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Gauss Integration Limits in Nearly Singular Integrals of BEM for Geometrically Linear Elements

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Abstract. *The most suitable and widely used numerical integration method for boundary integrals in the BEM method is Gauss-Legendre integration. But, this integration method is not appropriate for singular and nearly singular integrations in BEM. In this study, some criteria are introduced for recognizing nearly singular integrals in the integral form of the Laplace equation. At the first stage, a criterion is obtained for the constant element and, at the later stages, higher order elements are investigated. In the present research, the Romberg integration method is used for nearly singular integrals. The results of this numerical method have good agreement with analytical integration. The singular integrals are solved by composing the Romberg method and midpoint rule. Constant, linear and other interpolation functions of potentials over an element are a category of BEM elements. In those elements, the Gauss-Legendre integration will be accurate if the source point is placed out of the circle with a diameter equal to element length, and its center matched to the midpoint of the element.*

Keywords: *Boundary element method; Gauss-Legendre integration; Laplace equation; Nearly singular integrals; Romberg integration.*

INTRODUCTION

As is well known, the Gaussian quadratures are the most conventional methods for BEM integration, because it is accurate and calculation time is very short compared to other numerical integration methods. However, in the case of singular or nearly singular integrals, the ordinary Gaussian quadrature is not accurate [1].

Much research has investigated singular integrals in BEM. A new method known as the Direct Gauss quadrature formula was introduced by Smith [2] for singular integrals. Ozgener [3] verified a newly developed quadrature formula for singular integrals, Sladek [4] explained and defined the singularity in BEM and Zisis and Ladopoulos [5] introduced an exact solution for singular integrals in BEM.

Many researchers have focused on nearly singular integrands of an integral form of the Laplace equation in the boundary element method. Ma and Kamiya [6] have introduced a general algorithm for accurate integration of nearly singular integration known as the boundary layer effect in BEM. Niu et al. [7,8] have focused on the analytical integration of nearly singular integrands for some types of element of BEM by introducing relative distance.

As mentioned, all the above research introduced different methods for solving the nearly singular problem in BEM. This study attempts to answer the following important question: In what relative position between source point and element would we have a nearly singular integral? In this research, the position of the source point relative to the element has been determined, such that the Gauss-Legendre quadrature would be accurate for BEM integrals. Also, the

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Romberg integration method is used for nearly singular integrals.

GAUSS-LEGENDRE INTEGRATION

The Gauss-Legendre integration is based on the following equation [9-12]:

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i), \tag{1}$$

in which f is an arbitrary function, w_i are weight factors and x_i are Gauss points.

The Taylor series around zero (Maclaurin series) can be used for deriving Gauss-Legendre integration parameters [10]:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{m-1}}{(m-1)!}f^{(m-1)}(0) + \frac{x^m}{m!}f^{(m)}(\xi). \tag{2}$$

The left hand side of Equation 1 and the Maclaurin series results in the following equation:

$$\int_{-1}^{1} f(x)dx = xf(0) + \frac{x^2}{2!}f'(0) + \frac{x^3}{3!}f''(0) + \cdots + \frac{x^m}{m!}f^{(m-1)}(0) + \frac{x^{m+1}}{(m+1)!}f^{(m)}(\xi)\ |_{-1}^{1}. \tag{3}$$

By combining Equations 1 and 2, the following is obtained:

$$\int_{-1}^{1} f(x)dx = \sum_{j=1}^{2k}\left(\frac{4(j/2-[j/2])}{j!}f^{(j-1)}(0)\right) + \frac{2}{(2k+1)!}f^{(2k)}(\xi). \tag{4}$$

The right hand side of Equation 1 and the Maclaurin series provides Equation 5:

$$\sum_{i=1}^{n} w_i f(x_i) = \sum_{i=1}^{n} w_i \left(f(0) + x_i f'(0) + \frac{x_i^2}{2!}f''(0) + \cdots + \frac{x_i^{2k-1}}{(2k-1)!}f^{(2k-1)}(0) + \frac{x_i^{2k}}{(2k)!}f^{(2k)}(\xi)\right). \tag{5}$$

The closed form of the above equation is as follows:

$$\sum_{i=1}^{n} w_i f(x_i) = \sum_{j=1}^{2k}\left(\frac{f^{(j-1)}(0)}{(j-1)!}\sum_{i=1}^{n}(w_i x_i^{j-1})\right) + \sum_{i=1}^{n} w_i \frac{x_i^{2k}}{(2k)!}f^{(2k)}(\xi). \tag{6}$$

Relations 4 and 6 are equal by considering Equation 1, therefore, it could be written as:

$$\sum_{i=1}^{n}(w_i x_i^{j-1}) = \frac{4(j/2-[j/2])}{j}, \quad \text{for} \quad j=1 \text{ to } 2n, \tag{7}$$

or:

$$\begin{cases} w_1 + w_2 + \cdots + w_n = 2 \\ w_1x_1 + w_2x_2 + \cdots + w_nx_n = 0 \\ w_1x_1^2 + w_2x_2^2 + \cdots + w_nx_n^2 = \frac{2}{3} \\ \vdots \\ w_1x_1^{2n-2} + w_2x_2^{2n-2} + \cdots + w_nx_n^{2n-2} = \frac{2}{2n-1} \\ w_1x_1^{2n-1} + w_2x_2^{2n-1} + \cdots + w_nx_n^{2n-1} = 0 \end{cases} \tag{8}$$

Weight factors and Gauss points of n points Gauss integration are obtained by solving the system of Equation 7 or 8. Also, these values could be found in many numerical analysis books or by applying Legendre's polynomials [9-12].

Considering Equations 4 and 6, the error could be calculated by using the following equation:

$$E = |I - I_n| = \left(\frac{2}{2n+1} - (w_1x_1^{2n} + w_2x_2^{2n} + \cdots + w_nx_n^{2n})\right)\frac{|f^{(2n)}(\xi)|}{(2n)!}, \quad -1 \le \xi \le 1. \tag{9}$$

After some lengthy mathematical operations, the following well-known equation can be achieved [11]:

$$E = \frac{2^{2n+1}.(n!)^4}{(2n+1)((2n)!)^3}f^{(2n)}(\xi), \tag{10}$$

in which n is the number of points of Gauss integration.

The following formula is another error formula that is also used [12]:

$$E = |I - I_n| = \frac{1}{2n+1}\left[f(1) + f(-1) - I_n - \sum_{i=1}^{n} w_i x_i f'(x_i)\right]. \tag{11}$$

INTEGRALS IN BOUNDARY ELEMENT METHOD

In BEM for the Laplace equation there are two integrals in the integral equation that should be solved for each node and element. The integrands of these integrals (following equations) are singular and nearly singular for some cases in which the Gauss integration does not provide an accurate result.

$$H=\int_\Gamma uv_n ds=\int_\Gamma u\frac{\cos\Phi}{2\pi r}ds$$
$$=\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\hat{n}.\vec{r}(\xi)}{r(\xi)^2}.N_j(\xi).J(\xi)d\xi\right)u^j, \quad (12)$$

$$G=\int_\Gamma vu_n ds=\int_\Gamma \frac{\ln r}{2\pi}\frac{\partial u}{\partial n}ds$$
$$=\left(\frac{1}{2\pi}\int_{-1}^{1}\ln r(\xi).N_j(\xi).J(\xi)d\xi\right)u_n^j. \quad (13)$$

In general, the limits of these integrals are -1 and 1 and they fit the definition of Gauss-Legendre integration. But, there is a problem. According to Equation 1, if the Maclaurin series diverges, the truncation error will increase. In other words, if n is sufficiently close to infinity, the error according to Equation 9 or 10 will tend to approach infinity. Therefore, the numerical method will be incorrect. Then, it is necessary for the truncation error to be close to zero. Hence, the Maclaurin series of integrands of Equations 12 and 13 and also their convergence conditions should be derived.

