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Some considerations on higher order approximation of Duffing equation in the case of primary resonance

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Abstract. Here, the higher order approximation of forced Duffing equation is studied. First, using the renormalization group method, the modulation equations of Duffing equation in the case of primary resonance is determined. The resulting modulation equations are identical with those previously obtained by the method of multiple scales and generalized method of averaging. Second, the periodic steady state behavior of the solutions and the problem of spurious solutions in higher order approximation are considered. It is shown that depending on the truncation method of original phase and amplitude modulation equations, two types of frequency response equation may be obtained. One possesses spurious solutions for the case of softening nonlinearity, and the other for the case of hardening nonlinearity. Furthermore, it is shown that the truncation of the frequency response equation do not necessarily lead to more accurate results. Finally, by application of root classification of polynomials and Descartes' rule of signs, a criterion is presented to detect the existence of spurious solutions in any point of frequency response equation without solving it. This method is also applicable to other nonlinear systems.

1. Introduction

The perturbation methods are among the most powerful methods in applied mathematics and engineering [1,2], and have been applied in diverse problems (e.g. [3-9]). The problem of higher order approximation of nonlinear systems, using perturbation methods, has been studied previously by some researchers. Nayfeh in a series of papers [10-15] applied the Method of Multiple Scales (MSM) to obtain the higher order approximation of some nonlinear systems. He used the method of reconstitution [15] to determine the modulation equation. Only the solutions which were the continuation of first order solutions was considered, and the spurious solutions were discarded. Rahman and Burton [16,17] called this method the version I of MSM. They devised the version II of MSM in an attempt to obtain higher order approximations of nonlinear systems that are free from spurious solutions. Later, different types of nonlinear systems were studied by others, using this version of MSM [18-24]. Nayfeh [25] compared the MSM and Generalized Method of Averaging (GMA) to determine the higher order approximation of some nonlinear systems, and found that these methods produce the same modulation equations. Moreover, he showed that spurious solutions are not avoidable, in general, either in version I or in version II of the MMS. He concluded that in Duffing equation in the case of primary resonance, the spurious solutions do exist for the case of hardening nonlinearity.

In statistical mechanics and quantum field theory, Renormalization Group Method (RGM) extracts the features of system, which are insensitive to details [26]. So, this method was regarded as an asymptotic analysis [27]. The RGM, was first proposed by Chen et

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al. [27,28]. The RGM may be applied to a wide range of problems, which were treated before by MSM, GMA and WKB methods [29]. In contrast to these methods, RGM requires neither assumptions about the structure of perturbation series (e.g. time scales in MSM), nor the use of asymptotic matching [28]. Nevertheless, this method is less well-known in engineering community. Kunihiro [30,31] formulated the RGM, based on the classical theory of envelopes for both scalar and vector fields. Mudavanhu and O'malley [32] developed a simplified version of RGM to determine the higher order approximation. This version is valid on larger time intervals in comparison with GMA and MSM. Chiba [33] showed that the RGM could be used to determine the approximate center manifold and the approximate flow on it. Furthermore, he investigated higher order RG equations to refine the approximate solutions [34]. DeVille et al. [35] concluded that the RG method may be used to determine the normal forms of autonomous and non-autonomous perturbed differential equations. Hosseini [36] proposed a direct method based on the RGM for determining the analytical approximation of weakly nonlinear continuous systems. This method may be a suitable alternative method for multiple scales method in treating nonlinear continuous systems.

The Duffing equation has been extensively studied in the literature by different methods (e.g. [37,38]). In present paper, the higher order approximation of Duffing equation in the case of primary resonance is studied. The RGM is applied, and a modulation equation is found that is identical with those obtained by MSM and GMA in [25]. The analysis is important because although the RGM is very powerful and versatile, there is no use for this method in engineering problems, and it is relatively unknown for engineers. Furthermore, a few studies have been reported for the application of RGM to forced nonlinear oscillators [39]. Moreover, the present study shows that the results obtained by RGM are identical to the results of version I of MSM and not to the version II of MSM. On the other hand, one will observe that the application of RGM in higher order approximation is simpler than the other methods. To apply the higher order MSM, the method of reconstitution [15] is necessary, but in RGM, the higher order approximation is direct. A discussion on this case is presented at the end of Section 2. In Section 3, the periodic steady state behavior of these solutions is studied. Previously, Nayfeh [25] has concluded that in Duffing equation in the case of primary resonance, the spurious solutions exist for the case of softening nonlinearity. It is shown that this result is not always true. It is shown that depending on the truncation method of original phase and amplitude modulation equations, two types of frequency response equation may be obtained. One possesses spurious solutions for the case of softening nonlinearity and the other for the case of hardening nonlinearity. Furthermore, it is shown that in contrast to statement of [25], truncation of the frequency response equation does not necessarily lead to more accurate results. Finally, by application of root classification of polynomials and Descartes' rule of signs, a criterion is presented to detect the existence of spurious solutions in any point of frequency response equation without solving it. This procedure is also applicable to other nonlinear systems.

2. Application of the RGM to Duffing equation in the case of primary resonance

In this section, first the renormalization group method is reviewed, and then it is applied to the Duffing equation. Our presentation is brief; the related theoretical basis and rigorous proofs can be found in [33-35].

2.1. A brief review of the RGM

Here, the RGM is described and it is implemented to an initial value problem. The present description closely follows Chiba [34]. Consider the following differential equation:

$$\dot{\mathbf{x}} = \mathbf{H}\mathbf{x} + \varepsilon \mathbf{g_1}(t, \mathbf{x}) + \varepsilon^2 \mathbf{g}_2(t, \mathbf{x}), \tag{1}$$

where $\mathbf{H}$ is $n \times n$ diagonal matrix all of whose eigenvalues lie on the imaginary axis; $\mathbf{x}$ is the vector of dependent variables; t is independent variable, and ε is a small parameter. It is assumed that $\mathbf{g}_j(t, \mathbf{x}) j = 1, 2$ is periodic in t and polynomial in $\mathbf{x}$.

