j(\mu_{\alpha_{\sigma(j)}}) \leq \frac{2}{1+\prod_{j=1}^{n}\left(\frac{1-(\mu_{\alpha_{\sigma(j)}})}{1+(\mu_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} - 1 \leq \max_j(\mu_{\alpha_{\sigma(j)}}). \tag{40}$$

And Eq. (41) is shown in Box VI.

Let $g(y) = \frac{2-y}{y}$, $y \in (0,1]$, then $g'(y) = \frac{-2}{y^2} < 0$, i.e. $g(y)$ is a decreasing function. Since $\min_j(v_{\alpha_{\sigma(j)}}) \leq v_{\alpha_j} \leq \max_j(v_{\alpha_{\sigma(j)}})$, for all $j$, then:

$$f\left(\max_j(v_{\alpha_{\sigma(j)}})\right) \leq f\left(v_{\alpha_{\sigma(j)}}\right) \leq f\left(\min_j(v_{\alpha_{\sigma(j)}})\right), \tag{42}$$

i.e.:

$$\frac{2-\max_j(v_{\alpha_{\sigma(j)}})}{\max_j(v_{\alpha_{\sigma(j)}})} \leq \frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}} \leq \frac{2-\min_j(v_{\alpha_{\sigma(j)}})}{\min_j(v_{\alpha_{\sigma(j)}})}, \quad j=1,2,\cdots,n. \tag{43}$$

$$\Rightarrow \min_j(\mu_{\alpha_{\sigma(j)}}) \le \frac{\prod_{j=1}^n (1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}-\prod_{j=1}^n (1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^n (1+\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^n (1-\mu_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \le \max_j(\mu_{\alpha_{\sigma(j)}}). \tag{41}$$

**Box VI.**

Since $\rho(B_{\sigma(j)} - B_{\sigma(j-1)}) \ge 0$ for all $j$, then we have:

$$\left(\frac{2-\max_j(v_{\alpha_{\sigma(j)}})}{\max_j(v_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \left(\frac{2-\min_j(v_{\alpha_{\sigma(j)}})}{\min_j(v_{\alpha_{\sigma(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{44}$$

$$\Rightarrow \prod_{j=1}^n\left(\frac{2-\max_j(v_{\alpha_{\delta(j)}})}{\max_j(v_{\alpha_{\delta(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \prod_{j=1}^n\left(\frac{2-v_{\alpha_{\delta(j)}}}{v_{\alpha_{\delta(j)}}}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \prod_{j=1}^n\left(\frac{2-\min_j(v_{\alpha_{\delta(j)}})}{\min_j(v_{\alpha_{\delta(j)}})}\right)^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{45}$$

$$\Rightarrow \left(\frac{2-\max_j(v_{\alpha_{\sigma(j)}})}{\max_j(v_{\alpha_{\sigma(j)}})}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \left(\frac{2-\min_j(v_{\alpha_{\sigma(j)}})}{\min_j(v_{\alpha_{\sigma(j)}})}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})}, \tag{46}$$

$$\Rightarrow \frac{2-\max_j(v_{\alpha_{\sigma(j)}})}{\max_j(v_{\alpha_{\sigma(j)}})} \le \left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})} \le \frac{2-\min_j(v_{\alpha_{\sigma(j)}})}{\min_j(v_{\alpha_{\sigma(j)}})}, \tag{47}$$

$$\Rightarrow \frac{2}{\max_j(v_{\alpha_{\sigma(j)}})} \le \left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+1 \le \frac{2}{\min_j(v_{\alpha_{\sigma(j)}})}, \tag{48}$$

$$\Rightarrow \frac{\min_j(v_{\alpha_{\sigma(j)}})}{2} \le \frac{1}{\left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+1} \le \frac{\max_j(v_{\alpha_{\sigma(j)}})}{2}, \tag{49}$$

$$\Rightarrow \min_j(v_{\alpha_{\sigma(j)}}) \le \frac{2}{\left(\frac{2-v_{\alpha_{\sigma(j)}}}{v_{\alpha_{\sigma(j)}}}\right)^{\sum_{j=1}^n \rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+1} \le \max_j(v_{\alpha_{\sigma(j)}}), \tag{50}$$

and Eq. (51) is shown in Box VII. Let:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha = (\mu_\alpha, v_\alpha), \tag{52}$$

we have:

$$\min_j(\mu_{\alpha_{\sigma(j)}}) \le \mu_\alpha \le \max_j(\mu_{\alpha_{\sigma(j)}}), \tag{53}$$