Nearly Singular Condition for Constant Element

In the boundary of the BEM domain, the coordinates, x and y, for each element, and u (variable of Laplace equation) can be expressed by the polynomial of degree, n (shape function of degree n and $n>0$) and m (shape function of degree m), respectively. If $n=1$ and $m=0$, then, the element is called constant. In this study, Gauss-Legendre integration has been discussed in the constant element and then extended to higher order elements. H and G for the constant element are as below:

$$H=\left(\frac{L}{2\pi}\int_{-1}^{1}\frac{\hat{n}.\vec{r}(\xi)}{r(\xi)^2}d\xi\right)u^j, \quad (14)$$

$$G=\left(\frac{L}{2\pi}\int_{-1}^{1}\ln r(\xi)d\xi\right)u_n^j. \quad (15)$$

The integrand of G and H only depend on the position of node i relative to the element. The value of the u polynomial on the element is equal to 1 for the constant element. Therefore, if node i and the element together have translational and rotational movements, the integrals do not differ. Hence, the integrals H and

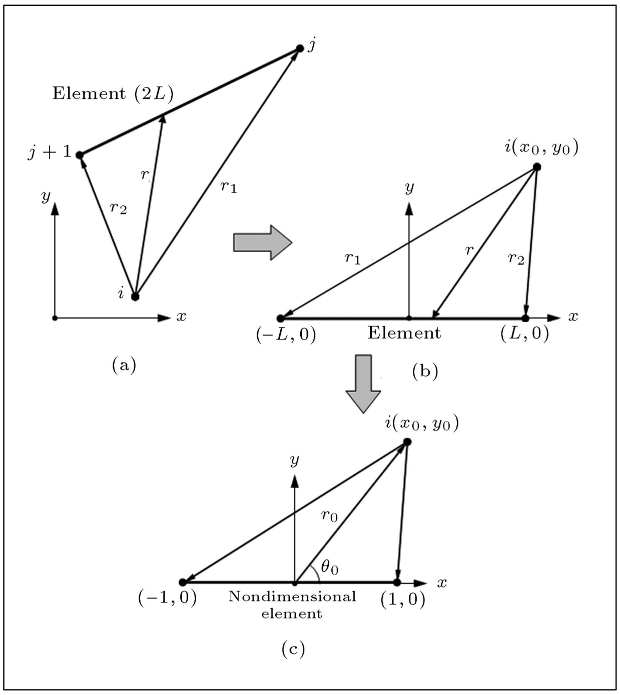


Figure 1. (a) An arbitrary linear element with node i. (b) Transportation of part (a) without change in lengths. (c) Non-dimensionalizing part (b) with respect to L.

G are equal for Figures 1a and 1b. Figure 1c is obtained for integrals H and G by non-dimensionalizing lengths, with respect to L, as follows:

$$x_0=\frac{X_0}{L},\qquad y_0=\frac{Y_0}{L},$$
$$(-L,0)\rightarrow(-1,0),$$
$$(L,0)\rightarrow(1,0). \quad (16)$$

This specification fits the definition of a local coordinate. By considering Figure 1c and using Equations 1, 14 and 15, the Maclaurin series are expressed as below:

$$\text{integrand}(H)=f(x)=\frac{\hat{n}.\vec{r}(x)}{r(x)^2}=\frac{y_0}{[(x-x_0)^2+y_0^2]}$$
$$\Rightarrow f(0)=\frac{y_0}{[x_0^2+y_0^2]},$$

$$f'(x)=\frac{-2y_0(x-x_0)}{[(x-x_0)^2+y_0^2]^2}\Rightarrow f'(0)=\frac{2y_0x_0}{[x_0^2+y_0^2]^2},$$

$$f''(x)=\frac{2y_0[3(x-x_0)^2-y_0^2]}{[(x-x_0)^2+y_0^2]^3}$$
$$\Rightarrow f''(0)=\frac{2y_0[3x_0^2-y_0^2]}{[x_0^2+y_0^2]^3},$$

$$f'''(x) = \frac{-24y_0(x-x_0)[(x-x_0)^2 - y_0^2]}{[(x-x_0)^2 + y_0^2]^4}$$

$$\Rightarrow f'''(0) = \frac{24y_0x_0[x_0^2 - y_0^2]}{[x_0^2 + y_0^2]^4},$$

$$f^{(4)}(x) = \frac{24y_0[5((x-x_0)^2 - y_0^2)^2 - 4y_0^2]}{[(x-x_0)^2 + y_0^2]^5}$$

$$\Rightarrow f^{(4)}(0) = \frac{24y_0[5(x_0^2 - y_0^2)^2 - 4y_0^2]}{[x_0^2 + y_0^2]^5},$$

$$\vdots \tag{17}$$

$$\text{integrand}(G) = g(x) = \ln r(x) = \frac{1}{2}\ln[(x-x_0)^2 + y_0^2]$$

$$\Rightarrow g(0) = \frac{1}{2}\ln(x_0^2 + y_0^2),$$

$$g'(x) = \frac{(x-x_0)}{[(x-x_0)^2 + y_0^2]} \Rightarrow g'(0) = \frac{-x_0}{(x_0^2 + y_0^2)},$$

$$g''(x) = \frac{-[(x-x_0)^2 - y_0^2]}{[(x-x_0)^2 + y_0^2]^2} \Rightarrow g''(0) = \frac{-(x_0^2 - y_0^2)}{[x_0^2 + y_0^2]^2},$$

$$g'''(x) = \frac{2(x-x_0)[(x-x_0)^2 - 3y_0^2]}{[(x-x_0)^2 + y_0^2]^3}$$

$$\Rightarrow g'''(0) = \frac{-2x_0[x_0^2 - 3y_0^2]}{[x_0^2 + y_0^2]^3},$$

$$g^{(4)}(x) = \frac{6[4x_0^2y_0^2 - ((x-x_0)^2 - y_0^2)^2]}{[(x-x_0)^2 + y_0^2]^4}$$

$$\Rightarrow g^{(4)}(x) = \frac{6[4x_0^2y_0^2 - (x_0^2 - y_0^2)^2]}{[x_0^2 + y_0^2]^4},$$

$$\vdots \tag{18}$$

Defining the following variables, better formulations are obtained for the terms of the Maclaurin series:

$$x_0 = r_0\cos\theta_0,$$

$$y_0 = r_0\sin\theta_0, \tag{19}$$

$$\frac{1}{0!}f(0) = \frac{\sin\theta_0}{r_0},$$

$$\frac{1}{1!}f'(0) = \frac{2\sin\theta_0.\cos\theta_0}{r_0^2} \Rightarrow \frac{1}{1!}f'(0) = \frac{\sin 2\theta_0}{r_0^2},$$

$$\frac{1}{2!}f''(0) = \frac{2\sin\theta_0[3\cos^2\theta_0 - \sin^2\theta_0]}{r_0^3}$$

$$\Rightarrow \frac{1}{2!}f''(0) = \frac{\sin 3\theta_0}{r_0^3},$$

$$\frac{1}{3!}f'''(0) = \frac{4\sin\theta_0.\cos\theta_0[\cos^2\theta_0 - \sin^2\theta_0]}{r_0^4}$$

$$\Rightarrow \frac{1}{3!}f'''(0) = \frac{\sin 4\theta_0}{r_0^4},$$

$$\frac{1}{4!}f^{(4)}(0) = \frac{\sin\theta_0[5\cos^2 2\theta_0 - 4\sin^4\theta_0]}{r_0^5}$$

$$\Rightarrow \frac{1}{4!}f^{(4)}(0) = \frac{\sin 5\theta_0}{r_0^5},$$

$$\vdots$$

$$\frac{1}{n!}f^{(n)}(0) = \frac{\sin(n+1)\theta_0}{r_0^{n+1}}, \tag{20}$$

$$\frac{1}{0!}g(0) = \ln(r_0),$$

$$\frac{1}{1!}g'(0) = \frac{-\cos\theta_0}{r_0} \Rightarrow \frac{1}{1!}g'(0) = -\frac{\cos\theta_0}{r_0},$$

$$\frac{1}{2!}g''(0) = \frac{-[\cos^2\theta_0 - \sin^2\theta_0]}{2r_0^2}$$

$$\Rightarrow \frac{1}{2!}g''(0) = -\frac{\cos 2\theta_0}{2r_0^2},$$

$$\frac{1}{3!}g'''(0) = \frac{-\cos\theta_0[\cos^2\theta_0 - 3\sin^2\theta_0]}{3r_0^3}$$

$$\Rightarrow \frac{1}{3!}g'''(0) = -\frac{\cos 3\theta_0}{3r_0^3},$$

$$\frac{1}{4!}g^{(4)}(0) = \frac{-[(\cos^2\theta_0 - \sin^2\theta_0)^2 - 4\sin^2\theta_0.\cos^2\theta_0]}{4r_0^4}$$

$$\Rightarrow \frac{1}{4!}g^{(4)}(0) = -\frac{\cos 4\theta_0}{4r_0^4}$$

$$\vdots$$

$$\frac{1}{n!}g^{(n)}(0) = -\frac{\cos n\theta_0}{nr_0^n}. \tag{21}$$

As mentioned in the previous section, the nth term of the Maclaurin series should approach zero when "n" is tending to approach infinity. Therefore, we arrive at the following condition:

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}\frac{1}{n!}f^{(n)}(0)=0\Rightarrow\lim\limits_{n\to\infty}\frac{\sin(n+1)\theta_0}{r_0^{n+1}}=0\\ -1\leq\sin(n+1)\theta_0\leq 1\end{array}\right\}\Rightarrow r_0>1. \tag{22}$$

This condition is the necessary condition for series convergence.

