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*Research Note*

# Comparison and successive iteration of approximate solution of ordinary differential equations with initial conditions by the new modified Krasnoselskii iteration method

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**Abstract.** In this paper, we used the Picard successive iteration method and the new modified Krasnoselskii iteration method in order to solve different types of ordinary linear differential equations having initial conditions. By applying the new modified Krasnoselskii iteration method, not only do we obtain the approximate solutions for the problem, but also establish the corresponding iterative schemes. Finally, it is shown that the accuracy of the new iteration method (called the new modified Krasnoselskii iteration method) is substantially improved by employing variable steps which adjust themselves to the solution of the differential equation.



## 1. Introduction

Iterative methods such as the Krasnoselskii method are increasingly being used for many mathematical models in science and engineering in order to solve the different types of ordinary differential equations. In fact, Krasnoselskii iteration method is considered an alternative solution to the linear differential equations having initial conditions. The theory of this iteration method has been extensively studied by several authors [1-3].

The authors [4,5] have used the fixed point theorem and also iteration to solve the differential equations. The fixed point theory on normed linear space was first presented by L.E.J. Brouwer in 1909-1913 [6]. Subsequently, several authors investigated the theorem for different types of spaces, such as metric [7], Banach [4,8] and Hilbert [9], respectively.

The fixed point theorem has become important, in recent years, as a mathematical model of phenomena in biology [10], electrical engineering [11,12], and so on.

There has been a significant development in this theory especially in the area of non-linear differential equations having boundary conditions. Recently, Sun [13] discussed the existence and successive iteration of positive solutions of boundary value problems, and He [14,15] proposed a new perturbation method using the homotopy technique. The presented method, requiring no parameters in the equation, can readily eliminate the limitations of the traditional perturbation methods.

Motivated by this work, we defined the new modified Krasnoselskii iteration method, in order to solve ordinary linear differential equations having initial conditions. Additionally, we compared the numerical

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results using the Euler Method [16,17], Runge-Kutta Method [16-20] and Picard iteration method [16,17] according to the exact solution. Comparison of the numerical results show that the new modified Krasnoselskii iteration method is effective and convenient for solving different types of linear differential equations.

On the other hand, the variational iteration method will be more fully explained and applied to the differential type of the linear and nonlinear problem in another paper where the relationship compared with other techniques will be given in detail.

## 2. Preliminaries

Some basic definitions and properties of the new modified Krasnoselskii iteration method used in this paper are required. $X$ is the metric linear spaces of the continuous function and $T : X \to Y$ is a given operator with $x \in X$.

Krasnoselskii [5] proved that the sequence of iteration $\{T^n(x_0)\}$, starting from a given point, $x_0 \in E$, does not converge necessarily to a fixed point of $T$, whereas the sequence $\{T_\lambda^n(x_0)\}$ where $T_\lambda = (1-\lambda)I + \lambda T, 0 < \lambda \leq 1$ may converges to a fixed point of $T$, as shown by Krasnoselskii [5], who assumes $\lambda = 1/2$. Here, $E$ is compact and $X$ is uniformly convex. This topic of research plays an important role in the stability problem of fixed point iterations. In 1995, Liu [21] initiated a study of fixed point iterations with errors. On the other hand, there are some attempts in the double sequence setting [22,23]. The fixed point theorems, presented in this paragraph, are all related to the Banach contraction principle, which asserts that every complete metric space is a fixed point space for the class of contractive mappings.

### *2.1. Banach contraction principle*

The Banach contraction principle is the simplest and one of the most versatile elementary results in fixed point theory. Based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map. It produces approximations of any required accuracy. Even, the number of iterations needed to get a specified accuracy can be determined [24].

**Theorem 1.** *(Banach contraction principle). Let $(Y, d)$ be a complete metric space and $T : Y \to Y$ be contractive. Then $T$ has a unique fixed point $u$, and $T^n(y) \to u$ for each $y \in Y$ (see [24]).*

The Banach principle has a useful local version that involves an open ball, $B$, in a complete metric space, $Y$, and a contractive map of $B$ into $Y$ which does not displace the center of the ball too far:

**Corollary 1.** *Let $(Y, d)$ be complete and $B = B(y_0, r) = \{y | d(y, y_0) < r\}$. Let $T : B \to Y$ be a contractive map with constant $\alpha < 1$. If $d(T(y_0), y_0) < (1-\alpha)r$, then $T$ has a fixed point (see [24]).*

**Proof.** Choose $\varepsilon < r$, such that $d(Ty_0, y_0) < (1-\alpha)\varepsilon < (1-\alpha)r$. We show that $T$ maps the closed ball, $K = \{y | d(y, y_0) \leq \varepsilon\}$, into itself: for, if $y \in K$, then:

$$d(Ty, y_0) \leq d(Ty, Ty_0) + d(Ty_0, y_0) \leq \alpha d(y, y_0) + (1-\alpha)\varepsilon \leq \varepsilon.$$

Since K is complete, then the conclusion of corollary 1 is proved by Banach contraction principle. □

**Definition 1.** *If the $\{x_n\}_{n=0}^\infty$ sequence provides the condition $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$, then this is called Picard iteration [25].*

**Definition 2.** *If $x_0 \in X$, $\lambda \in [0, 1]$ and also the $\{x_n\}_{n=0}^\infty$ sequence provides the condition $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n$ for $n = 0, 1, 2, \cdots$ then this is called Krasnoselskii iteration [26].*

**Definition 3.** *If $\lambda \in [0, 1]$, $x_0 \in X$ and $T$ is defined as the contraction mapping with regards to Picard iteration, and also the $\{x_n\}_{n=0}^\infty$ sequence provides the conditions:*