Substituting the naïve (straightforward) expansion:

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \varepsilon^2 \mathbf{x}_2, \tag{2}$$

into Eq. (1), the following is found:

$$\dot{\mathbf{x}}_0 + \varepsilon \dot{\mathbf{x}}_1 + \varepsilon^2 \dot{\mathbf{x}}_2 = \mathbf{H}(\mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \varepsilon^2 \mathbf{x}_2) + \sum_{j=1}^{2} \varepsilon^j \mathbf{g}_j(t, \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \varepsilon^2 \mathbf{x}_2). \tag{3}$$

Expanding Eq. (3) with respect to ε, and equating the like coefficients of ε^j, the following is obtained:

$$\dot{\mathbf{x}}_0 = \mathbf{H}(\mathbf{x}_0), \tag{4}$$

$$\dot{\mathbf{x}}_1 = \mathbf{H}\mathbf{x}_1 + G_1(t, \mathbf{x_0}), \tag{5}$$

$$\dot{\mathbf{x}}_2 = \mathbf{H}\mathbf{x}_2 + G_2(t, \mathbf{x_0}, \mathbf{x}_1), \tag{6}$$

where:

$$G_1(t, \mathbf{x_0}) = g_1(t, \mathbf{x_0}),$$

$$G_2(t, \mathbf{x_0}, \mathbf{x}_1) = \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}(t, \mathbf{x_0})\mathbf{x}_1 + g_2(t, \mathbf{x_0}). \tag{7}$$

The solution of Eq. (4) is:

$$\mathbf{x}_0 = \mathbf{X}(t)\mathbf{A}, \tag{8}$$

where $\mathbf{X}(t) = e^{\mathbf{H}t}$ is the fundamental matrix, and $\mathbf{A}$ is the vector of initial value. Substituting Eq. (8) in Eq. (5), and solving the result, the following is found:

$$\mathbf{x}_1 = \mathbf{X}(t)\mathbf{X}(\tau)\mathbf{h} + \mathbf{X}(t)\int_\tau^t \mathbf{X}(t)^{-1}\mathbf{G}_1(s, \mathbf{X}(s)\mathbf{A})ds, \tag{9}$$

where $\mathbf{h}$ is the vector of initial value at initial time τ. Now, substituting Eq. (9) into Eq. (6) and solving the result, one may obtain a solution for $\mathbf{x}_2$. It is noted that the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are the functions of τ and $\mathbf{A}$. Finally, applying the *RG condition*:

$$\left.\frac{d}{d\tau}\right|_{\tau=t} \{\mathbf{x}_0 + \varepsilon\mathbf{x}_1(t, \tau, \mathbf{A}(\tau)) + \varepsilon^2\mathbf{x}_2(t, \tau, \mathbf{A}(\tau))\} = 0, \tag{10}$$

a differential equation is obtained in term of $\mathbf{A}(t)$. This differential equation is called *RG equation* or modulation equation. As Kunihiro [30,31] stated, with the application of RG condition (Eq. (10)), an envelope for the family (Eq. (2)) parameterized by τ is constructed. The other interpretation is that [40] "the naïve perturbation is independent of the parameter τ, hence the approximate solution should not depend on τ". It is observed that RGM requires neither assumptions about the structure of perturbation series (e.g. time scales in MSM) nor the use of asymptotic matching.

In the next subsection, the present general approach is applied to Duffing equation to study the problem of spurious solutions in the higher order approximation of this equation.

2.2. Application of the RGM to the Duffing equation

Following [16,17,25], the primary resonance of Duffing equation is considered:

$$\Omega^2\frac{d^2u}{dt^2} + u + 2\varepsilon\mu\Omega\frac{du}{dt} + \varepsilon\alpha u^3 = \varepsilon F\cos(t), \tag{11}$$

where u is displacement, Ω excitation frequency, μ damping coefficient, F excitation amplitude, t time and ε positive ordering parameter. The parameter α is the coefficient of nonlinearity term. All variables and parameters are in dimensionless form. Using transformations $u = \zeta + \bar{\zeta}$, $\frac{du}{dt} = i(\zeta - \bar{\zeta})$, the above equation in complex variable form becomes:

$$\Omega^2\frac{d\zeta}{dt} = \frac{1}{2}i\Omega^2(\zeta - \bar{\zeta}) + \frac{1}{2}i(\zeta + \bar{\zeta}) - \varepsilon\mu\Omega(\zeta - \bar{\zeta}) + \frac{1}{2}i\varepsilon\alpha(\zeta + \bar{\zeta})^3 - \frac{1}{4}iF\varepsilon(e^{it} + e^{-it}), \tag{12}$$

where an overbar denotes a complex conjugate, and $i = \sqrt{-1}$.

Here, only primary resonance is considered. So [25]:

$$\Omega^2 = 1 + \sigma\varepsilon,$$

$$\Omega = 1 + \frac{1}{2}\sigma\varepsilon - \frac{1}{8}\sigma^2\varepsilon^2 + \cdots \tag{13}$$

where σ is a detuning parameter.