The procedure below completes the convergence of the series by using the following correct inequality.

$$\frac{|\sin(n+1)\theta_0|}{r_0^{n+1}}\leq\frac{1}{r_0^{n+1}}. \tag{23}$$

The comparison test theorem for the convergence of series shows that if $\sum_{n=0}^{\infty}\frac{1}{r_0^{n+1}}$ converges, then the absolute values of the Maclaurin series for the H integrand will also converge. The ratio test for series convergence is applied for convergence proofing using the condition of Equation 22 [13].

$$\lim_{n\to\infty}\frac{1}{r_0^{n+2}}/\frac{1}{r_0^{n+1}}=\frac{1}{r_0}<1. \tag{24}$$

Equation 24 and other mentioned convergence theorems show the convergence of the $\sum_{n=0}^{\infty}\frac{|\sin(n+1)\theta_0|}{r_0^{n+1}}$ series. Therefore, the nth term of the Maclaurin series of the H integrand ($n\to\infty$) is sufficiently close to zero and the Gauss-Legendre integration can be applied with good precision. The same procedure can be used for the G integrand and the same result is obtained.

Equation 10 shows the error of the n points Gauss integration that can be expressed as the following for the H integrand:

$$E=B(n)f^{(2n)}(\xi)=B(n)\left(f^{(2n)}(0)+\xi f^{(2n+1)}(0)\right.$$
$$\left.+\frac{\xi^2}{2!}f^{(2n+2)}(0)+\cdots+\frac{\xi^k}{k!}f^{(2n+k)}(0)+\cdots\right). \tag{25}$$

The error can be written in the form of the following equation by using Equation 20:

$$E=B(n)\frac{2n!}{r_0^{2n+1}}\left(\sin(2n+1)\theta_0\right.$$
$$+\frac{(2n+1)\xi\sin(2n+2)\theta_0}{r_0}+\cdots$$
$$\left.+\frac{(2n+k)!\xi^k\sin(2n+k+1)\theta_0}{2n!k!r_0^k}+\cdots\right). \tag{26}$$

The following inequality can be written for using the comparison test theorem for convergence of the series:

$$E<B(n)\frac{2n!}{r_0^{2n+1}}\left(1\right.$$
$$\left.+\frac{(2n+1)\xi}{r_0}+\cdots+\frac{(2n+k)!\xi^k}{2n!k!r_0^k}+\cdots\right). \tag{27}$$

Now, the comparison test theorem for convergence of the series is applied as follows [13]:

$$\left.\begin{array}{l}\lim\limits_{k\to\infty}\frac{(2n+k+1)(2n+k)\cdots(k+2)\xi}{(2n+k)(2n+k-1)\cdots(k+1)r_0}=\frac{\xi}{r_0}\\ -1<\xi<1;\qquad r_0>1\end{array}\right\}\Rightarrow\frac{\xi}{r_0}<1. \tag{28}$$

It could be shown that the truncation error is a limited value and becomes close to zero by increasing n.

The same procedure can be used for the G integrand and the same results are obtained. The final result of these calculations is:

$$r_0>1\Rightarrow x_0^2+y_0^2>1\Rightarrow\left[\left(X_0-\frac{X_j+X_{j+1}}{2}\right)/L\right]^2$$
$$+\left[\left(Y_0-\frac{Y_j+Y_{j+1}}{2}\right)/L\right]^2>1$$
$$\Rightarrow(X_j-2X_0+X_{j+1})^2+(Y_j-2Y_0+Y_{j+1})^2>4L^2. \tag{29}$$

This means that if the ith node is located out of the represented circle in Figure 2, then, the Gauss integration can be applied with acceptable precision.

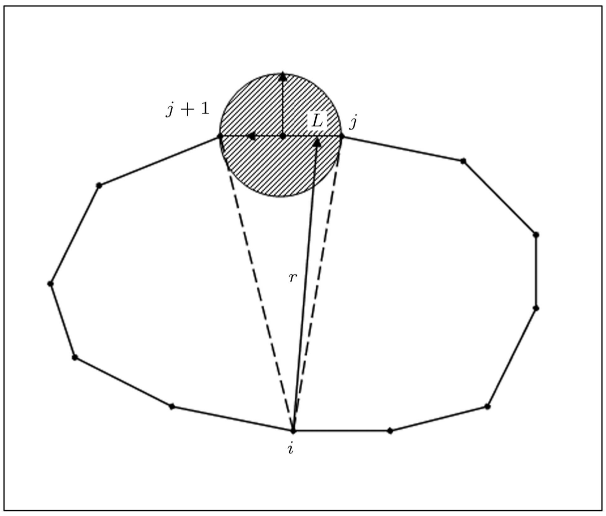


Figure 2. Boundary of BEM domain and the limits of precise Gauss-Legendre integration.

Higher Order Shape Functions of Potentials over an Element

In this section, it is assumed that the shape function of geometry over an element is linear, and potential interpolation over an element is a polynomial with order k.

In the BEM integrations, the following integrals are found by defining local coordinates:

$$H = \int_{-1}^{1} f(\xi)d\xi = \frac{1}{2\pi}\int_{-1}^{1} \frac{\hat{n}.\vec{r}(\xi)}{r(\xi)^2}.\xi^k J(\xi)d\xi,$$

$$G = \int_{-1}^{1} g(\xi)d\xi = \frac{1}{2\pi}\int_{-1}^{1} \ln r(\xi).\xi^k J(\xi)d\xi. \qquad (30)$$

In which J is Jacobean, whose value is L (half of the element length) for the geometrically linear element.

Equations 17 to 21 can be extended for Equation 30 after some similar and lengthy operations. Final formulations for the Maclaurin series of integrands in Equation 30 are as below:

$$\frac{1}{n!}f^{(n)}(0) = \begin{cases} 0; & n < k \\ \frac{\sin(n-k+1)\theta_0}{2\pi r_0^{n-k+1}}; & n \geq k \end{cases} \qquad (31)$$

$$\frac{1}{n!}g^{(n)}(0) = \begin{cases} 0; & n < k \\ \ln r_0; & n = k \\ -\frac{\cos(n-k)\theta_0}{2\pi(n-k)r_0^{n-k}}; & n > k \end{cases} \qquad (32)$$

The shape function with order k can be written as follows:

$$N_i(\xi) = a_k^i \xi^k + a_{k-1}^i \xi^{k-1} + \cdots + a_0^i. \qquad (33)$$

The following equation is obtained by using Equations 30 to 32:

$$\frac{1}{n!}f^{(n)}(0) = \sum_{j=0}^{J} a_j^i \frac{\sin(n-j+1)\theta_0}{2\pi r_0^{n-j+1}},$$

$$\begin{cases} J = k; & k < n \\ J = n; & k > n \end{cases} \qquad (34)$$

$$\frac{1}{n!}g^{(n)}(0) = a_n^i A - \sum_{j=0}^{J} a_j^i \frac{\cos(n-j)\theta_0}{2\pi(n-j)r_0^{n-j}},$$

$$\begin{cases} J = k \quad \& \quad A = 0; & k < n \\ J = n \quad \& \quad A = \ln r_0; & k \geq n \end{cases} \qquad (35)$$

The same condition ($r_0 > 1$) is achieved for these formulations by applying a similar convergence procedure to that used in Equations 22 to 28.

Integration Methods of BEM Singular and Nearly Singular Integrals

In this study, the Romberg method is used for nearly singular integrals. This numerical integration was selected due to its fine precision and its rapid convergence [9-12].

In the present research, the trapezoidal rule was used as the base of Romberg integration. At the first step, the distance between integral bounds is divided into 10 sub-distances. Then, the successive steps of Romberg integration are continued to 4 steps. The results show that the Romberg method is significantly more accurate than the Gauss method in the nearly singular integral. However, the CPU time increases using the Romberg method. The results in the next section show the modification of errors in the Romberg method.