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt \qquad n = 0, 1,$$

$$y_{n+1} = (1-\lambda)y_n + \lambda Ty_{n-2} \qquad n = 2, 3, \cdots$$

$$Ty_{n-1} = y_n \qquad 0 < \lambda < 1,$$

*then this is called a modified Krasnoselskii iteration.*

## 3. Application of methods

**Example 1.** *Let us consider the initial value problem:*

$$y' = \sqrt{|y|} \qquad y(0) = 1. \tag{1}$$

By Theorem 1 and Corollary 1, since $T = \int_{x_0}^{x} F(t, y_n(t))$, then:

$$|T(x) - T(y)| = \left| \int_0^x \sqrt{t} - \int_0^y \sqrt{t} \right| = \left| \leq \frac{2}{3} \right| \sqrt{x^3} - \sqrt{y^3} \left| \leq \frac{2}{3} \right| x - y|,$$

is obtained. So:

$$|T(x) - T(y)| \leq \frac{2}{3}|x - y|,$$

is found. Thus, $T$ has a unique fixed point, which is the unique solution of integral equation $T = \int\limits_{x_0}^{x} F(t, y_n(t))$ or the differential equation, $y' = \sqrt{|y|}\ y(0) = 1$. Firstly, we obtained the exact solution of the equation as $|y| = \frac{1}{4}(x+2)^2 = 1 + x + \frac{x^2}{4}$. Then we approached the approximate solution using by the Picard iteration method. Thus the followings are obtained:

$$y_1 = 1 + x,$$

$$y_2 = \frac{1}{3} + \frac{2}{3}(x+1)^{3/2},$$

If we use the Maclauren series expansion for the seventh term of $y_2$ then:

$$y_2 = 1 + x + \frac{x^2}{4} + \frac{x^3}{24} + \frac{x^4}{64} + \frac{x^5}{128} + \frac{7x^6}{1536},$$

is found. Now, applying the modified Krasnoselskii iteration method to Eq. (1) for $\lambda = 0,5$ the followings are obtained:

$$y_1 = 1 + x,$$

$$y_2 = 1 + x + \frac{x^2}{4} - \frac{x^3}{24} + \frac{x^4}{64} - \frac{x^5}{128} + \frac{7x^6}{1536},$$

$$y_3 = 1 + x + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{128} - \frac{x^5}{256} + \frac{7x^6}{3072},$$

$$\begin{aligned} y_4 =& 1 + x + 0.1875x^2 - 0.03125x^3 \\ &+ 0.01171875x^4 - 0.005859375x^5 \\ &+ 0.00341796875x^6, \end{aligned}$$

$$\begin{aligned} y_5 =& 1 + x + 0.15625x^2 - 0.026041666x^3 \\ &+ 0.009765625x^4 - 0.0048828125x^5 \\ &+ 0.002848307292x^6, \end{aligned}$$

$$\begin{aligned} y_6 =& 1 + x + 0.171875x^2 - 0.028645833x^3 \\ &+ 0.010742187x^4 - 0.00537109375x^5 \\ &+ 0.003133138021x^6, \end{aligned}$$

$$\begin{aligned} y_7 =& 1 + x + 0.1640625x^2 - 0.027343749x^3 \\ &+ 0.010253906x^4 - 0.005126953125x^5 \\ &+ 0.002990722657x^6, \end{aligned}$$

and for $\lambda = 0.4$, the followings are calculated:

$$y_1 = 1 + x,$$

$$y_2 = 1 + x + \frac{x^2}{4} - \frac{x^3}{24} + \frac{x^4}{64} - \frac{x^5}{128} + \frac{7x^6}{1536},$$

$$\begin{aligned} y_3 =& 1 + x + 0.15x^2 - 0.025x^3 + 0.009375x^4 \\ &- 0.0046875x^5 + 0.002734375x^6, \end{aligned}$$

$$\begin{aligned} y_4 =& 1 + x + 0.19x^2 - 0.03167x^3 + 0.011875x^4 \\ &- 0.0059375x^5 + 0.003463541667x^6, \end{aligned}$$

$$\begin{aligned} y_5 =& 1 + x + 0.174x^2 - 0.029002x^3 + 0.010875x^4 \\ &- 0.00454375x^5 + 0.013015625x^6, \end{aligned}$$

$$\begin{aligned} y_6 =& 1 + x + 0.1804x^2 - 0.0300692x^3 + 0.011275x^4 \\ &- 0.0056375x^5 + 0.009194791667x^6. \end{aligned}$$

On the other hand, for $\lambda = 0.9$ the followings are founds:

$$y_1 = 1 + x,$$

$$y_2 = 1 + x + \frac{x^2}{4} - \frac{x^3}{24} + \frac{x^4}{64} - \frac{x^5}{128} + \frac{7x^6}{1536},$$

$$\begin{aligned} y_3 =& 1 + x + 0.025x^2 - 0.004166666667x^3 \\ &+ 0.0015625x^4 - 0.0078125x^5 \\ &+ 0.0004557291667x^6, \end{aligned}$$

$$\begin{aligned} y_4 =& 1 + x + 0.2275x^2 - 0.037916666x^3 \\ &+ 0.01421875x^4 - 0.007109375x^5 \\ &+ 0.004147135417x^6, \end{aligned}$$

$$\begin{aligned} y_5 =& 1 + x + 0.04525x^2 - 0.014291666x^3 \\ &+ 0.002828125x^4 - 0.0014140625x^5 \\ &+ 0.0008248697917x^6. \end{aligned}$$

At last, for $\lambda = 0.0001$, the followings are obtained:

$$y_1 = 1 + x,$$

$$y_2 = 1 + x + \frac{x^2}{4} - \frac{x^3}{24} + \frac{x^4}{64} - \frac{x^5}{128} + \frac{7x^6}{1536},$$

$$y_3 = 1 + x + 0.029975x^2 - 0.00416625x^3 + 0.015623437x^4 - 0.00781171875x^5 + 0.0004556835938x^6,$$

$$y_4 = 1 + x + 0.249975002x^2 - 0.0416625x^3 + 0.015623437x^4 - 0.007811718828x^5 + 0.00455683598x^6.$$

Now we tend the approximate solution, using by the Euler method. Firstly, we use formula:

$$y_{n+1} = y_n + hF(x_n, y_n),$$

with $F(x,y) = \sqrt{|\, y \,|}$, $h = 0.2$ and $x_0 = 0$ $y_0 = 1$.

From the initial condition, $y(0) = 1$, we have $F(0,1) = 1$. We now proceed with the calculations coomputed as follows:

$$y_1 = y_0 + hF(y_0, x_0) = 1 + 0.2 = 1.200,$$

$$x_1 = x_0 + h = 1.000 + 0.200 = 1.200,$$

$$y_2 = y_1 + hF(y_1, x_1) = 1.2 + 0.2 \cdot 1.095445115 = 1.419089023,$$

$$x_2 = x_1 + h = 1.200 + 0.200 = 1.400,$$

$$y_3 = y_2 + hF(y_2, x_2) = 1.419089023 + 0.2 \cdot 1.19125523 = 1.657340069,$$

$$x_3 = x_2 + h = 1.400 + 0.200 = 1.600.$$

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results each step to four such places.