The RGM is applied to Eq. (12). Substituting the naive (straightforward) expansion:

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2, \tag{14}$$

into Eq. (12) and using Eq. (13), the following is obtained:

$O(\varepsilon^0)$:

$$\frac{d\zeta_0}{dt} - i\zeta_0 = 0, \tag{15}$$

$O(\varepsilon^1)$:

$$\frac{d\zeta_1}{dt} - i\zeta_1 = \mu(\bar{\zeta}_0 - \zeta_0) - \frac{1}{2}i\sigma(\bar{\zeta}_0 + \zeta_0) + \frac{1}{2}i\alpha(\zeta_0 + \bar{\zeta}_0)^3 - \frac{1}{4}iF(e^{it} + e^{-it}), \tag{16}$$

$O(\varepsilon^2)$:

$$\begin{aligned}\frac{d\zeta_2}{dt} - i\zeta_2 = {}& \mu(\bar{\zeta}_1 - \zeta_1) - \frac{1}{2}i\sigma(\bar{\zeta}_1 + \zeta_1) + \frac{1}{2}\mu\sigma(\zeta_0 - \bar{\zeta}_0) + \frac{1}{2}i\sigma^2(\zeta_0 + \bar{\zeta}_0) \\ & + \frac{3}{2}i\alpha(\zeta_1 + \bar{\zeta}_1)(\zeta_0 + \bar{\zeta}_0)^2 - \frac{1}{2}i\sigma\alpha(\bar{\zeta}_0 + \zeta_0)^3 \\ & + \frac{1}{4}i\sigma F(e^{it} + e^{-it}).\end{aligned} \tag{17}$$

The solution of Eq. (15) with initial time τ is:

$$\zeta_0 = Ae^{i(t-\tau)}, \tag{18}$$

where A is a complex constant.

Substituting Eq. (18) into Eq. (16) and solving the outcome, it is found:

$$\begin{aligned}\zeta_1 = {}& \frac{1}{4}i(4i\mu A + 6\alpha A^2\bar{A} - 2\sigma A - F)(t - \tau)e^{i(t-\tau)} \\ & + \frac{1}{8}(4i\mu\bar{A} + 2\sigma + F - 6\alpha A\bar{A}^2 - \alpha\bar{A}^3 + 2\alpha A^3)e^{i(t-\tau)} \\ & + \frac{1}{8}(4i\mu + 2\sigma\bar{A} + F - 6\alpha A\bar{A}^2)e^{-i(t-\tau)} \\ & - \frac{1}{8}\alpha\bar{A}^3e^{-3i(t-\tau)} + \frac{1}{4}\alpha A^3e^{3i(t-\tau)}.\end{aligned} \tag{19}$$

In the above equation, the homogenous parts of the solution are chosen so that $\zeta_1(\tau)=0$. The integration constant A is renormalized and the homogenous parts of solutions are absorbed into it. Consequently, a new integration constant $A=A(\tau)$ is generated. Removing the *non-secular* resonance terms at first order [35], the following is found:

$$\begin{aligned}\tilde{\zeta}_1 =&\frac{1}{4}i(4i\mu A+6\alpha A^2\bar{A}-2\sigma A-F)(t-\tau)e^{i(t-\tau)}\\&+\frac{1}{8}(4i\mu\bar{A}+2\sigma\bar{A}+F-6\alpha A\bar{A}^2)e^{-i(t-\tau)}\\&-\frac{1}{8}\alpha\bar{A}^3e^{-3i(t-\tau)}+\frac{1}{4}\alpha A^3e^{3i(t-\tau)}.\end{aligned}\tag{20}$$

Substituting Eqs. (18) and (20) into Eq. (17), solving the outcome with initial time τ and removing the non-secular resonance terms, the following is obtained:

$$\begin{aligned}\tilde{\zeta}_2 =&\frac{1}{16}(6i\sigma^2A-8i\mu^2A+8\mu\sigma A-51i\alpha^2A^3\bar{A}^2\\&+3iF\sigma+2F\mu+6iF\alpha A\bar{A}\\&+3iF\alpha A^2)(t-\tau)e^{i(t-\tau)}+\frac{1}{16}i(-12i\alpha\sigma A^2\bar{A}\\&-48\mu\alpha A^2\bar{A}+2i\sigma^2A-8i\mu^2A+8\mu\sigma A\\&+18i\alpha^2A^3\bar{A}^2+F\sigma i+2F\mu-6iF\alpha A\bar{A}\\&+3iF\alpha A^2)(t-\tau)^2e^{i(t-\tau)}+\frac{1}{32}(-12F\alpha A\bar{A}\\&-2F\sigma-12\alpha\sigma A\bar{A}^2-48i\mu\alpha A\bar{A}^2-4\sigma^2\bar{A}\\&+69\alpha^2A^2\bar{A}^3)e^{-i(t-\tau)}-\frac{1}{16}(-2i\sigma^2\bar{A}+8\mu\sigma\bar{A}\\&+8i\mu^2\bar{A}+12i\alpha\sigma A\bar{A}^2-48\mu\alpha A\bar{A}^2\\&-18i\alpha^2A^2\bar{A}^3-F\sigma i+2F\mu-3iF\alpha\bar{A}^2\\&+6iF\alpha A\bar{A})(t-\tau)e^{-i(t-\tau)}+\frac{1}{32}(-18i\mu\alpha A^3\\&+3\alpha\sigma A^3-30\alpha^2A\bar{A}^4+6\alpha FA^2)e^{3i(t-\tau)}\\&-\frac{3}{16}i(2\sigma A-4i\mu A-6\alpha A^2\bar{A}\\&+F)(t-\tau)e^{3i(t-\tau)}-\frac{1}{128}(24i\mu\alpha\bar{A}^3\\&-42\alpha^2A\bar{A}^4+9\alpha F\bar{A}^2)e^{-3i(t-\tau)}-\frac{3}{32}i(2\sigma\bar{A}\\&+4i\mu\bar{A}-6\alpha\bar{A}^2A+F)(t-\tau)e^{-3i(t-\tau)}\\&+\frac{3}{64}\alpha^2A^5e^{5i(t-\tau)}-\frac{1}{32}\alpha^2\bar{A}^5e^{-5i(t-\tau)}.\end{aligned}\tag{21}$$