If the source point places on the element, the BEM integrals will be singular [1,4]; in this situation the Gauss integration is not applicable. Also, the procedure of the Romberg method includes indefinite (division by zero) status and is also not usable.

In this research, composition of the midpoint rule and Romberg method is used to overcome this problem [10]. Results show that this method is suitable for integration in BEM.

RESULTS

As obtained in Equations 29, the Maclaurin series converge when r_0 is greater than 1. Figure 3 shows this result for $r_0 = 0.75$, 1 and 1.25 for different θ_0 ($\theta_0 = \pi/2, \pi/3, \pi/4$ and $\pi/6$).

Figures 4, 5 and 6 show error estimation using Schaum's formula (Equation 11), the conventional formula (Equation 10) and by comparing them with the analytical solution for the constant element, respectively. The analytical solution is obtained from the research of Zisis and Ladopoulos [5] and Nui et al. [8]. These figures confirm the condition obtained in Equation 22. The errors after $r_0 = 1$ are very close to zero. Also, Table 1 shows the errors are very close to zero for 6 point Gauss integration in $r_0 = 1$.

Also, Figures 4 to 6 show an acceptable agreement between Schaum's formula and error estimation by comparing analytical integration. Therefore, Schaum's formula will be used for other elements that do not have exact solutions.

By comparing some degrees of Gauss integration, we conclude that the 6 point Gauss integration is accurate enough for BEM integrations. The maximum error of 6 point Gauss integration is 1.285×10^{-4}

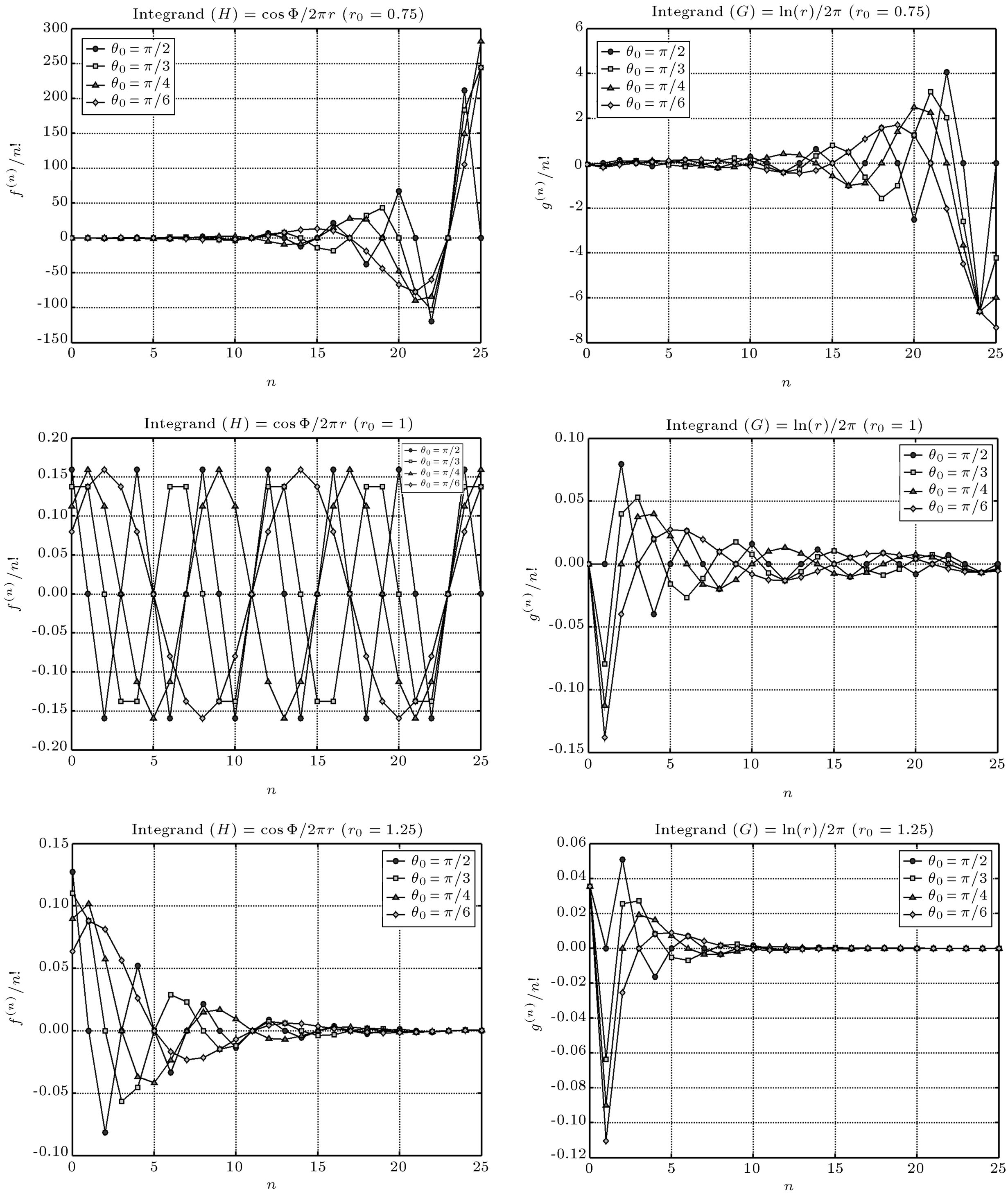


Figure 3. Twenty five terms of Maclaurin series in several r_0 and θ_0 for H and G integrands.

for H, where the analytical integral value is 0.25 and 1.484×10^{-5} for G where the analytical integral value is -0.0118 according to Figures 7 and 8.

In this study, the errors have also been estimated for various angles θ_0 and a critical amount of r_0 ($r_0 = 1$). As shown in Figure 9, the errors decrease by increasing angle θ_0.

As mentioned in the previous section, the condition of Equation 22 is applicable for higher order of u. Figures 10 to 12 show that this condition is true for the linear element of the potential function. Higher order potential functions also have the same results. The analytical solution is obtained from the research of Zisis and Ladopoulos [5] and Nui et al. [8].