Here, $F(x,y) = \sqrt{|y|}$, $x_0 = 0$ and $y_0 = 1$, and we are to use $h = 0.2$. Using these quantities, we calculated, successively, $k_1$, $k_2$, $k_3$, $k_4$ and $K_0$ defined by:

$$k_1 = hg(y_0, x_0),$$

$$k_2 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right),$$

$$k_3 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right),$$

$$k_4 = hg(y_0 + h, x_0 + k_3),$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)y_{n+1} = y_n + K_0$. Thus we find $k_1$, $k_2$, $k_3$ and $k_4$ for $n = 0$ as:

$$k_1 = hF(x_0, y_0) = 0.20000000,$$

$$k_2 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.209761769,$$

$$k_3 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2102266628,$$

$$k_4 = hF(x_0 + h, y_0 + k_3) = 0.220020601.$$

So, $y_1 = 1.209999565$ is obtained for $x_1 = 0.2$. On the other hand, we calculate $k_1$, $k_2$, $k_3$ and $k_4$ for $n = 1$ as:

$$k_1 = hF(x_1, y_1) = 0.21999996,$$

$$k_2 = hF\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.229782466,$$

$$k_3 = hF\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.230207801,$$

$$k_4 = hF(x_1 + h, y_1 + k_3) = 0.240017279.$$

Hence, $y_2 = 1.4399999194$ is calculated for $x_2 = 0.4$.

Finally, we get $k_1$, $k_2$, $k_3$ and $k_4$ for $n = 2$ as:

$$k_1 = hF(x_2, y_2) = 0.239999993,$$

$$k_2 = hF\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.249799913,$$

$$k_3 = hF\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.2250191916,$$

$$k_4 = hF(x_2 + h, y_2 + k_3) = 0.260014756.$$

Thus, $y_3 = 1.689999654$ is obtained for $x_3 = 0.6$.

After the necessary calculations shown above, a comparison is shown, schematically, in Figure 1.

On the other hand we show Tables 1 and 2 concerning the Picard iteration method, Euler method, Runge-Kutta method and the modified Krasnoselskii iteration method for different values of *lambda*. iteration method for different values of $\lambda$, Picard iteration method, Euler method and Runge-Kutta method.

**Table 1.** Comparison of the solutions obtained by the modified Krasnoselskii iteration method, Picard iteration method, Runge-Kutta method, and Euler method with the exact solution for different values of $\lambda$.

| Modified Krasnoselskii iteration | | | | |
|---|---|---|---|---|
| $x$ | $\lambda = 0.0001$ | $\lambda = 0.4$ | $\lambda = 0.5$ | $\lambda = 0.9$ |
| 0.2 | $y_1 = 1.21$ | $y_1 = 1.2$ | $y_1 = 1.2$ | $y_1 = 1.2$ |
| | $y_2 = 1.2097$ | $y_2 = 1.2097$ | $y_2 = 1.2097$ | $y_2 = 1.2097$ |
| | $y_3 = 1.209688489$ | $y_3 = 1.205813675$ | $y_3 = 1.204844729$ | $y_3 = 1.200966916$ |
| | $y_4 = 1.209688489$ | $y_4 = 1.207363962$ | $y_4 = 1.207267094$ | $y_4 = 1.208817407$ |
| | | $y_5 = 1.206744477$ | $y_5 = 1.206055911$ | $y_5 = 1.201483792$ |
| | | $y_6 = 1.206992271$ | $y_6 = 1.206661503$ | |
| | | | $y_7 = 1.206358707$ | |
| 0.4 | $y_1 = 1.4$ | $y_1 = 1.4$ | $y_1 = 1.4$ | $y_1 = 1.4$ |
| | $y_2 = 1.437672$ | $y_2 = 1.437672$ | $y_2 = 1.437672$ | $y_2 = 1.437672$ |
| | $y_3 = 1.437668233$ | $y_3 = 1.4226032$ | $y_3 = 1.418836$ | $y_3 = 1.4036952$ |
| | $y_4 = 1.437668233$ | $y_4 = 1.428630507$ | $y_4 = 1.428254$ | $y_4 = 1.43428152$ |
| | | $y_5 = 1.426259904$ | $y_5 = 1.423545$ | $y_5 = 1.404644632$ |
| | | $y_6 = 1.427208145$ | $y_6 = 1.4258995$ | |
| | | | $y_7 = 1.42472225$ | |
| 0.6 | $y_1 = 1.6$ | $y_1 = 1.6$ | $y_1 = 1.6$ | $y_1 = 1.6$ |
| | $y_2 = 1.682630125$ | $y_2 = 1.682630125$ | $y_2 = 1.682630125$ | $y_2 = 1.682630125$ |
| | $y_3 = 1.6826218662$ | $y_3 = 1.649578075$ | $y_3 = 1.641315063$ | $y_3 = 1.607716263$ |
| | $y_4 = 1.682621863$ | $y_4 = 1.662797175$ | $y_4 = 1.661972594$ | $y_4 = 1.675193414$ |
| | | $y_5 = 1.657969405$ | $y_5 = 1.651643828$ | $y_5 = 1.607666053$ |
| | | $y_6 = 1.659900913$ | $y_6 = 1.656808211$ | |
| | | | $y_7 = 1.65422602$ | |
| $x$ | **Picard** | **Runge-Kutta** | **Euler** | **Exact solution** |
| 0.2 | $y_1 = 1.2$<br>$y_2 = 1.2097$ | $y_1 = 1.209999565$ | $y_1 = 1.2$ | $y = 1.21$ |
| 0.4 | $y_1 = 1.4$<br>$y_2 = 1.437672$ | $y_1 = 1.4399999194$ | $y_1 = 1.419089023$ | $y = 1.44$ |
| 0.6 | $y_1 = 1.6$<br>$y_2 = 1.682630125$ | $y_1 = 1.689999654$ | $y_1 = 1.657340069$ | $y = 1.69$ |

**Table 2.** Absolute error of Example 1 for different values of $\lambda$ ($x = 0.2$, $x = 0.4$ and $x = 0.6$, respectively).