With the application of the RG condition:

$$\left.\frac{d(\zeta_0+\varepsilon\tilde{\zeta}_1+\varepsilon^2\tilde{\zeta}_2)}{d\tau}\right|_{\tau=t}=0,\tag{22}$$

and bearing in mind that $A=A(\tau)$, the following is found:

$$\begin{aligned}\frac{dA}{dt} =&\frac{1}{4}(6i\alpha A^2\bar{A}-iF-2i\sigma A-4\mu A)\varepsilon\\&+\frac{1}{16}(8\mu\sigma A+6i\sigma^2A+2\mu F-8i\mu^2A\\&+3i\sigma F+6i\alpha FA\bar{A}-51i\alpha^2A^3\bar{A}^2\\&+3iF\alpha A^2)\varepsilon^2+O(\varepsilon^3).\end{aligned}\tag{23}$$

Neglecting the higher order terms $O(\varepsilon^3)$, the RG equation [34] is found as:

$$\begin{aligned}\frac{dA}{dt} =&\frac{1}{4}(6i\alpha A^2\bar{A}-iF-2i\sigma A-4\mu A)\varepsilon\\&+\frac{1}{16}(8\mu\sigma A+6i\sigma^2A+2\mu F-8i\mu^2A\\&+3i\sigma F+6i\alpha FA\bar{A}-51i\alpha^2A^3\bar{A}^2\\&+3iF\alpha A^2)\varepsilon^2.\end{aligned}\tag{24}$$

Substituting the polar form:

$$A=\frac{1}{2}ae^{\theta i},\tag{25}$$

into Eq. (24) and separating real and imaginary parts, the phase and amplitude modulation equations are obtained as:

$$\begin{aligned}\frac{da}{dt}=&-\left(\mu a+\frac{1}{2}F\sin\theta\right)\varepsilon+\left(\frac{1}{2}\mu\sigma a+\frac{1}{4}\mu F\cos\theta\right.\\&\left.+\frac{3}{8}\sigma F\sin\theta+\frac{3}{32}\alpha Fa^2\sin\theta\right)\varepsilon^2,\end{aligned}\tag{26}$$

$$\begin{aligned}\frac{d\theta}{dt}=&\left(\frac{3}{8}\alpha a^2-\frac{F\cos\theta}{2a}-\frac{\sigma}{2}\right)\varepsilon+\left(\frac{3\sigma F\cos\theta}{8a}\right.\\&-\frac{\mu F\sin\theta}{4a}+\frac{9}{32}F\alpha a\cos\theta-\frac{51}{256}\alpha^2a^4\\&\left.-\frac{\mu^2}{2}+\frac{3\sigma^2}{8}\right)\varepsilon^2.\end{aligned}\tag{27}$$

Eqs. (26) and (27) are identical with Eq. (26) of [25] obtained by MSM (version I) and GMA. So, the three methods RGM, MSM, and GMA give the same higher order approximation for primary resonance of Duffing Eq. (11). A comprehensive discussion of Eqs. (26) and (27) was presented in [18]. In addition, a complete discussion in versions I and II of MSM can be found in [17].

Let us compare the application of two methods MSM and RGM in higher order approximation of Duffing equation. It seems that the application of RGM in higher order approximation is simpler than the MSM. In the application of MSM, only particular solution of higher order problems is often included [1,2]. But to obtain Eq. (24), Nayfeh [25] included the homogeneous solution of the higher order problem. Furthermore, in MSM, two or more complex-valued partial differential equations are obtained for modulation of amplitude and phase. Instead of solving theses equations directly, Nayfeh [15] combined these partial differential equations (i.e. the method of reconstitution) to obtain an ordinary differential equation. On the other hand, some authors [20-23] used the original complex-valued partial differential equations of modulation to treat the problem. But in RGM, the modulation equation (i.e. Eq. (24)) is determined directly without any extra computations and assumptions. Furthermore, the RGM requires no assumption about the structure of perturbation series, i.e. time scales. Consequently, in comparison to MSM, higher order approximation in RGM is more natural and may be efficiently used for other problems in engineering.

3. Steady state periodic solution

Periodic solution of Eq. (11) corresponds to the equilibrium solution of Eqs. (26) and (27). So, the following algebraic equations are found for the periodic solution of Eq. (11):

$$-\mu a-\frac{1}{2}F\sin\theta+\left(\frac{1}{2}\mu\sigma a+\frac{1}{4}\mu F\cos\theta+\frac{3}{8}\sigma F\sin\theta+\frac{3}{32}\alpha F a^2\sin\theta\right)\varepsilon=0, \tag{28}$$

$$\frac{3}{8}\alpha a^2-\frac{F\cos\theta}{2a}-\frac{\sigma}{2}+\left(\frac{3\sigma F\cos\theta}{8a}-\frac{\mu F\sin\theta}{4a}+\frac{9}{32}F\alpha a\cos\theta-\frac{51}{256}\alpha^2a^4-\frac{\mu^2}{2}+\frac{3\sigma^2}{8}\right)\varepsilon=0. \tag{29}$$

One may solve Eqs. (28) and (29) simultaneously for a and θ, using numerical methods [10-14]. Other approach is eliminating phase θ and obtaining an equation in amplitude a, i.e. frequency response equation. In the later approach, first Eqs. (28) and (29) are solved for $\cos(\theta)$ and $\sin(\theta)$ as:

$$\sin\theta=\frac{f_1}{g},\qquad \cos\theta=\frac{f_2}{g}, \tag{30}$$

where:

$$\begin{aligned}
f_1=&-\mu a[(96\sigma^2+51\alpha^2a^4+128\mu^2+144\sigma\alpha a^2)\varepsilon^2\\
&-(512\sigma+384\alpha a^2)\varepsilon+512],\\
f_2=&\frac{1}{8}[(612\alpha^2\sigma a^5-288\alpha\sigma^2a^3+384\alpha\mu^2a^3\\
&+512\sigma\mu^2a+153\alpha^3a^7-1152\sigma^3a)\varepsilon^2\\
&+(-768\alpha\sigma a^3-1104\alpha^2a^5+3072\sigma^2a)\varepsilon\\
&+(-2048\sigma a+1536\alpha a^3)],\\
g=&F[(144\sigma^2+27\alpha^2a^4+64\mu^2+144\sigma\alpha a^2)\varepsilon^2\\
&-(384\sigma+192\alpha a^2)\varepsilon+256].
\end{aligned} \tag{31}$$