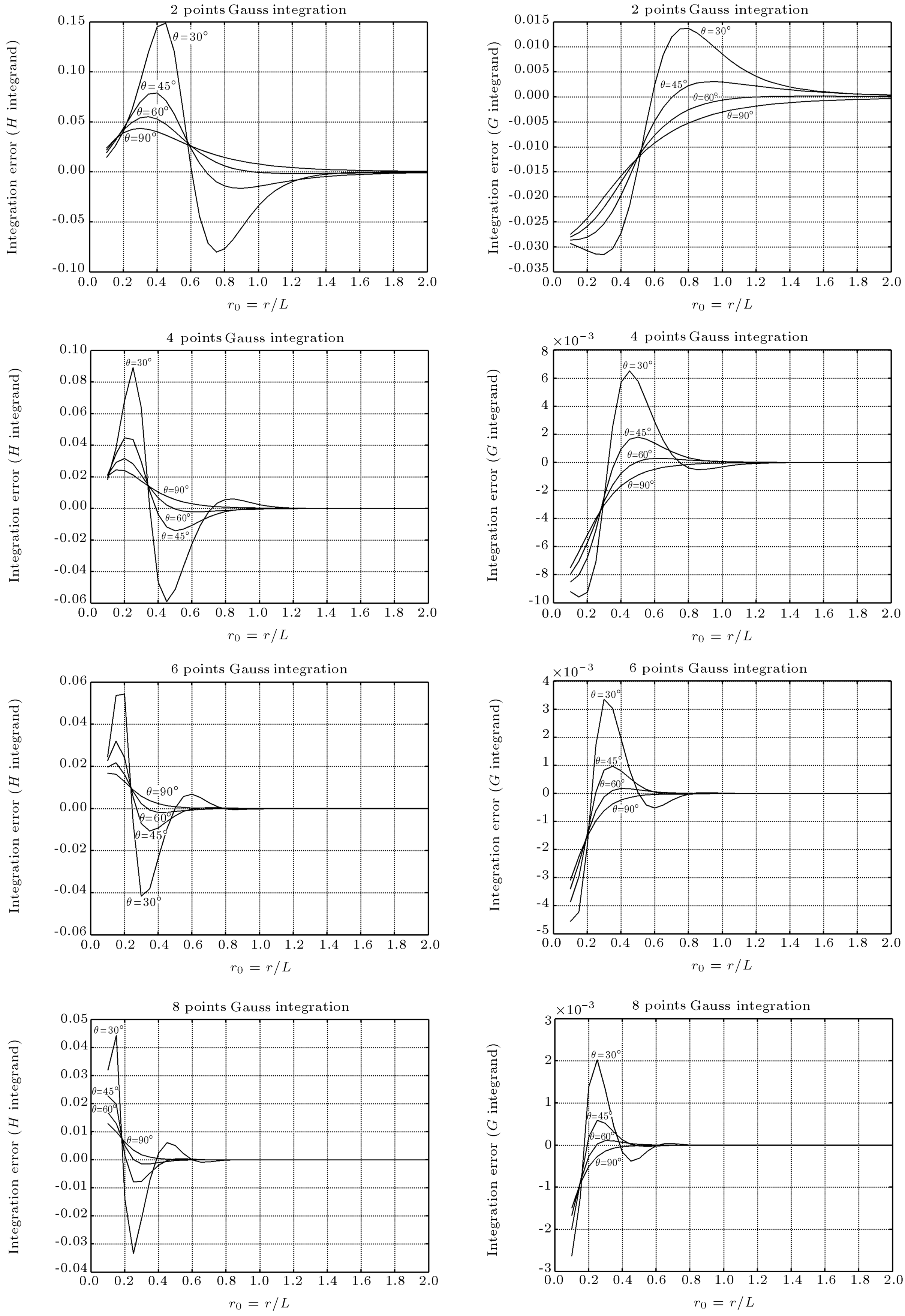


Figure 4. The error estimation of H and G integration by using Schaum's formula (Equation 11).

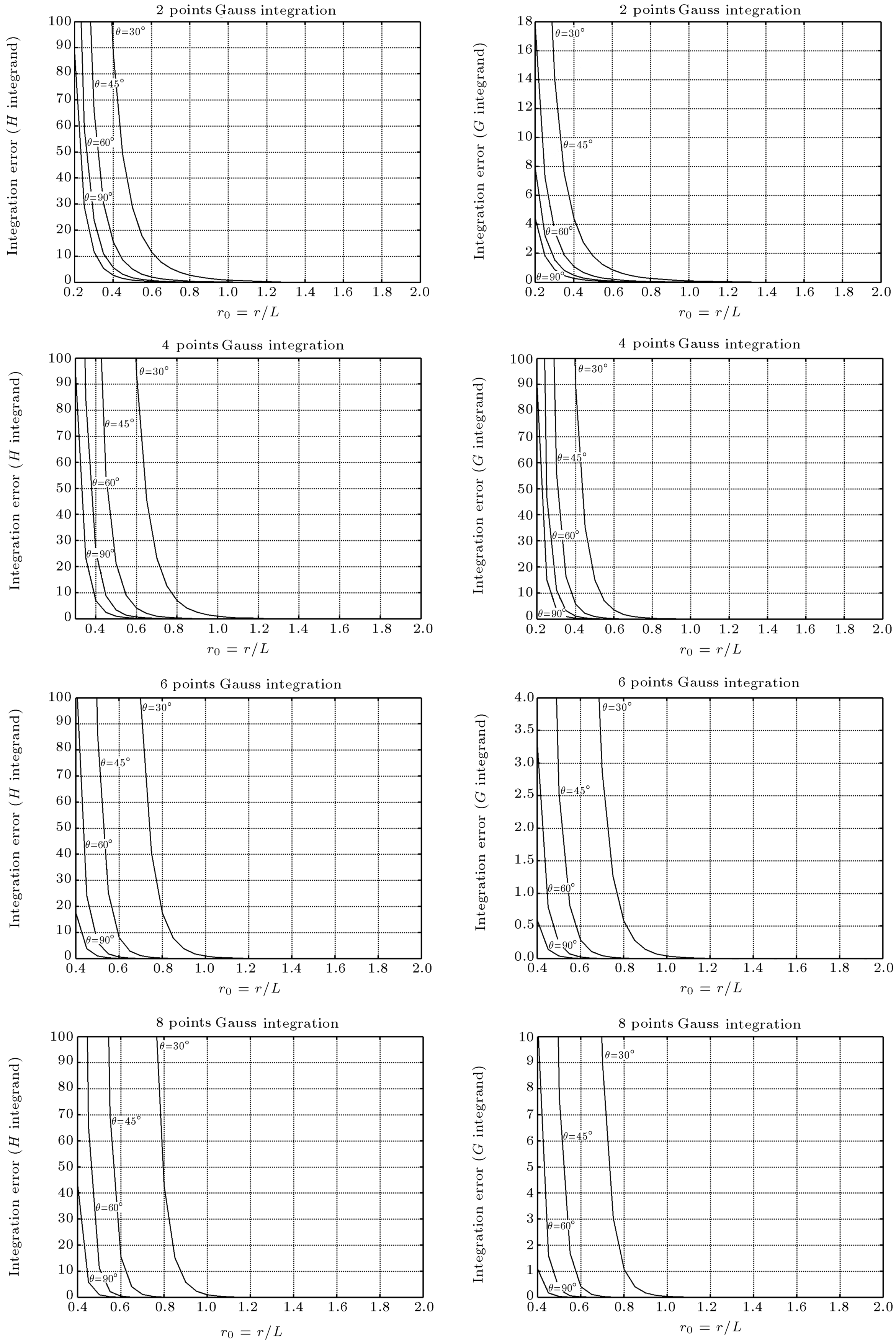


Figure 5. The error estimation of H and G integration by using Equation 10.

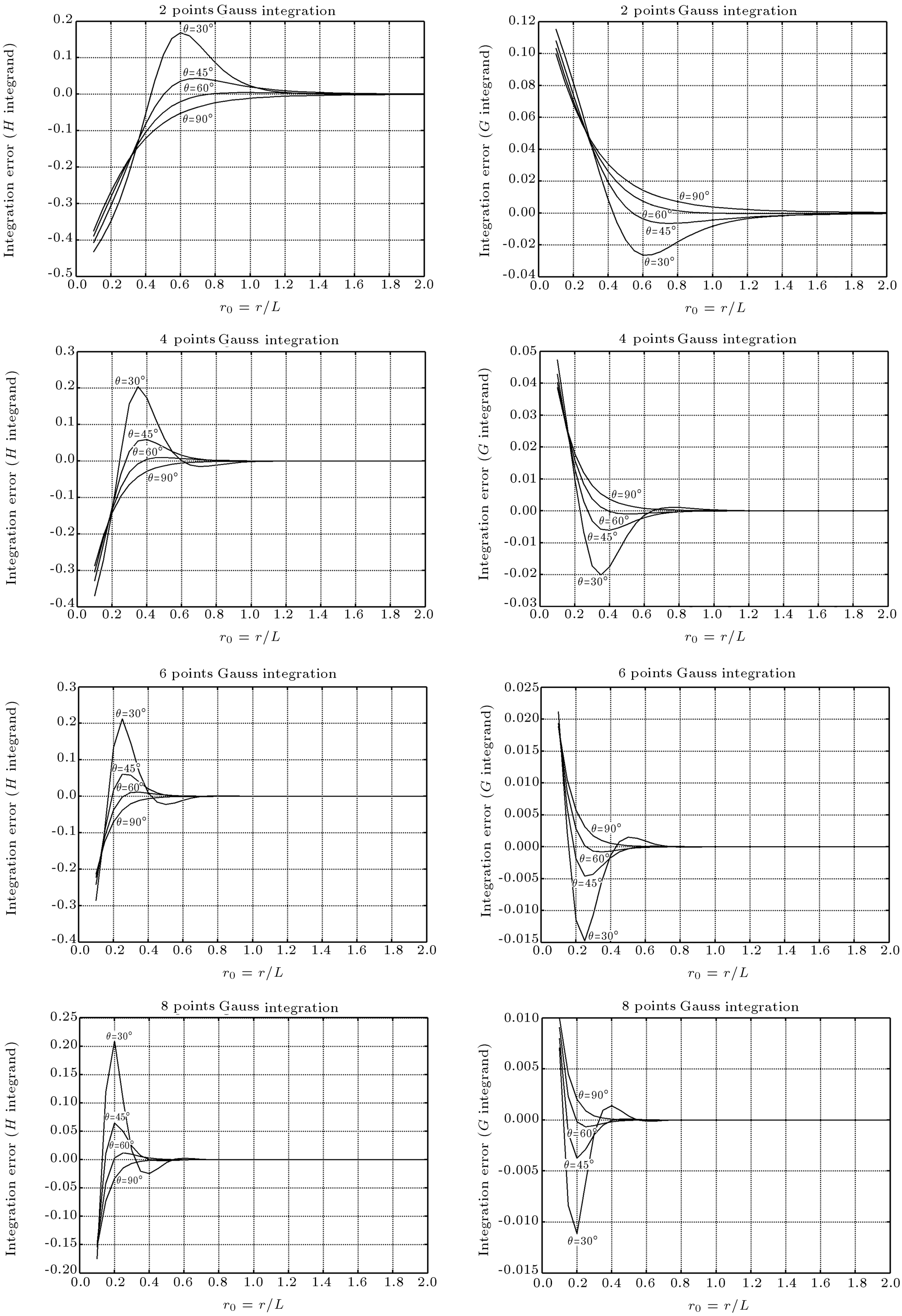


Figure 6. The error estimation of H and G integration by comparing with analytical integration.