| Absolute error table<br>Modified Krasnoselskii iteration | | | | |
|---|---|---|---|---|
| $x$ | $\lambda = 0.0001$ | $\lambda = 0.4$ | $\lambda = 0.5$ | $\lambda = 0.9$ |
| 0.2 | $3.11511 \times 10^{-4}$ | $3.007729 \times 10^{-3}$ | $3.641293 \times 10^{-3}$ | $8.516208 \times 10^{-3}$ |
| 0.4 | $2.331767 \times 10^{-3}$ | $1.2791855 \times 10^{-2}$ | $1.527775 \times 10^{-2}$ | $3.5355368 \times 10^{-2}$ |
| 0.6 | $7.378137 \times 10^{-3}$ | $3.0099087 \times 10^{-2}$ | $3.577398 \times 10^{-2}$ | $8.2333947 \times 10^{-2}$ |

| $x$ | Picard | Runge-Kutta | Euler |
|---|---|---|---|
| 0.2 | $3 \times 10^{-4}$ | $4.35 \times 10^{-7}$ | 0.01 |
| 0.4 | $2.328 \times 10^{-3}$ | $8.1 \times 10^{-8}$ | $2.0910977 \times 10^{-2}$ |
| 0.6 | $7.369875 \times 10^{-3}$ | $4.6 \times 10^{-7}$ | $3.2659931 \times 10^{-2}$ |

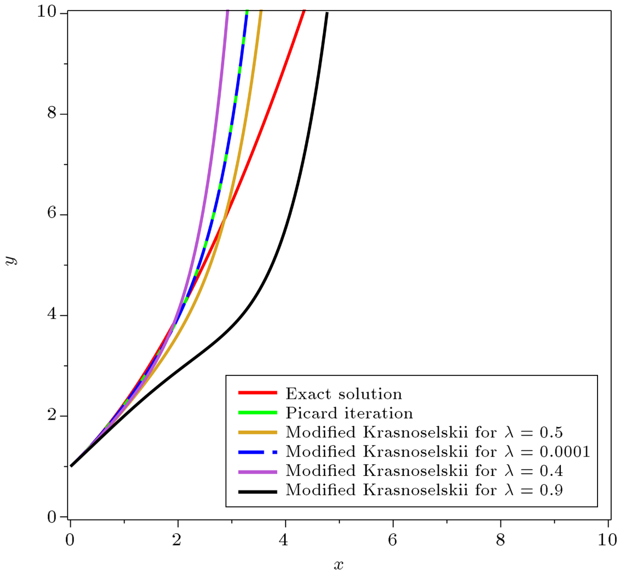


**Figure 1.** Comparison of the exact solution with approximation solution of Example 1 for different values of $\lambda$.

**Corollary 2.** *If the approximate solution compares with the different values of* $\lambda$, *then the conclusion may be indicated, by Table 1.*

*The best approximation is obtained taking different values of* $\lambda$ *using the modified Krasnoselskii iteration method for* $x = 0.2$ *getting* $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = 0.4$ *and* $\lambda = 0.0001$, *respectively, in accordance with the solution using the Picard iteration method, Runge-Kutta method, Euler iteration method and the exact solution.*

*Once, we obtain the solution for* $x = 0.4$, *then, the approximation is obtained to be more sensitive for* $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = 0.4$ *and* $\lambda = 0.0001$, *respectively, using the modified Krasnoselskii iteration method.*

*Similarly, we calculated the solution for* $x = 0.6$. *Then the approximation is found to be more sensitively, for* $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = 0.4$ *and* $\lambda = 0.0001$, *respectively, using the modified Krasnoselskii iteration method.*

*Consequently, the solution, using the modified Krasnoselskii iteration method, gives more accurate results than the solution of the Picard iteration method, Runge-Kutta method, and Euler method for different values of* $\lambda$.

**Corollary 3.** *Absolute error of the modified Krasnoselskii iteration method is computed for different values of* $\lambda$ *which is more effective than that of the Euler method, but not better than the Runge-Kutta method and Picard iteration method, in accordance with Table 2.*

**Example 2.** *Let us consider the differential equation:*

$$y' = \frac{y}{x + \ln y} \qquad y(1) = 1, \tag{2}$$

*subject to the initial condition.*

In accordance with the nature of the given differential equation, we define:

$$f(x, y) = \frac{y}{x + \ln y}, \qquad f(1, 1) = 1 \neq 0, \tag{3}$$

and $\frac{df}{dx} = -\frac{y}{(x+\ln y)^2}$. Thus, the function $\frac{df}{dx}$ is bounded in the rectangular domain including point $(1, 1)$. In this case, if $y = y(x)$ is the local solution of this problem, then its inverse function is also the solution of:

$$x' = g(y, x), \qquad x(y_0) = x_0. \tag{4}$$

Now, we state the method of successive approximation with $g(y, x) = \frac{1}{f(x,y)}$. Hence the solution $x = x(y)$ of Problem (4) is also the solution of the problem:

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{5}$$

which is the inverse solution of Problem (4). Therefore:

$$x_{n+1}(y) = x_0 + \int_{y_0}^{y} g(t, x_n(t))dt.$$

Thus:

$$x_{n+1}(y) = 1 + \int_{1}^{y} g(t, x_n(t))dt$$

$$= 1 + \ln y + \int_{1}^{y} \frac{x_n(t) + \ln(t)}{t} dt,$$

for $x_0 = 1$ and $y_0 = 1$. So, we may write:

$$x_{n+1}(y) = 1 + \ln y + \int_{1}^{y} \frac{x_n(t)}{t} dt.$$

Using Theorem 1 and Corollary 1, since $T = \int_{y_0}^{y} F(t, x_n(t))dt$, then $T$ has a unique fixed point which is the unique solution of the differential equation $y' = \frac{y}{x+\ln y}$, having the initial condition $y(1) = 1$.

Hence we approach the approximate solution, using the Picard iteration method. Thus:

$$x_1(y) = 1 + \ln y + \frac{\ln^2 y}{2},$$

$$x_2(y) = 1 + \ln y + \ln^2 y + \frac{\ln^3 y}{6},$$

$$x_3(y) = 1 + \ln y + \ln^2 y + \frac{\ln^3 y}{3} + \frac{\ln^4 y}{24}.$$

Consequently, solution $x = 2y - 1 - \ln y$ is obtained as $n \to \infty$.