$$\min_j(v_{\alpha_{\sigma(j)}}) \le v_\alpha \le \max_j(v_{\alpha_{\sigma(j)}}), \tag{54}$$

$$s(\alpha)=\mu_\alpha - v_\alpha \le \max_j(\mu_{\alpha_{\sigma(j)}}) - \min_j(v_{\alpha_{\sigma(j)}}) = s(\alpha^+), \tag{55}$$

$$s(\alpha)=\mu_\alpha - v_\alpha \ge \min_j(\mu_{\alpha_{\sigma(j)}}) - \max_j(v_{\alpha_{\sigma(j)}}) = s(\alpha^-). \tag{56}$$

$$\Rightarrow \min_j(v_{\alpha_{\sigma(j)}}) \leq \frac{2\prod_{j=1}^n v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}}{\prod_{j=1}^n (2-v_{\alpha_{\sigma(j)}})^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}+\prod_{j=1}^n v_{\alpha_{\sigma(j)}}^{\rho(B_{\sigma(j)}-B_{\sigma(j-1)})}} \leq \max_j(v_{\alpha_{\sigma(j)}}). \quad (51)$$

**Box VII.**

If $s(\alpha) < s(\alpha^+)$ and $s(\alpha) > s(\alpha^-)$, then by order relation between two IFVs which was described in Section 2, we have:

$$\alpha^- < \text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) < \alpha^+. \quad (57)$$

If $s(\alpha) = s(\alpha^+)$, i.e. $\mu_\alpha - v_\alpha = \max_j(\mu_{\alpha_{\delta(j)}}) - \min_j(v_{\alpha_{\delta(j)}})$, we have:

$$\mu_\alpha = \max_j(\mu_{\alpha_{\sigma(j)}}), \qquad v_\alpha = \min_j(v_{\alpha_{\sigma(j)}}). \quad (58)$$

Therefore:

$$h(\alpha)=\mu_\alpha+v_\alpha=\max_j(\mu_{\alpha_{\sigma(j)}})+\min_j(v_{\alpha_{\sigma(j)}})=h(\alpha^+), \quad (59)$$

then:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^+. \quad (60)$$

If $s(\alpha)=s(\alpha^-)$, i.e. $\mu_\alpha - v_\alpha = \min_j(\mu_{\alpha_{\sigma(j)}}) - \max_j(v_{\alpha_{\sigma(j)}})$, then we have:

$$\mu_\alpha = \min_j(\mu_{\alpha_{\sigma(j)}}), \qquad v_\alpha = \max_j(v_{\alpha_{\sigma(j)}}). \quad (61)$$

Therefore:

$$h(\alpha)= \mu_\alpha+\nu_\alpha=\min_j(\mu_{\alpha_{\sigma(j)}})+\max_j(\mu_{\alpha_{\sigma(j)}})=h(\alpha^-). \quad (62)$$

Thus, it follows that:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^-. \quad (63)$$

From Eqs. (57), (60) and (63), we know that Eq. (30) always holds. It should be noted that the proof of Theorems 2 and 4 are referred to the proved methods provided by Wang and Liu [44].

The Monotonicity of the EIFCA operator can be obtained by a similar proving method.

**Theorem 5 (Monotonicity).** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j^* = (\mu_{\alpha_j^*}, v_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFVs, $\alpha_{\sigma(j)}$ be the $j$th largest of $\alpha_j (j = 1, 2, \cdots, n)$ and $\alpha^*_{\sigma(j)}$ be the $j$th largest of $\alpha_j^*$ $(j = 1, 2, \cdots, n)$, $\lambda > 0$. If $\mu_{\alpha_{\sigma(j)}} \leq \mu_{\alpha^*_{\sigma(j)}}$ and $v_{\alpha_{\sigma(j)}} \geq v_{\alpha^*_{\sigma(j)}}$, for all $j$, then:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \text{EIFCA}(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*). \quad (64)$$

**Theorem 6.** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs and $\alpha_{\sigma(j)}$ be the $j$th largest of them. If$(\alpha_1', \alpha_2', \cdots, \alpha_n')$ is any permutation of $\alpha_j$ $(j = 1, 2, \cdots, n)$, then:

$$\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{EIFCA}(\alpha_1', \alpha_2', \cdots, \alpha_n'). \quad (65)$$