Using identity $\sin^2(\theta)+\cos^2(\theta)-1=0$ and Eqs. (30) and (31), the following is obtained:

$$f_1^2+f_2^2-g^2=0. \tag{32}$$

Substituting Eq. (31) into Eq. (32), the frequency response equation is found as:

$$\begin{aligned}
&\left(\frac{23409\alpha^6a^{14}}{4194304F^2}+P_{12}\right)\varepsilon^4+\left(\frac{10557\alpha^5a^{12}}{131027F^2}+P_{10}\right)\varepsilon^3\\
&+\left(\frac{6597\alpha^4a^{10}}{16384F^2}+P_8\right)\varepsilon^2+\left[-\frac{207\alpha^3a^8}{256F^2}\right.\\
&+\frac{33\sigma\alpha^2a^6}{64F^2}-\frac{6\alpha\mu^2a^4}{F^2}+\frac{3\alpha\sigma^2a^4}{F^2}-\frac{8\mu^2\sigma a^2}{F^2}\\
&\left.-\frac{3\sigma^3a^2}{F^2}+\frac{3\alpha a^2}{2}+3\sigma\right]\varepsilon+\frac{9\alpha^2a^6}{16F^2}-\frac{3\alpha\sigma a^4}{2F^2}\\
&+\frac{4\mu^2a^2}{F^2}+\frac{\sigma^2a^2}{F^2}-1,
\end{aligned} \tag{33}$$

where P_i is polynomial of degree i in a that for the sake of brevity, it is not presented. It is stated in [25] that the terms $O(\varepsilon^2)$ and higher in Eq. (33) are incomplete and must be discarded, i.e.:

$$\begin{aligned}
&\left(-\frac{207\alpha^3a^8}{256F^2}+\frac{33\sigma\alpha^2a^6}{64F^2}-\frac{6\alpha\mu^2a^4}{F^2}+\frac{3\alpha\sigma^2a^4}{F^2}\right.\\
&\left.-\frac{8\mu^2\sigma a^2}{F^2}-\frac{3\sigma^3a^2}{F^2}+\frac{3\alpha a^2}{2}+3\sigma\right)\varepsilon+\frac{9\alpha^2a^6}{16F^2}\\
&-\frac{3\alpha\sigma a^4}{2F^2}+\frac{4\mu^2a^2}{F^2}+\frac{\sigma^2a^2}{F^2}-1.
\end{aligned} \tag{34}$$

This truncated frequency response equation is not equivalent to Eqs. (28) and (29). Consequently, if one can solve Eqs. (28) and (29) in a and β [10-14], he/she can also solve Eq. (33) without truncating it. One may think that the number of spurious solutions in truncated Eq. (34) is necessarily less than the original Eq. (33). This is generally not the case. For example, in the case of hardening nonlinearity, $\alpha > 0$, even for very small value of ε, Eq. (34) has spurious solution for any σ, but Eq. (33) may not. To investigate this case more thoroughly, the amplitude a versus detuning parameter σ are presented in Figures 1 and 2. These figures are plotted for hardening case with the following data:

$$\varepsilon = 0.005, \qquad \alpha = 1, \qquad F = 1, \qquad \mu = 0.1.$$

In both figures, the first order approximation (the case $\varepsilon = 0$ in Eq. (34)) is plotted. Second order approximation corresponding to Eqs. (33) and (34) are shown in Figures 1 and 2, respectively. It is observed that Eq. (34) for these data does not possess the spurious solution, but Eq. (33), even for the small value, $\varepsilon = 0.005$, has spurious solution. For softening case ($\varepsilon = 0.005$, $\alpha = -1$, $F = 1$, $\mu = 0.1$.), Figures 3 and 4 are plotted. Again, it is observed that Eq. (34), for these data, does not possess the spurious solution, but Eq. (33) has spurious solution. Therefore, the statement of [25], that to obtain a consistent expansion (i.e. a solution without spurious solution), the terms $O(\varepsilon^2)$ and higher in frequency response equation are incomplete and must be discarded, is not always true.

Frequency response Eq. (34), when $\varepsilon \to 0$, has spurious solution $\frac{1}{\sqrt{23\alpha\varepsilon}}$ for the case $\alpha > 0$, i.e. in the case of hardening nonlinearity. Also, Eq. (33) does possess spurious solution $\frac{4\sqrt{2346}}{51\sqrt{\alpha\varepsilon}}$ in the case of hardening nonlinearity, when $\varepsilon \to 0$.

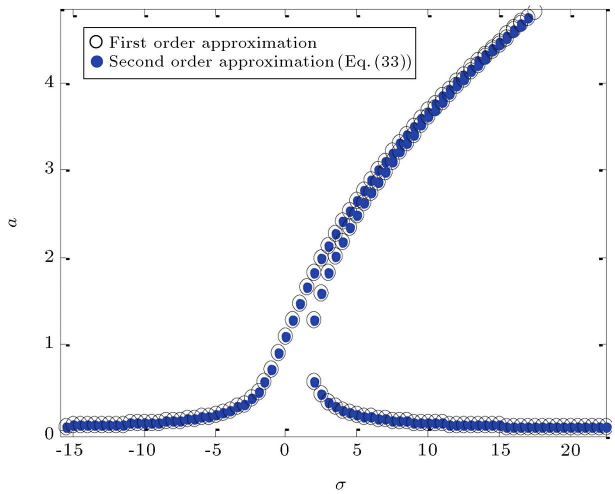

Figure 1. Amplitude versus detuning parameter for hardening case $\alpha = 1$.