On the other hand the exact solution of the equation is $x = 2y - 1 - \ln y$, which coincides with the approximate solution.

Now, applying the modified Krasnoselskii iteration method to the equation for $\lambda = 0,5$, then:

$$x_1(y) = 1 + \ln y + \frac{\ln^2 y}{2},$$

$$x_2(y) = 1 + \ln y + \ln^2 y + \frac{\ln^3 y}{3},$$

$$x_3(y) = 1 + \ln y + \frac{3}{4}\ln^2 y + \frac{1}{6}\ln^3 y,$$

$$x_4(y) = 1 + \ln y + \frac{7}{8}\ln^2 y + \frac{1}{4}\ln^3 y,$$

$$x_5(y) = 1 + \ln y + \frac{13}{16}\ln^2 y + \frac{5}{24}\ln^3 y,$$

$$x_6(y) = 1 + \ln y + \frac{27}{32}\ln^2 y + \frac{11}{48}\ln^3 y,$$

are obtained and also for $\lambda = 0.9$,

$$x_1(y) = 1 + \ln y + \frac{\ln^2 y}{2},$$

$$x_2(y) = 1 + \ln y + \ln^2 y + \frac{\ln^3 y}{3},$$

$$x_3(y) = 1 + \ln y + 0.55\ln^2 y + 0.033\ln^3 y,$$

$$x_4(y) = 1 + \ln y + 0.955\ln^2 y + 0.3033\ln^3 y,$$

$$x_5(y) = 1 + \ln y + 05905\ln^2 y + 0.06003\ln^3 y,$$

$$x_6(y) = 1 + \ln y + 0.91855\ln^2 y + 0.278973\ln^3 y,$$

are calculated. At last, for $\lambda = 0.01$:

$$x_1(y) = 1 + \ln y + \frac{\ln^2 y}{2},$$

$$x_2(y) = 1 + \ln y + \ln^2 y + \frac{\ln^3 y}{3},$$

$$x_3(y) = 1 + \ln y + 0.995\ln^2 y + 0.33\ln^3 y,$$

$$x_4(y) = 1 + \ln y + 0.99505\ln^2 y + 0.330033333\ln^3 y,$$

$$x_5(y) = 1 + \ln y + 0.99504495\ln^2 y + 0330033\ln^3 y,$$

$$x_6(y) = 1 + \ln y + 0.995049505\ln^2 y + 0330033003\ln^3 y,$$

are found.

Now, we tend the approximate solution using the Euler method. Firstly, we use formula:

$$x_{n+1} = x_n + hg(y_n, x_n),$$

with $g(y,x) = \frac{x+\ln y}{y}$ and $h = 0.2$, such that $f(x,y) = \frac{y}{x+\ln y}$.

From the initial condition $y(1) = 1$, we have $x_0 = 1$, $y_0 = 1$. We now proceed with the calculations starting with $g(y_0, x_0) = g(1,1) = 1.000$, then:

a) $x_1 = x_0 + hg(y_0, x_0) = 1.200$,

$y_1 = y_0 + h = 1.000 + 0.2 = 1.200$.

b) $x_2 = x_1 + hg(y_1, x_1) = 1.430386926$,

$y_2 = y_1 + h = 1.2000 + 0.2 = 1.400$.

c) $x_3 = x_2 + hg(y_2, x_2) = 1.682795378$,

$y_3 = y_2 + h = 1.400 + 0.2 = 1.600$.

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results at each step to four such places.

Here, $g(y,x) = \frac{x+\ln y}{y}$, $x_0 = 1, y_0 = 1$, and we are to use $h = 0.2$. Using these quantities, we calculate, successively, $k_1$, $k_2$, $k_3$, $k_4$ and $K_0$ defined by:

$$k_1 = hg(y_0, x_0),$$

$$k_2 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}),$$

$$k_3 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}),$$

$$k_4 = hg(y_0 + h, x_0 + k_3),$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), y_{n+1} = y_n + K_0$. Thus, we find $k_1$, $k_2$, $k_3$, $k_4$ for $n = 0$ starting with $g(1,1) = 0.20000000$, then:

$$k_1 = hg(y_0, x_0) = 0.20000000,$$

$$k_2 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}) = 0.217329123,$$

$$k_3 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}) = 0.218904498,$$

$$k_4 = hg(y_0 + h, x_0 + k_3) = 0.233537675.$$

So, $x_1 = 1.217667486$ is obtained for $y_1 = 1.20000000$. On the other hand, we calculate $k_1$, $k_2$, $k_3$, $k_4$ for $n = 1$, as:

$$k_1 = hg(y_1, x_1) = 0.23333507,$$

$$k_2 = hg(y_1 + \frac{h}{2}, x_1 + \frac{k_1}{2}) = 0.245645769,$$

$$k_3 = hg(y_1 + \frac{h}{2}, x_1 + \frac{k_2}{2}) = 0.24659302,$$

$$k_4 = hg(y_1 + h, x_1 + k_3) = 0.257247534.$$

Hence $x_2 = 1.463510256$ is calculated for $y_2 = 1.40000000$.

Finally, we get $k_1$, $k_2$, $k_3$, $k_4$ for $n = 2$ as:

$$k_1 = hg(y_2, x_2) = 0.257140356,$$

$$k_2 = hg(y_2 + \frac{h}{2}, x_2 + \frac{k_1}{2}) = 0.266339405,$$

$$k_3 = hg(y_2 + \frac{h}{2}, x_2 + \frac{k_2}{2}) = 0.266952675,$$

$$k_4 = hg(y_2 + h, x_2 + k_3) = 0.27505832.$$

Thus, $x_3 = 1.729974062$ is obtained for $y_3 = 1.60000000$.

After the necessary calculations, the comparison is shown schematically in Figure 2.

Now we show Tables 3 and 4 concerning the Picard iteration method, Euler method, Runge-Kutta method and the modified Krasnoselskii iteration method for different values of *lambda*. method for different values of $\lambda$, the Picard iteration method, Euler method and Runge-Kutta method.