Table 1. The errors estimation by different methods for 6 points Gauss integration in $r_0 = 1$.

	Integral H ($r_0 = 1$)			
θ_0 (Deg.)	**Error According to Figure 6**	**Error According to Figure 7**	**Error According to Figure 8**	**Exact Solution**
90	7.2830E-06	0.00011747	1.0285E-05	0.25
60	1.4069E-06	0.00076212	3.2643E-06	0.25
45	2.8359E-05	0.01061588	1.8623E-05	0.25
30	0.00017828	0.95677845	0.00012851	0.25
	Integral G ($r_0 = 1$)			
θ_0 (Deg.)	**Error According to Figure 6**	**Error According to Figure 7**	**Error According to Figure 8**	**Exact Solution**
90	8.823E-07	9.79E-06	1.18502E-06	0.042008
60	5.462E-07	5.50E-05	2.3228E-07	0.02933365
45	3.619E-06	0.0006257	3.3627E-06	0.0128151
30	1.499E-05	0.039899	1.4842E-05	-0.011791

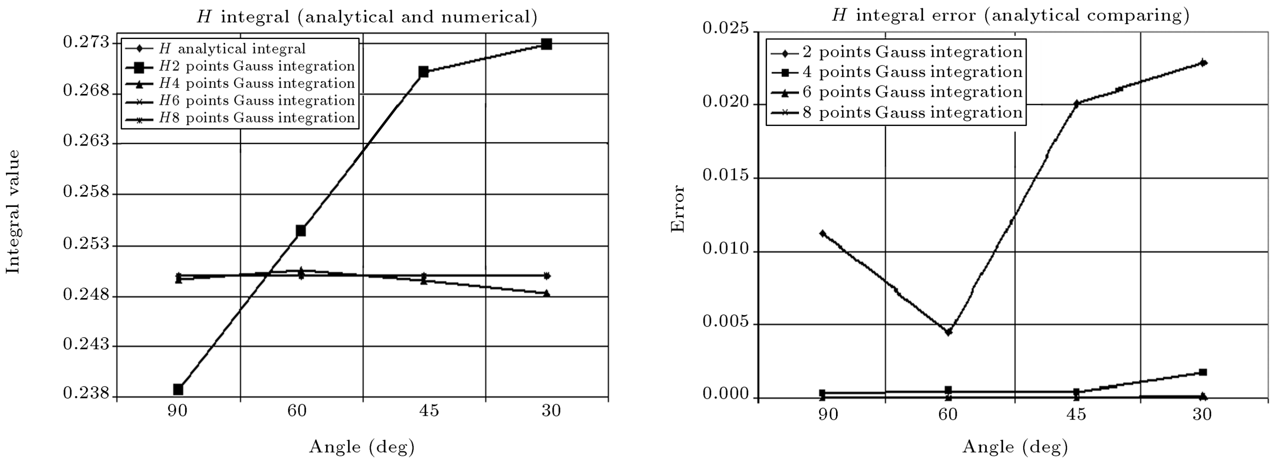


Figure 7. The integral value of H integral by analytical and some Gauss integration (left) and the error of numerical integrations by subtracting numerical value from analytical value (right).

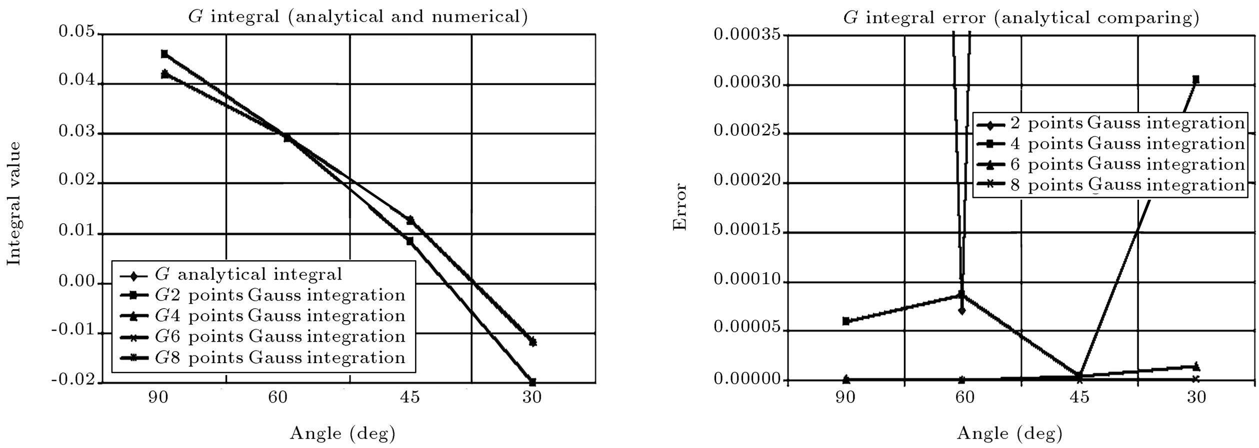


Figure 8. The integral value of G integral by analytical and some Gauss integration (left) and the error of numerical integrations by subtracting numerical value from analytical value (right).

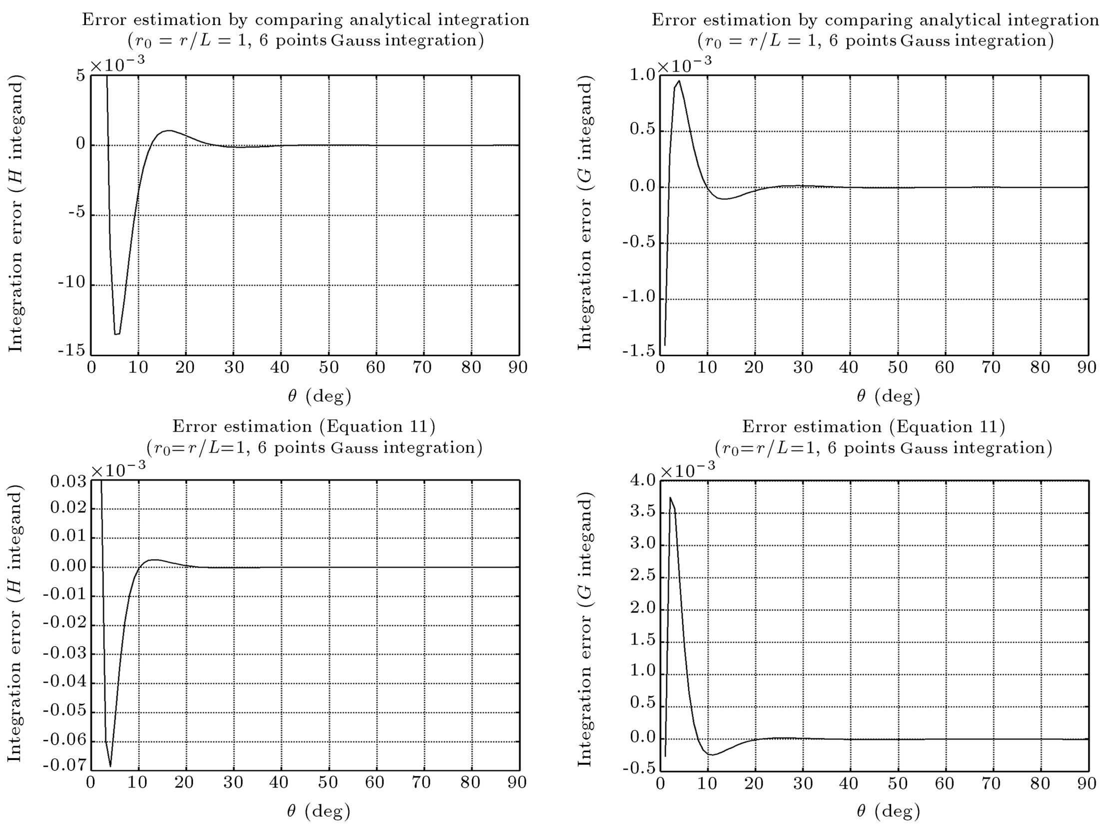


Figure 9. Error estimation with respect to angle θ_0.