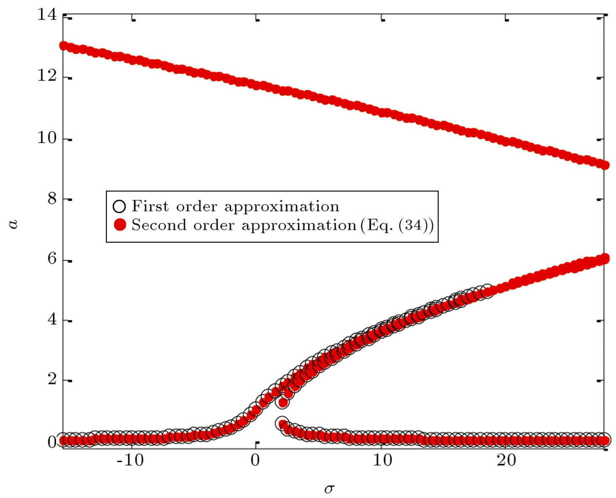

Figure 2. Amplitude versus detuning parameter for hardening case $\alpha = 1$.

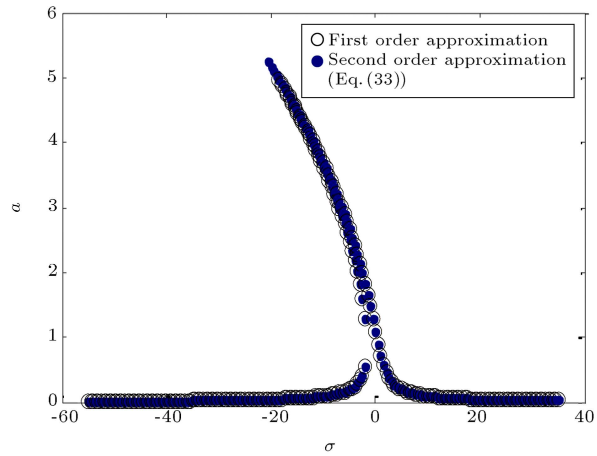

Figure 3. Amplitude versus detuning parameter for softening case $\alpha = -1$.

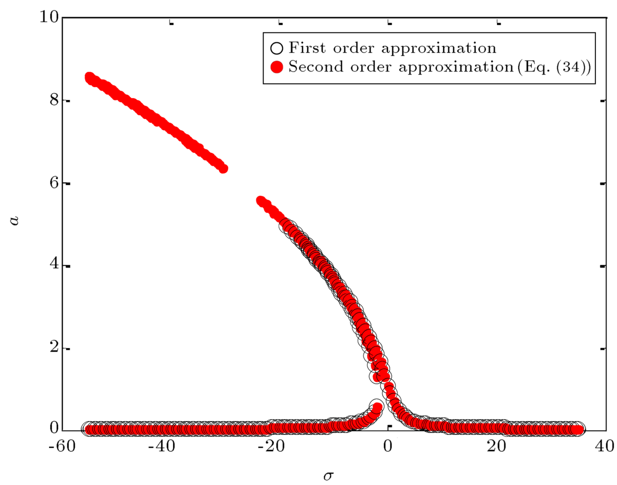

Figure 4. Amplitude versus detuning parameter for softening case $\alpha = -1$.

There is another method to compute the truncated frequency response equation. In this approach, first, $\cos(\theta)$ and $\sin(\theta)$ defined in Eqs. (30) and (31) are truncated, i.e. $\frac{f_1}{g}$ and $\frac{f_2}{g}$ in Eq. (31) are expanded in Taylor series with respect to ε up to the first order as:

$$\sin(\theta) = -\frac{\mu a(2+\varepsilon\sigma)}{F}, \tag{35}$$

$$\cos(\theta) = \frac{a(3\varepsilon\alpha^2 a^4 + 96\alpha a^2 - 128\sigma)}{128F}. \tag{36}$$

Using identity:

$$\sin^2(\theta) + \cos^2(\theta) - 1 = 0$$

and Eqs. (35) and (36), and neglecting higher order terms, the following is obtained:

$$\left(\frac{9\alpha^3 a^8}{256F^2} - \frac{3\sigma\alpha^2 a^6}{64F^2} + \frac{4\mu^2\sigma a^2}{F^2}\right)\varepsilon + \frac{9\alpha^2 a^6}{16F^2} - \frac{3\alpha\sigma a^4}{2F^2} + \frac{4\mu^2 a^2}{F^2} + \frac{\sigma^2 a^2}{F^2} - 1. \tag{37}$$

Frequency response Eq. (37) is identical to Eq. (59) of [25], if higher order terms are neglected there. Although Eq. (37) was previously obtained by Nayfeh [25], the initial amplitude and phase equations that have been used in Nayfeh [25] are not the same as Eqs. (28) and (29).

Furthermore, his method to obtain Eq. (37) is not similar to the procedure used in Eqs. (36) and (37). Rather, it is like the procedure used in Eqs. (30)-(34). But surprisingly, the final solution is identical. Eq. (37) possesses spurious solution $\frac{16}{3\sqrt{-\alpha\varepsilon}}$ when $\varepsilon \to 0$ for $\alpha < 0$, i.e. in the case of softening nonlinearity. This result was previously obtained by Nayfeh [25]. In summary, frequency response Eq. (33) and truncated frequency response Eq. (34) possess a spurious solution in the case of hardening nonlinearity, but truncated frequency response Eq. (37) possesses a spurious solution in the case of softening nonlinearity. As an example, Figure 5 shows the amplitude a versus detuning parameter σ for the case of softening nonlinearity ($\varepsilon = 0.005$, $\alpha = -1$, $F = 1$, $\mu = 0.1$.). Second order approximation is obtained by Eq. (37). It is observed in Figure 5 that in the case of softening nonlinearity, Eq. (37) has spurious solutions. Previously, Nayfeh [25] concluded that in Duffing equation, in the case of primary resonance, the spurious solutions do not exist for the case of hardening nonlinearity. The present analysis showed that this is not always true. A comparison between Figures 2 and 5 shows that depending on the truncation method of original phase and amplitude modulation equations, two types of frequency response equation may be obtained. One possesses spurious solutions for the case of softening nonlinearity (Eq. (37) and Figure 5) and the other (Eq. (34) and Figure 2) for the case of hardening nonlinearity.