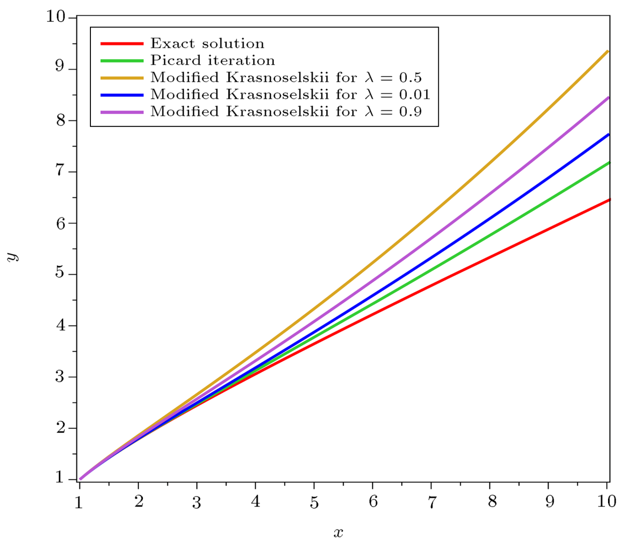


**Figure 2.** Comparison of the exact solution with the approximation solution of Example 2 for different values of $\lambda$.

**Corollary 4.** *If the approximate solution compares with the different values of $\lambda$, then the conclusion may be presented using Table 3.*

*The best approximation may be obtained for different values of $\lambda$, using the modified Krasnoselskii iteration method for $x = 1.2$, getting $\lambda = 0.9$, $\lambda = 0.5$ and $\lambda = 0.001$, respectively, in accordance with the solution of the Picard iteration method, Runge-Kutta method, Euler method and exact solution.*

*We obtained the solution for $y = 1.4$, then, taking $\lambda = 0.9$, $\lambda = 0.5$ and $\lambda = 0.01$, respectively, using the modified Krasnoselskii iteration method.*

*Similarly, we calculated the solution for $y = 1.6$, then, the approximation is found to be more sensitive than for $\lambda = 0.9$, $\lambda = 0.5$ and $\lambda = 0.01$, respectively, using the modified Krasnoselskii iteration method.*

*Consequently, the solution, using the modified Krasnoselskii iteration method, gives more accurate results than the solutions of the Picard iteration method, Runge-Kutta method and Euler method for different values of $\lambda$.*

**Corollary 5.** *The absolute error of the modified Krasnoselskii iteration method, calculated for different values of $\lambda$, is more effective than the Euler method but not better than the Runge-Kutta method and Picard iteration method, according to Table 4.*

**Example 3.** *Let us consider the differential equation*

$$y' = 2x(y + 1), \tag{6}$$

*subject to the initial condition:*

$$y(0) = 0.$$

Using Theorem 1 and Corollary 1, since $T = \int_{x_0}^{x} F(t, y_n(t))dt$, then $T$ has a unique fixed point which is the unique solution of the differential equation $y' = 2x(y + 1)$, having the initial condition $y(0) = 0$.

Firstly, we obtain the exact solution of the equation as $y = e^{x^2} - 1$. Then, we approach the approximate solution, using the Picard iteration method as follows:

$$y_1 = x^2,$$

$$y_2 = x^2 + \frac{x^4}{2},$$

$$y_3 = x^2 + \frac{x^4}{2} + \frac{x^6}{6},$$

$$y_4 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}.$$

**Table 3.** Comparison of the solutions obtained by the modified Kranoselskii iteration method, Picard iteration method, Runge-Kutta method, and the Euler method with the exact solution for different values of $\lambda$.

| **Modified Krasnoselskii iteration** | | | |
|---|---|---|---|
| $y$ | $\lambda = 0.01$ | $\lambda = 0.5$ | $\lambda = 0.9$ |
| 1.2 | $x_1 = 1.198942132$ | $x_1 = 1.198942132$ | $x_1 = 1.198942132$ |
| | $x_2 = 1.2175829$ | $x_2 = 1.2175829$ | $x_2 = 1.2175829$ |
| | $x_3 = 1.217396492$ | $x_3 = 1.208262515$ | $x_3 = 1.200786007$ |
| | $x_4 = 1.217398356$ | $x_4 = 1.212922707$ | $x_4 = 1.215738823$ |
| | $x_5 = 1.217398337$ | $x_5 = 1.21059261$ | $x_5 = 1.202314272$ |
| | $x_6 = 1.217398338$ | $x_6 = 1.211757659$ | $x_6 = 1.214545953$ |
| 1.4 | $x_1 = 1.39307902$ | $x_1 = 1.39307902$ | $x_1 = 1.39307902$ |
| | $x_2 = 1.462383543$ | $x_2 = 1.462383543$ | $x_2 = 1.462383543$ |
| | $x_3 = 1.461690498$ | $x_3 = 1.427731281$ | $x_3 = 1.399882495$ |
| | $x_4 = 1.461697428$ | $x_4 = 1.445057412$ | $x_4 = 1.456144866$ |
| | $x_5 = 1.461697359$ | $x_5 = 1.436394347$ | $x_5 = 1.405611583$ |
| | $x_6 = 1.46169736$ | $x_6 = 1.44072588$ | $x_6 = 1.451091538$ |
| 1.6 | $x_1 = 1.580455335$ | $x_1 = 1.580455335$ | $x_1 = 1.580455335$ |
| | $x_2 = 1.725515509$ | $x_2 = 1.725515509$ | $x_2 = 1.725515509$ |
| | $x_3 = 1.724064907$ | $x_3 = 1.652985422$ | $x_3 = 1.594615268$ |
| | $x_4 = 1.724079413$ | $x_4 = 1.689250466$ | $x_4 = 1.712456633$ |
| | $x_5 = 1.724079268$ | $x_5 = 1.671117944$ | $x_5 = 1.606679733$ |
| | $x_6 = 1.72407927$ | $x_6 = 1.680184205$ | $x_6 = 1.701878943$ |

| $y$ | Picard | Runge-Kutta | Euler | Exact solution |
|---|---|---|---|---|
| 1.2 | $x_1 = 1.198942132$<br>$x_2 = 1.2175829$<br>$x_3 = 1.21762894$ | $x = 1.217667486$ | $x = 1.2$ | $x = 1.217678443$ |
| 1.4 | $x_1 = 1.39307902$<br>$x_2 = 1.462383543$<br>$x_3 = 1.462917598$ | $x = 1.463510256$ | $x = 1.430386926$ | $x = 1.463527763$ |
| 1.6 | $x_1 = 1.580455335$<br>$x_2 = 1.725515509$<br>$x_3 = 1.727548772$ | $x = 1.729974062$ | $x = 1.682795378$ | $x = 1.729996371$ |