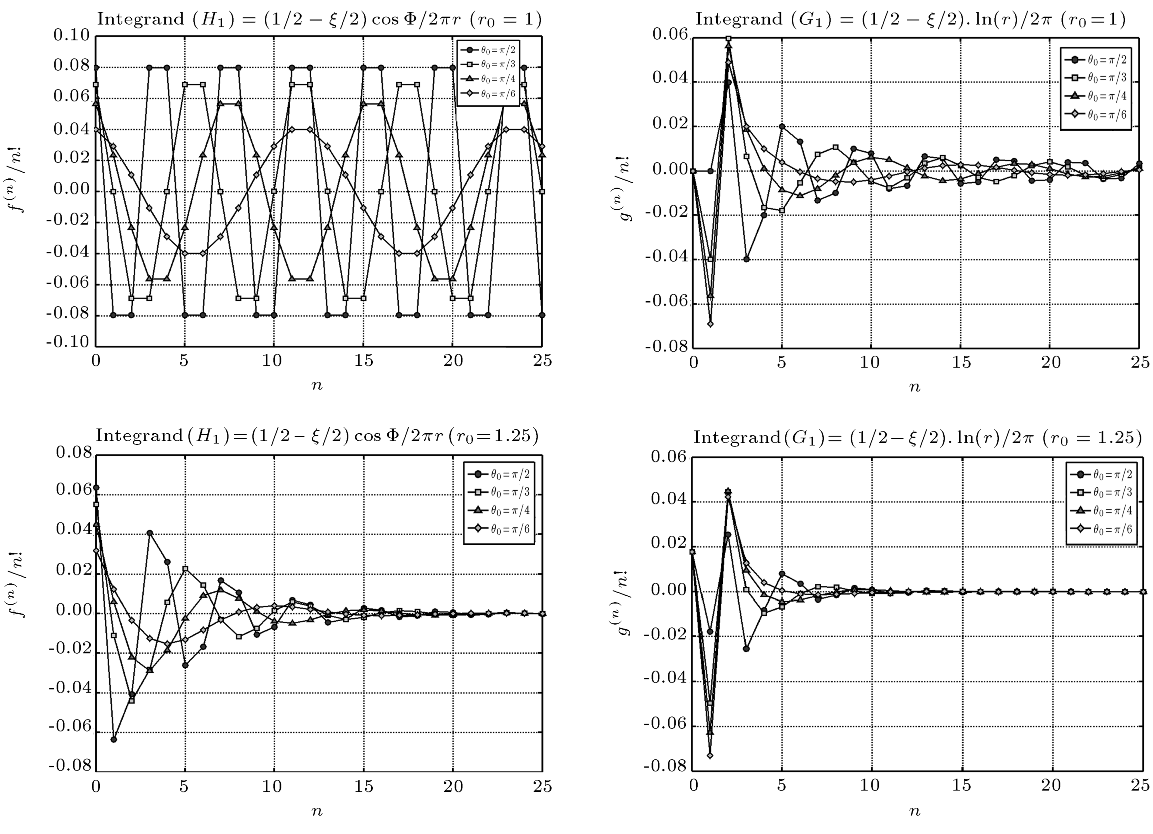


Figure 10. Nineteen terms of Maclaurin series in several r_0 and θ_0 for H and G integrands in linear shape function of potential function.

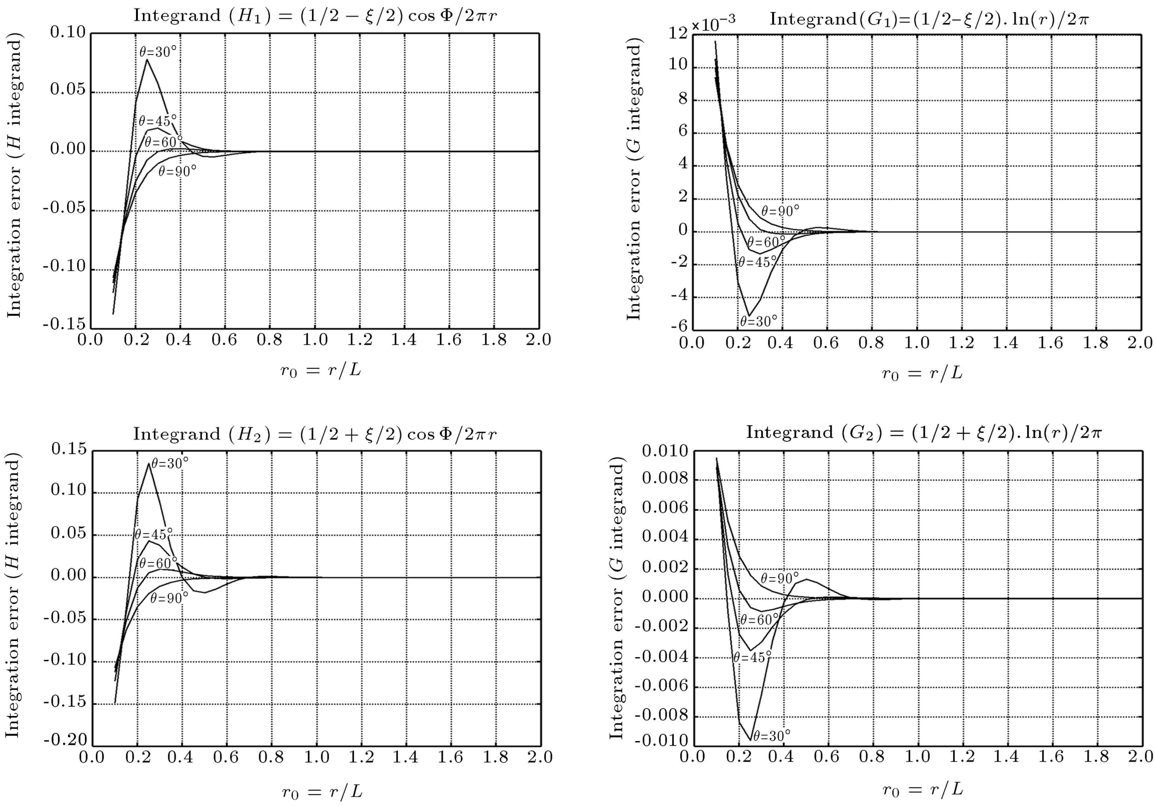


Figure 11. The error estimation of H and G integration by comparing with analytical integration in linear shape function of potential function.