**Proof.** According to Definition 4, we have:

$$\text{EIFCA}(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)) = \overset{n}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha(x_{\sigma(j)})\right), \quad (66)$$

and:

$$\text{EIFCA}(\alpha'(x_1), \alpha'(x_2), \cdots, \alpha'(x_n)) = \overset{n}{\underset{j=1}{\oplus}} \left(\rho(B_{\sigma(j)} - B_{\sigma(j-1)})\alpha'(x_{\sigma(j)})\right). \quad (67)$$

Since $(\alpha_1', \alpha_2', \cdots, \alpha_n')$ is any permutation of $\alpha_j$, we have:

$$\alpha(\sigma_{\sigma(i)}) = \alpha'(\sigma_{\sigma(i)} i = 1, 2, \cdots, n. \quad (68)$$

Therefore, $\text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{EIFCA}(\alpha_1', \alpha_2', \cdots, \alpha_n')$, which completes the proof of Theorem 6.

From Definition 4, the following property of the EIFCA operator can be obtained easily.

**Property 1 (Shift-invariance).** Let $\rho$ be a fuzzy measure on $X$ and $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs. If $\beta = (\mu_\beta, \nu_\beta)$ is an intuitionistic fuzzy value on $X$, then:

$$\text{EIFCA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \cdots, \alpha_n \oplus \beta) = \text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) \oplus \beta. \quad (69)$$

**Property 2 (Homogeneity).** Let $\rho$ be a fuzzy measure on $X$ and $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs. If $\lambda > 0$, then:

$$\text{EIFCA}(\lambda\alpha_1, \lambda\alpha_2, \cdots, \lambda\alpha_n) = \lambda\ \text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n). \quad (70)$$

**Property 3.** Let $\rho$ be a fuzzy measure on $X$ and $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFVs. If $\beta = (\mu_\beta, \nu_\beta)$ is an intuitionistic fuzzy value on $X$ and if $\lambda > 0$, then:

$$\text{EIFCA}(\lambda\alpha_1 \oplus \beta, \lambda\alpha_2 \oplus \beta, \cdots, \lambda\alpha_n \oplus \beta)$$
$$= \lambda\ \text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n) \oplus \beta. \tag{71}$$

**Property 4.** Let $\rho$ be a fuzzy measure on $X$, $\alpha_j = (\mu_{\alpha_j}, v_{\alpha_j})$ and $\beta_j = (\mu_{\beta_j}, v_{\beta_j})$ $(j = 1, 2, \cdots, n)$ be two collections of IFVs, then:

$$\text{EIFCA}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_n \oplus \beta_n)$$
$$= \text{EIFCA}(\alpha_1, \alpha_2, \cdots, \alpha_n)$$
$$\oplus\, \text{EIFCA}(\beta_1, \beta_2, \cdots, \beta_n). \tag{72}$$

## 4. An approach to multi-criteria decision making under intuitionistic fuzzy environment

The multi-criteria decision making is a very practical method in the real world, and have very significant effect on both theory and practical. It aims to find the best alternatives from a finite number of options. Multi-criteria decision making, using IFVs, is an important variety of decision making theory [15-18]. In the following, we utilize the EIFCA operator to develop an approach for multi-criteria decision making using IFVs, which includes the following steps:

**Step 1.** The intuitionistic fuzzy decision making matric $B = (b_{ij})_{m\times n}$ always needs to be standardized, since there are cost attributes (the smaller the attribute values the better). Xu and Hu [51] have proposed a standardization method.

$$r_{ij} = (\mu_{ij}, v_{ij}) = \begin{cases} b_{ij}, & \text{for benefit attribute} \quad C_j \\ \bar{b}_{ij}, & \text{for cos}\,t\ \text{attribute} \quad C_j \end{cases}$$

$$i = 1, 2, \cdots, m \qquad j = 1, 2, \cdots, n, \tag{73}$$

where $\bar{b}_{ij}$ is the complement of $b_{ij}$ such that:

$$\bar{b}_{ij} = (f_{ij}, t_{ij}),$$

$$i = 1, 2, \cdots, m, \qquad j = 1, 2, \cdots, n. \tag{74}$$

Based on Eqs. (94) and (95), the standardized decision making matric $R = (r_{ij})_{m\times n}$ can be obtained.

**Step 2.** By the order relation between IFVs, $r_{ij}$ is reordered such that $r_{i(1)} \geq r_{i(2)} \geq \cdots \geq r_{i(n)}$.

**Step 3.** Based on Choquet integral, calculate the correlations between the criteria, using the method given in Section 2 [10,11].