Applying the modified Krasnoselskii iteration method to the equation $\lambda = 0,5$, then:

$$y_1 = x^2,$$

$$y_2 = x^2 + \frac{x^4}{2},$$

$$y_3 = x^2 + \frac{x^4}{4},$$

$$y_4 = x^2 + \frac{3x^4}{8},$$

$$y_5 = x^2 + \frac{5x^4}{16},$$

$$y_6 = x^2 + \frac{11x^4}{32},$$

are found. And, also, for $\lambda = 0,01$:

$$y_1 = x^2,$$

$$y_2 = x^2 + \frac{x^4}{2},$$

$$y_3 = x^2 + 0.0495x^4,$$

$$y_4 = x^2 + 0.049505x^4,$$

$$y_5 = x^2 + 0.04950495x^4,$$

$$y_6 = x^2 0.0495049505x^4,$$

**Table 4.** Absolute error of Example 2 for different values of $\lambda$ ($y = 1.2$, $y = 1.4$ and $y = 1.6$, respectively).

| | | | |
|---|---|---|---|
| **Absolute error table** | | | |
| **Modified Krasnoselskii iteration** | | | |
| $y$ | $\lambda = 0.01$ | $\lambda = 0.5$ | $\lambda = 0.9$ |
| 1.2 | 0.000280105 | 0.005920784 | 0.00313249 |
| 1.4 | 0.0018330403 | 0.022801883 | 0.012436265 |
| 1.6 | 0.005917101 | 0.049854321 | 0.028117428 |
| $y$ | **Picard** | **Runge-Kutta** | **Euler** |
| 1.2 | 0.000049503 | 0.000010957 | 0.017678443 |
| 1.4 | 0.000610165 | 0.00017507 | 0.033140837 |
| 1.6 | 0.0002447599 | 0.00002230898 | 0.047200993 |

are calculated. At last, for $\lambda = 0,9$, then:

$$y_1 = x^2,$$

$$y_2 = x^2 + \frac{x^4}{2},$$

$$y_3 = x^2 + 0.05x^4,$$

$$y_4 = x^2 + 0.0455x^4,$$

$$y_5 = x^2 + 0.0905x^4,$$

are obtained.

Now we tend the approximate solution, using the Euler method. Firstly, we use formula:

$$y_{n+1} = y_n + hF(x_n, y_n),$$

with $F(x, y) = 2x(y + 1)$, $h = 0.2$ and $x_0 = 0$ $y_0 = 0$.

From the initial condition $y(0) = 0$, we have $F(0, 0) = 0$. We now proceed where the calculations:

$$y_1 = y_0 + hF(y_0, x_0) = 0 + 0.2 = 0.0000,$$

$$x_1 = x_0 + h = 0.000 + 0.200 = 0.2000,$$

$$y_2 = y_1 + hF(y_1, x_1) = 0.0 + 0.2 \cdot 0.4 = 0.0800,$$

$$x_2 = x_1 + h = 0.200 + 0.200 = 0.4000,$$

$$y_3 = y_2 + hF(y_2, x_2) = 0.08 + 0.2 \cdot 0.864 = 0.2528,$$

$$x_3 = x_2 + h = 0.400 + 0.200 = 0.6000.$$

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point, and round off the final results at each step to four such places.

Here, $F(x, y) = 2x(y + 1)$, $x_0 = 0$, $y_0 = 0$, $x_{n+1} = x_n + h$ and we are to use $h = 0.2$. Using these quantities, we calculated, successively, $k_1$, $k_2$, $k_3$, $k_4$ and $K_0$ defined by:

$$k_1 = hg(y_0, x_0),$$

$$k_2 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}),$$

$$k_3 = hg(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}),$$

$$k_4 = hg(y_0 + h, x_0 + k_3),$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)y_{n+1} = y_n + K_0$. Thus we find $k_1$, $k_2$, $k_3$ and $k_4$ for $n = 0$, as:

$$k_1 = hF(x_0, y_0) = 0.00000000,$$

$$k_2 = hF(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.04,$$

$$k_3 = hF(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.0408,$$

$$k_4 = hF(x_0 + h, y_0 + k_3) = 0.083264.$$

So, $y_1 = 0.040810666$ is obtained for $x_1 = 0.2$.

On the other hand, we calculated $k_1$, $k_2$, $k_3$ and $k_4$ for $n = 1$ as:

$$k_1 = hF(x_1, y_1) = 0.083264853,$$

$$k_2 = hF(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.129893171,$$

$$k_3 = hF(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.13269087,$$

$$k_4 = hF(x_1 + h, y_1 + k_3) = 0.187760245.$$

Hence, $y_2 = 0.173509529$ is calculated for $x_2 = 0.4$

Finally, we get $k_1$, $k_2$, $k_3$, $k_4$ for $n = 2$, as:

$$k_1 = hF(x_2, y_2) = 0.187761524,$$

$$k_2 = hF(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}) = 0.253478058,$$

$$k_3 = hF(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}) = 0.260049711,$$

$$k_4 = hF(x_2 + h, y_2 + k_3) = 0.344054217.$$

Hence, $y_3 = 0433321409$ is obtained for $x_3 = 0.6$.

After the necessary calculations done above, the comparison is shown schematically in Figure 3.

We may present the results given in Tables 5 and 6.