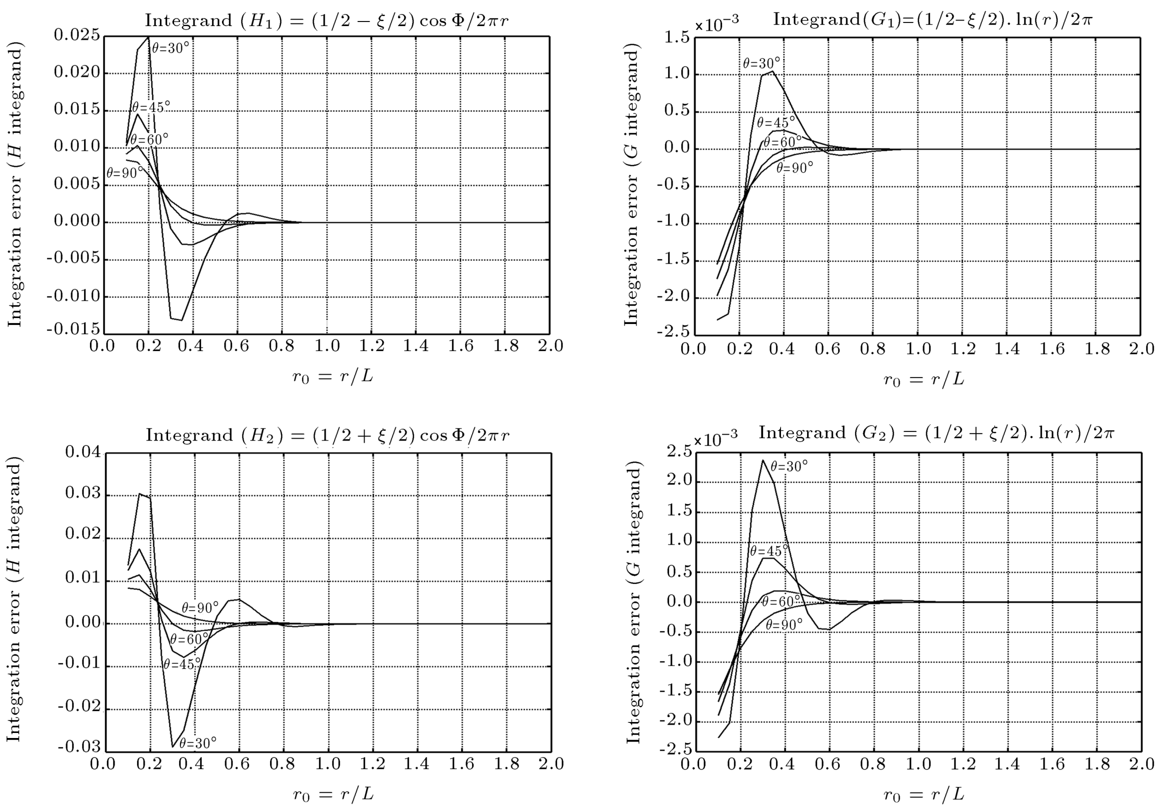


Figure 12. The error estimation of H and G integration using Schaum's formula (Equation 11) in linear shape function of potential function.

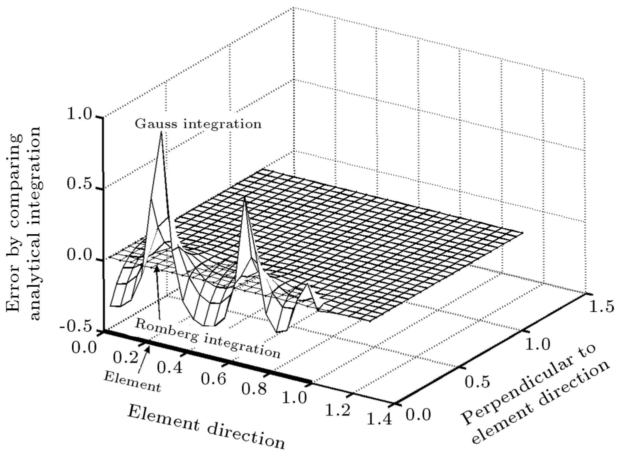
Table 2. Comparing Romberg and Gauss integration error in nearly singular integrals.

Integration Method (Integrand)	Romberg (*H* Integrand)	Gauss (*H* Integrand)	Romberg (*G* Integrand)	Gauss (*G* Integrand)
Error	0.00209	0.97251	2.67E-05	0.05208

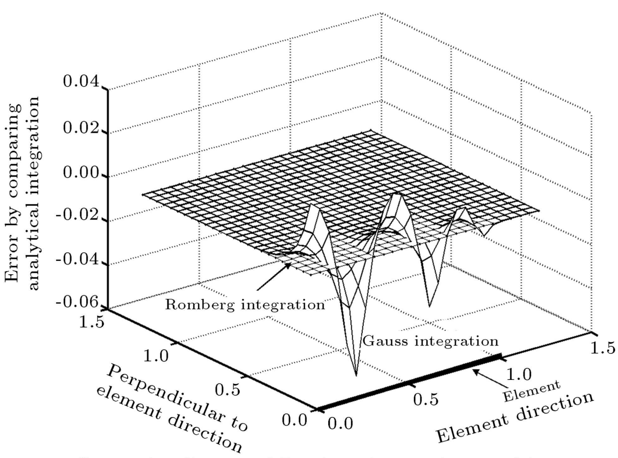
In the same section it was mentioned that the Romberg method is used for nearly singular integrals in the constant element. To ensure that this method is accurate, Romberg is compared with 6 point Gauss integration in nearly singular integrals. Maximum absolute errors by comparing analytical integrations are shown in Table 2.

Figure 13 shows that the Romberg method is more accurate than Gauss integration (6 point) for the source point near the element. But, the error decreases significantly at other source points for both methods.

Figure 14 illustrates the errors of singular integrals (G integrand) in the geometrically linear element and some interpolations of potential functions. The value of the H integral is zero for the geometrically linear element and, therefore, the errors are zero too.



Comparing Gauss and Romberg integration precision in nearly singular integral (H)



Comparing Gauss and Romberg integration precision in nearly singular integral (G)

Figure 13. Error estimation by comparing analytical solution for (a) H integrand and (b) G integrand.

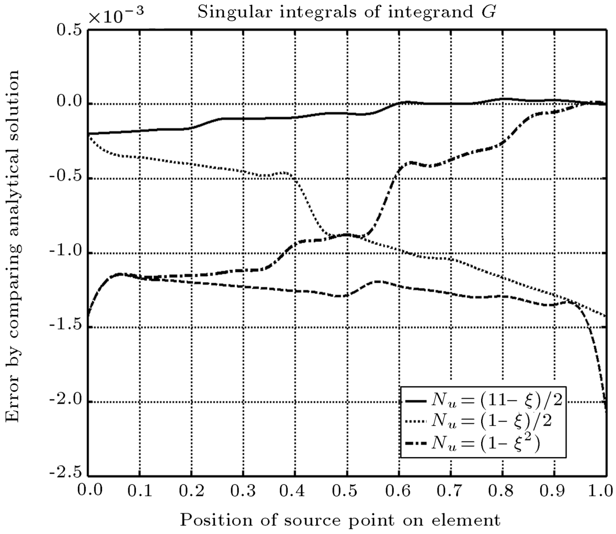


Figure 14. The error estimation of singular integrals by comparing composing Romberg and midpoint rule with analytical solution.

CONCLUSIONS

This study has been focused on obtaining nearly singular criteria for the source point position in BEM integrals. The nearly singular integrals in BEM are not a new subject, but the conditions of near singularity have not been completely discussed in other literature.

It is necessary to provide some constraint for the type of integration method used in computer programming, such that in one case the 6 point Gauss integration method is used and in another the Romberg method is applied.

In cases of geometrically linear elements, the condition $r_0 > 1$ is used for applying Gauss integration. The opposite of this situation (situations in which Gauss integration is not accurate as mentioned in this article) occurs in the following cases: (1) The angle between two adjacent elements is tight; (2) the boundary of domain comes nearly together at one or some points; (3) the boundary includes thin walls with

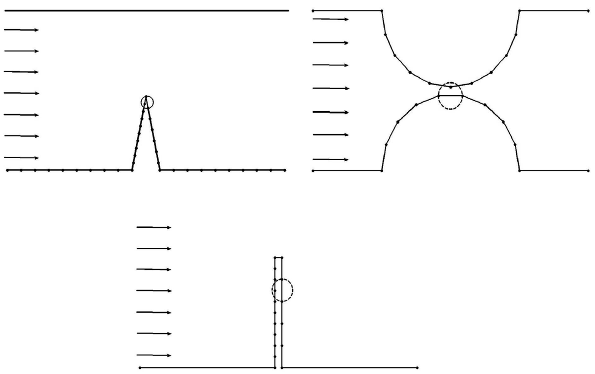


Figure 15. Some examples in which Gauss integration is not accurate for some elements and source points.

thicknesses less than element length etc. Examples of these cases are:

1. Inviscid flow over a triangle, according to Figure 15.
2. Inviscid flow in a ventury, according to Figure 15.
3. Inviscid flow over a thin wall, according to Figure 15.
4. Some large amplitude water waves.

As mentioned above, the points which may violate the conditions are usually corner points. Therefore, it is better to initially identify these points, because adding the condition line to the code and checking that for each node and element also increases CPU time. As is well known, BEM decreases CPU time, but this condition can weaken this property. By indicating the near singular element and applying these criteria, it is possible to prevent the increment of CPU time and increase the accuracy of the results.

Using the Romberg method is not the best method, but it is very accurate and simple. These procedures can extend to other types of element.

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BIOGRAPHIES

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