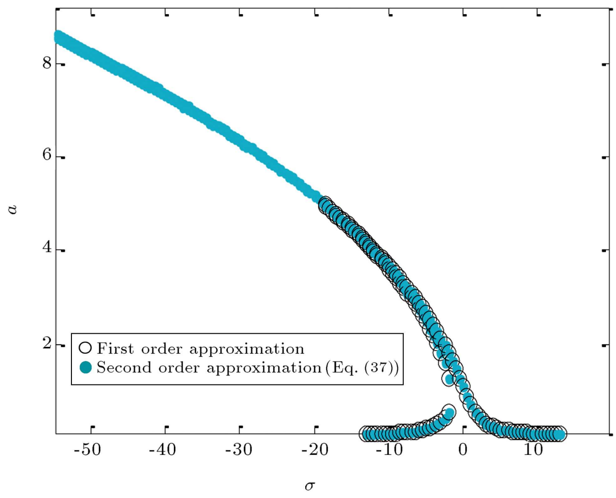

Figure 5. Amplitude versus detuning parameter for softening case $\alpha = -1$.

It is interesting to note that in numerical simulation of Duffing equation, in the case of primary resonance, and in the first order approximation of perturbation method (the case $\varepsilon = 0$ in Eq. (34)), the amplitude of steady state periodic solution is invariant under $\alpha \to -\alpha$, $\sigma \to -\sigma$ transformations. In other words, backbone curves in the cases of hardening and softening nonlinearity are mirror images of each other. But higher order approximations, i.e. Eqs. (33), (34) or (37), do not show this invariant property. It means that in reduction of forced Duffing oscillator (Eq. (11)) by higher order approximation of the perturbation methods, the invariant property of original equation destroys.

4. Criteria for the existence of spurious solution in the frequency response equation

Eqs. (34) and (37) possess additional and spurious solution in some region. To detect the existence of spurious solution in a specified point of frequency response equation, one should numerically solve the equation. A criterion is given that without solving the frequency response equation, one will be able to detect the existence of spurious solution in any point of it. Frequency response Eqs. (34) or (37) are degree 4 in a^2 and when $\varepsilon = 0$, they are degree 3 in a^2. So, a combination of root classification of cubic and quartic polynomials and Descartes' rule of signs is used to obtain the criteria (see Appendix). Additional and spurious solutions exist when the number of real roots of higher order frequency response equation, i.e. Eqs. (34) or (37), is greater than the number of real roots of the first order frequency response equation, i.e., Eqs. (34) or (37) for $\varepsilon = 0$.

Parameters Δ, D_2, D_3, D_4 and E are defined as:

$$\Delta = \frac{243}{4}\alpha^3\sigma F^2\mu^2 + \frac{27}{16}\alpha^3\sigma^3F^2 - \frac{2187}{256}\alpha^4F^4 - 72\alpha^2\sigma^2\mu^4 - 9\alpha^2\sigma^4\mu^2 - 144\alpha^2\mu^6,$$

$$D_2 = 3a_1^2 - 8a_0a_2,$$

$$D_3 = 16a_0^2a_4a_2 - 18a_0^2a_3^2 - 4a_0a_2^3 + 14a_0a_1a_2a_3 - 6a_0a_1^2a_4 + a_2^2a_1^2 - 3a_3a_1^3,$$

$$D_4 = 256a_0^3a_4^3 - 27a_0^2a_3^4 - 192a_0^2a_1a_3a_4^2 - 27a_1^4a_4^2 - 6a_0a_1^2a_3^2a_4 + a_1^2a_2^2a_3^2 - 4a_0a_2^3a_3^2 + 18a_1^3a_2a_3a_4 + 144a_0a_1^2a_2a_4^2 - 80a_0a_1a_2^2a_3a_4 + 18a_0a_1a_2a_3^3 - 4a_1^2a_2^3a_4 - 4a_1^3a_3^3 + 16a_0a_2^4a_4 - 128a_0^2a_2^2a_4^2 + 144a_0^2a_2a_4a_3^2,$$

$$E = 8a_0^2a_3 + a_1^3 - 4a_0a_1a_2, \tag{38}$$

where for the case of Eq. (34):

$$a_0 = -\frac{207\varepsilon\alpha^3}{256}, \qquad a_1 = \frac{33\varepsilon\sigma\alpha^2}{64} + \frac{9\alpha^2}{16},$$

$$a_2 = -6\varepsilon\alpha\mu^2 + 3\varepsilon\alpha\sigma^2 - \frac{3\alpha\sigma}{2}$$

$$a_3 = -8\mu^2\varepsilon\sigma - 3\varepsilon\sigma^3 + \frac{3F^2\alpha\varepsilon}{2} + 4\mu^2 + \sigma^2,$$

$$a_4 = -F^2 + 3F^2\varepsilon\sigma, \tag{39}$$

and for Eq. (37):

$$a_0 = \frac{9\varepsilon\alpha^3}{256}, \qquad a_1 = -\frac{3\varepsilon\sigma\alpha^2}{64} + \frac{9\alpha^2}{16},$$

$$a_2 = -\frac{3\alpha\sigma}{2}, \qquad a_3 = 4\mu^2\varepsilon\sigma + 4\mu^2 + \sigma^2,$$

$$a_4 = -F^2. \tag{40}$$

Now, Eqs. (34) or (37) possess spurious solution if one of the following conditions is satisfied:

$$\begin{aligned}
&(1) \quad \Delta>0, \{D_4>0\wedge D_3>0\wedge D_2>0\}, \qquad \Lambda = 4,\\
&(2) \quad \Delta<0, \{D_4>0\wedge D_3>0\wedge D_2>0\}, \qquad \Lambda>1,\\
&(3) \quad \Delta = 0, \{D_4 = 0 \wedge D_3 > 0\}, \qquad \Lambda > 1,\\
&(4) \quad \Delta < 0, \{D_4 < 0\}, \qquad \Lambda > 1,\\
&(5) \quad \Delta > 0, \{D_4 = 0 \wedge D_3 < 0\},\\
&(6) \quad \Delta > 0, \{D_4 = 0 \wedge D_3 = 0 \wedge D_2 > 0 \wedge E = 0\},\\
&(7) \quad \{D_4=0\wedge D_3=0\wedge D_2>0\wedge E\neq 0\}, \qquad \Lambda=4,
\end{aligned} \tag{41}$$

where Λ is the number of sign changes in the $a_i (i = 0 - 4)$, defined in Eq. (39) or (40). Bifurcation occurs when $\Delta = 0$. For example, with the following data:

$$\sigma = 2, \quad \alpha = 1, \quad \mu = 0.1, \quad \varepsilon = 0.1, \quad F = 1, \tag{42}$$

for Eq. (34), it is found that:

$$\Delta = 4.7, \quad D_2 = 0.16, \quad D_3 = 0.0037, \quad D_4 = 0.0001, \quad E = -0.001, \quad \Lambda = 4. \tag{43}$$

So, it is case 1 in Eq. (41) and consequently Eq. (34) possesses spurious solution. This procedure is also applicable to other nonlinear systems.

5. Conclusion

The higher order approximation of forced Duffing equation was studied. First, the modulation equations are determined using RGM, which were identical with those obtained previously by MSM and GMA. It seems that the application of RGM in higher order approximation is simpler than the other methods. Second, the periodic steady state behavior of the solutions and the problem of spurious solutions in higher order approximation were considered. It was shown that depending on the truncation method of the original phase and amplitude modulation equations, two types of frequency response equation may be obtained. One possesses spurious solutions for the case of softening nonlinearity, and the other for the case of hardening nonlinearity. Furthermore, it was shown that truncating the frequency response equation did not necessarily lead to more accurate results. Finally, by application of root classification of polynomials and Descartes' rule of signs, a criterion was presented to detect the existence of spurious solution in any point of frequency response equation without solving it. This procedure is also applicable to other nonlinear systems.

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42. http://en.wikipedia.org/wiki/Cubic_function

43. http://en.wikipedia.org/wiki/Descartes

Appendix A

A.1. Root classification of quartic polynomial

For quartic polynomial:

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4(a_0 \neq 0), \tag{A.1}$$

the numbers of real and imaginary roots and multiplicities of repeated roots in all cases are summarized as [41]:

$$\begin{aligned}
&(1) \quad D_4 > 0 \wedge D_3 > 0 \wedge D_2 > 0 \qquad \{1,1,1,1\},\\
&(2) \quad D_4 > 0 \wedge (D_3 \leq 0 \vee D_2 \leq 0) \qquad \{\},\\
&(3) \quad D_4 < 0 \qquad \{1,1\},\\
&(4) \quad D_4 = 0 \wedge D_3 > 0 \qquad \{2,1,1\},\\
&(5) \quad D_4 = 0 \wedge D_3 < 0 \qquad \{2\},\\
&(6) \quad D_4 = 0 \wedge D_3 = 0 \wedge D_2 > 0 \wedge E = 0 \qquad \{2,2\},\\
&(7) \quad D_4 = 0 \wedge D_3 = 0 \wedge D_2 > 0 \wedge E \neq 0 \qquad \{3,1\},\\
&(8) \quad D_4 = 0 \wedge D_3 = 0 \wedge D_2 < 0 \qquad \{\},\\
&(9) \quad D_4 = 0 \wedge D_3 = 0 \wedge D_2 = 0 \qquad \{4\}.
\end{aligned} \tag{A.2}$$

D_2, D_3, D_4 and E were defined earlier in Eq. (38).

The numbers in the brace in Eq. (A.2) describe the situations of the roots. For example, $\{1,1,1,1\}$ means four real simple roots and $\{2,1,1\}$ means one real double root plus two real simple roots.

A.2. Root classification of cubic polynomial

For cubic equation:

$$a_0x^3 + a_1x^2 + a_2x + a_3 \quad (a_0 \neq 0), \tag{A.3}$$

three following cases exist [42]:

$$\begin{aligned}
&(1) \quad \Delta > 0 \qquad \{1,1,1\},\\
&(2) \quad \Delta = 0 \qquad \{3\},\\
&(3) \quad \Delta < 0 \qquad \{1\},
\end{aligned} \tag{A.4}$$

where:

$$\Delta = 18a_0a_1a_2a_3 - 4a_1^3a_3 + a_1^2a_2^2 - 4a_0a_2^3 - 27a_0^2a_3^2. \tag{A.5}$$

A.3. Descartes' rule of signs

Descartes' rule of signs is a technique for determining the number of positive or negative real roots of a polynomial [43]. It states that the number of positive real roots of a polynomial with real coefficients is bounded by the number of changes of sign in its coefficients.

For example, the polynomial x^3+x^2-x-1 has one sign change and therefore it has exactly one positive root.

Biography

Seyyed Ali Asghar Hosseini was born in Qom, Iran, in 1978. He received his BS degree from Iran University of Science and Technology, and MS and PhD degrees from Tarbiat Modares University, Tehran, Iran. He is currently Assistant Professor in the Mechanical Engineering Department of Kharazmi University, Tehran, Iran. His research interests include: nonlinear vibrations, rotor dynamics and random vibrations.