**Step 4.** Utilize the EIFCA operator to aggregate all the performance values $r_{ij} (j = 1, 2, \cdots, n)$ of the $i$th line, and get the overall performance value $r_i$ corresponding to the alternative $A_i$.

**Step 5.** Calculate the score function and accuracy function of the overall values $r_i$ $(i = 1, 2, \cdots, m)$, then utilize order relation between IFVs described in Section 2 to rank the overall performance values $r_i$ and select the best one.

**Example 1.** Suppose a multinational corporation in China is planning its financial strategy for the next year, according to the group strategy objective. After preliminary screening, the four alternatives are produced.

$A_1$: To invest in the Southeast Asian markets;

$A_2$: To invest in the Eastern European market;

$A_3$: To invest in the North American market;

$A_4$: To invest in the local market.

This evaluation proceeds from the following four aspects, such as the growth analysis ($c_1$), the risk analysis ($c_2$), the sociopolitical impact analysis ($c_3$) and the environmental impact analysis ($c_4$). The four alternatives $A_i$ $(i = 1, 2, \cdots, 4)$ are to be evaluated by corresponding experts, using the IFVs, as shown in Table 1.

In order to choose the most appropriate investment program, the main steps are as follows (based on the EIFCA operator):

**Step 1:** Since the criteria $c_2$ and $c_4$ are the cost criteria, the decision matrix need normalization. Normalized decision matrix is shown in Table 2.

**Step 2:** According to Table 2, by the order relation between IFVs, the evaluation $r_{ij}$ of the candidate $A_i$ such that $r_{i(1)} \geq r_{i(2)} \geq \cdots \geq r_{i(n)}$ $(i = 1, 2, \cdots, 4)$, is

**Table 1.** The evaluation information on the projects.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $A_1$ | (0.6,0.2) | (0.2,0.8) | (0.8,0.1) | (0.5,0.3) |
| $A_2$ | (0.4,0.1) | (0.1,0.6) | (0.5,0.2) | (0.2,0.8) |
| $A_3$ | (0.7,0.3) | (0.2,0.8) | (0.6,0.3) | (0.1,0.4) |
| $A_4$ | (0.5,0.5) | (0.1,0.9) | (0.8,0.1) | (0.2,0.7) |

**Table 2.** The normalized evaluation information on the projects.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $A_1$ | (0.6,0.2) | (0.8,0.2) | (0.8,0.1) | (0.3,0.5) |
| $A_2$ | (0.4,0.1) | (0.6,0.1) | (0.5,0.2) | (0.8,0.2) |
| $A_3$ | (0.7,0.3) | (0.8,0.2) | (0.6,0.3) | (0.4,0.1) |
| $A_4$ | (0.5,0.5) | (0.9,0.1) | (0.8,0.1) | (0.7,0.2) |

reordered as follows:

$$r_{1(1)} = (0.8, 0.1), \qquad r_{1(2)} = (0.8, 0.2),$$

$$r_{1(3)} = (0.6, 0.2), \qquad r_{1(4)} = (0.3, 0.5),$$

$$r_{2(1)} = (0.8, 0.2), \qquad r_{2(2)} = (0.6, 0.1),$$

$$r_{2(3)} = (0.5, 0.2), \qquad r_{2(4)} = (0.4, 0.1),$$

$$r_{3(1)} = (0.8, 0.2), \qquad r_{3(2)} = (0.7, 0.3),$$

$$r_{3(3)} = (0.6, 0.3), \qquad r_{3(4)} = (0.4, 0.1),$$

$$r_{4(1)} = (0.9, 0.1), \qquad r_{4(2)} = (0.8, 0.1),$$

$$r_{4(3)} = (0.7, 0.2), \qquad r_{4(4)} = (0.5, 0.5).$$

**Step 3.** Suppose the fuzzy measures of criteria of $C$ and criteria sets of $C$ are as follows:

$$\rho(\emptyset) = 0, \qquad \rho(c_1) = 0.38, \qquad \rho(c_2) = 0.27,$$

$$\rho(c_3) = 0.36, \qquad \rho(c_4) = 0.21,$$

$$\rho(c_1, c_2) = 0.77, \qquad \rho(c_1, c_3) = 0.64,$$

$$\rho(c_1, c_4) = 0.51, \qquad \rho(c_2, c_3) = 0.44,$$

$$\rho(c_2, c_4) = 0.31, \qquad \rho(c_3, c_4) = 0.45,$$

$$\rho(c_1, c_2, c_3) = 0.83, \qquad \rho(c_1, c_2, c_4) = 0.69,$$

$$\rho(c_1, c_3, c_4) = 0.78, \qquad \rho(c_2, c_3, c_4) = 0.55,$$

$$\rho(c_1, c_2, c_3, c_4) = 1.$$

**Step 4.** Utilize the EIFCA operator to aggregate all the performance values $r_{ij} (j = 1, 2, \cdots, 4)$ of the $i$th line, and get the overall performance value $r_i$ corresponding to the alternative $A_i$ $(i = 1, 2, 3, 4)$.$r_1$ is calculated and shown in Box VIII. Similarly:

$$r_2 = (0.5804, 0.1372), \quad r_3 = (0.6879, 0.2251),$$

$$r_4 = (0.7291, 0.2318),$$

**Step 5.** Calculate the scores of $r_i$ $(i = 1, 2, 3, 4)$, respectively:

$$S_1 = 0.4819, \qquad S_2 = 0.4432,$$

$$S_3 = 0.4628, \qquad S_4 = 0.4973.$$

Since:

$$S_4 > S_1 > S_3 > S_2,$$

we have:

$$A_4 \succ A_1 \succ A_3 \succ A_2.$$

Hence, the best financial strategy is $A_4$, i.e. to invest in the local market.

If we use the IFCA operator proposed by Xu [42] and Tan and Chen [10] (i.e., Eq. (2) described in Section 2) to get the overall values $r_i''$ of the options $A_i$ $(i = 1, 2, 3, 4)$ then:

$$r_1'' = (0.6757, 0.1821), \qquad r_2'' = (0.5915, 0.1366),$$

$$r_3'' = (0.6922, 0.2231), \qquad r_4'' = (0.7381, 0.2227).$$

According to the overall values $r_i''$ $(i = 1, 2, 3, 4)$, by the order relations of IFVs, we can obtain that:

$$r_4'' > r_1'' > r_3'' > r_2'',$$

i.e.:

$$A_4 \succ A_1 \succ A_3 \succ A_2.$$

Hence, the best financial strategy is $A_4$, i.e. to invest in the local market. The optimal financial strategy and the ranking of the financial strategies are the same as the ones obtained by EIFCA Operator. Furthermore, we find that:

$$r_1 = (0.6676, 0.1857) < (0.6757, 0.1821) = r_1'',$$

$$r_2 = (0.5804, 0.1372) < (0.5915, 0.1366) = r_2'',$$

$$r_3 = (0.6879, 0.2251) < (0.6922, 0.2231) = r_3'',$$

$$r_4 = (0.7291, 0.2318) < (0.7381, 0.2227) = r_4''.$$

These results satisfy Theorem 2.

## 5. Concluding remarks

In this paper, we have proposed the Einstein-based Intuitionistic Fuzzy Choquet Averaging (EIFCA) operator. Various properties of EIFCA operator have been studied in this paper. Furthermore, the relationships between EIFCA operator and some existing operator such as IFCA have been discussed. The EIFCA operator were distinguished from the existing operators

$$r_1=\left(\frac{(1+0.8)^{0.36}(1+0.8)^{0.44-0.36}(1+0.6)^{0.83-0.44}(1+0.3)^{1-0.83}-(1-0.8)^{0.36}(1-0.8)^{0.44-0.36}(1-0.6)^{0.83-0.44}(1-0.3)^{1-0.83}}{(1+0.8)^{0.36}(1+0.8)^{0.44-0.36}(1+0.6)^{0.83-0.44}(1+0.3)^{1-0.83}+(1-0.8)^{0.36}(1-0.8)^{0.44-0.36}(1-0.6)^{0.83-0.44}(1-0.3)^{1-0.83}},\right.$$
$$\left.\frac{2\times0.1^{0.36}0.2^{(0.44-0.36)}0.2^{(0.83-0.44)}0.5^{(1-0.83)}}{(2-0.1)^{0.36}(2-0.2)^{(0.44-0.36)}(2-0.2)^{(0.83-0.44)}(2-0.5)^{(1-0.83)}+0.1^{0.36}0.2^{(0.44-0.36)}0.2^{(0.83-0.44)}0.5^{(1-0.83)}}\right)$$
$$=(0.6676,0.1857)$$

**Box VIII.**

not only due to the EIFCA operator, using the Einstein operations, but also due to the consideration of the inter-dependent phenomena among the evaluated criteria, which makes the method proposed in this paper to have more wide practical application potentials. The results of this paper can be extended to the dual hesitant fuzzy environment and applied to supplier selection, personnel evaluation and so on.

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