**Table 5.** Comparison of the solutions obtained by the modified Krasnoselskii iteration method, Picard iteration method, Runge-Kutta method, Euler method with the exact solution for different values of $\lambda$.

| Modified Krasnoselskii iteration | $\boldsymbol{\lambda = 0.01}$ | $\boldsymbol{\lambda = 0.5}$ | $\boldsymbol{\lambda = 0.9}$ |
|---|---|---|---|
| $x = 0.2$ | $y_1 = 0.04$ | $y_1 = 0.04$ | $y_1 = 0.04$ |
| | $y_2 = 0.0408$ | $y_2 = 0.0408$ | $y_2 = 0.0408$ |
| | $y_3 = 0.040792$ | $y_3 = 0.0404$ | $y_3 = 0.04008$ |
| | $y_4 = 0.04079208$ | $y_4 = 0.0406$ | $y_4 = 0.040728$ |
| | $y_5 = 0.040792079$ | $y_5 = 0.0405$ | $y_5 = 0.0401448$ |
| | $y_6 = 0.040792079$ | $y_6 = 0.04055$ | |
| $x = 0.4$ | $y_1 = 0.16$ | $y_1 = 0.16$ | $y_1 = 0.16$ |
| | $y_2 = 0.1728$ | $y_2 = 0.1728$ | $y_2 = 0.1728$ |
| | $y_3 = 0.172672$ | $y_3 = 0.1664$ | $y_3 = 0.16128$ |
| | $y_4 = 0.17267328$ | $y_4 = 0.1696$ | $y_4 = 0.171648$ |
| | $y_5 = 0.172673267$ | $y_5 = 0.168$ | $y_5 = 0.1623168$ |
| | $y_6 = 0.172673267$ | $y_6 = 0.1688$ | |
| $x = 0.6$ | $y_1 = 0.36$ | $y_1 = 0.36$ | $y_1 = 0.36$ |
| | $y_2 = 0.4248$ | $y_2 = 0.4248$ | $y_2 = 0.4248$ |
| | $y_3 = 0.424152$ | $y_3 = 0.3924$ | $y_3 = 0.36648$ |
| | $y_4 = 0.42415848$ | $y_4 = 0.4086$ | $y_4 = 0.418968$ |
| | $y_5 = 0.424158415$ | $y_5 = 0.4005$ | $y_5 = 0.3717288$ |
| | $y_6 = 0.42415845$ | $y_6 = 0.40455$ | |

| | **Picard** | **Runge-Kutta** | **Euler** | **Exact solution** |
|---|---|---|---|---|
| $x = 0.2$ | $y_1 = 0.04$ | $y_1 = 0.040810666$ | $y_1 = 0$ | $y = 0.040810774$ |
| | $y_2 = 0.0408$ | | | |
| | $y_3 = 0.040810666$ | | | |
| | $y_4 = 0.040810772$ | | | |
| $x = 0.4$ | $y_1 = 0.16$ | $y_1 = 0.181332873$ | $y_1 = 0.08$ | $y = 0.173510871$ |
| | $y_2 = 0.1728$ | | | |
| | $y_3 = 0.173482666$ | | | |
| | $y_4 = 0.173509972$ | | | |
| $x = 0.6$ | $y_1 = 0.36$ | $y_1 = 0.44287682$ | $y_1 = 0.252$ | $y = 0.433329414$ |
| | $y_2 = 0.4248$ | | | |
| | $y_3 = 0.432576$ | | | |
| | $y_4 = 0.43327584$ | | | |

**Corollary 6.** *If the approximate solution compares with the different values of $\lambda$, then the conclusion may be given using Table 5.*

*The best approximation may be obtained for different values of $\lambda$, such as $\lambda = 0.9$, $\lambda = 0.5$ and $\lambda = 0.01$, respectively, using the modified Krasnoselskii iteration method for $x = 0.2$, $x = 0.4$ and $x = 0.6$, in accordance with the solution, using the Picard iteration method Runge-Kutta method, Euler method and the exact solution.*

*As seen in Table 5, if numerical methods, such as the Runge-Kutta method, Euler method, Picard iteration method and modified Krasnoselskii iteration method are used in order to get the best approximation of each for different values of $\lambda$ such as $\lambda = 0.9$, $\lambda = 0.5$ and $\lambda = 0.01$, respectively, then it is concluded that the modified Krasnoselskii iteration method is more effective than the Picard iteration method and Runge-Kutta method, but not the Euler method, in accordance with the exact solution.*

**Corollary 7.** *The absolute error of the modified Krasnoselskii iteration method is computed for different values of $\lambda$ and is found to be more effective than*

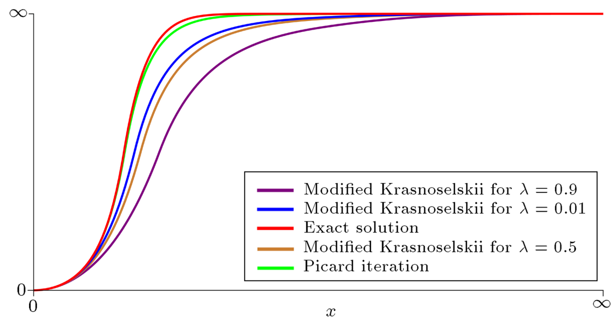


**Figure 3.** Comparison of the exact solution with the approximation solution of Example 3 for different values of $\lambda$.

**Table 6.** Absolute error of Example 3 for different values of $\lambda$ ($x = 0.2$, $x = 0.4$ and $x = 0.6$, respectively).

| Absolute error table | | | |
|---|---|---|---|
| | Modified Krasnoselskii iteration | | |
| | $\lambda = 0.01$ | $\lambda = 0.5$ | $\lambda = 0.9$ |
| $x=0.2$ | $1.9984\times10^{-5}$ | $2.60774\times10^{-4}$ | $6.65974\times10^{-4}$ |
| $x=0.4$ | $8.37604\times10^{-4}$ | $4.710871\times10^{-3}$ | $1.1194071\times10^{-2}$ |
| $x=0.6$ | $9.13569\times10^{-3}$ | $2.874414\times10^{-2}$ | $6.156534\times10^{-2}$ |
| | **Picard** | **Runge-Kutta** | **Euler** |
| $x=0.2$ | $2\times10^{-9}$ | $1.08\times10^{-7}$ | $4.0810774\times10^{-2}$ |
| $x=0.4$ | $8.99\times10^{-7}$ | $1.342\times10^{-5}$ | $9.3510871\times10^{-2}$ |
| $x=0.6$ | $1.83\times10^{-5}$ | $0.8005\times10^{-5}$ | $1.8049414\times10^{-1}$ |

*the Euler method, but not better than the Runge-Kutta method and the Picard iteration method, in accordance with Table 6.*

## 4. Conclusion

In this paper, we applied Picard iteration and modified Krasnoselskii iteration methods, selecting different types of example and also compared the results using the Runge-Kutta method and the Euler method, with the exact solution. In the conclusion, the comparisons indicate that there is very good agrement between the numerical solution and the exact solution in terms of accuracy.

The result shows that the modified Krasnoselskii iteration method is very effective and convenient for solving different types of equations having initial conditions, with respect to other methods in the literature.

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