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An Analytic Variable Limit np Control Chart

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Abstract. *The Shewhart np control chart is often used to monitor the quantity of nonconforming, but it is slow in detecting small deviations. This paper proposes an efficient approach to monitor the quantity of nonconforming. The novelty of the paper is utilization of an initial belief to construct an analytic variable limit np control chart. The proposed method uses all gathered data, sequentially. This approach is significantly faster than some existent effective approaches in detecting small deviations. These charts are mainly used for evaluation of the initial setup in the process. The simulated results for the average run length profiles demonstrate the superiority of the new approach against the standard np chart, binomial CUSUM, binomial EWMA and moving average approach.*

Keywords: *np chart; Variable limit; Quantity of nonconforming; Average run length.*

INTRODUCTION

The main purpose of statistical process control is to improve quality and productivity. One of the instruments that form a quality tool set is a control chart. Control charts are efficient instruments for checking changes or variations in processes [1].

The power or effectiveness of a control chart is usually measured by Average Run Length (ARL), which is the average number of inspected samples required to signal an out-of-control condition after it has occurred. The users of control charts desire the signal for an out-of-control condition to be as fast as possible, when a process is out-of-control, i.e. the out-of-control Average Run Length (ARL$_1$) to be as small as possible. Conversely, when the process is in control, the users are happy to see false alarms as little as possible, i.e. to have a large in-control Average Run Length (ARL$_0$) [2].

The traditional attribute control charts or the Shewhart np charts have been widely used in industries to control the quantity of nonconforming of the process. A nonconforming unit is a product or service that fails to meet at least one specified specification. Although the simplicity of using the Shewhart np charts is an advantage, its disadvantage merely lies in the informa-

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tion enclosed about the process in the last plotted point and the information of the whole sequence of all sample points is ignored. Hence, Shewhart np control charts are insensitive to small changes in the process, for a 1.5 standard deviation or less [1].

Control charting methods, based on attribute (or count) data, were first proposed by Shewhart [3]. One of the major weaknesses of Shewhart np charts is the ignorance of information given by the sequence of all points. To remedy this problem, Shewhart control charts are efficiently complemented by cumulative sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) control charts, when there is interest in detecting small changes in the process [1].

"The CUSUM control chart was initially proposed in England by Page and has been studied by many authors" [1]. Montgomery [4] mentioned that it is possible to project CUSUM procedures for Binomial and Poisson variables for modeling nonconformities and continuous processes. Reynolds and Stoumbos [5] dealt with a Bernoulli CUSUM chart, which is based on Bernoulli individual observations. Gan [6] proposed a CUSUM chart for binomial counts, based on the CUSUM statistics. This CUSUM chart intends to detect upward shifts in p. It is noticeable that the CUSUM control charts are more effective than the Shewhart np charts in detecting small and moderate p shifts, but they are difficult to understand and be used by the operators [5].

The EWMA control chart has been introduced by Roberts [7]. From the viewpoint of statistical process

control, the EWMA control chart is comparable to a CUSUM control chart in its capacity to monitor a process and detect the presence of assignable causes, which result in changes [4]. Hunter [8] claimed that the differences between Shewhart, CUSUM and EWMA control charts, depend on the way each charting technique uses the data generated by the production process. Some authors, such as Crowder [9,10], Lucas and Saccucci [11], claimed that the EWMA control chart is a good tool for detecting small and moderate changes in the process average.

Bernoulli EWMA has been introduced to monitor the outlying process by Somerville et al. [12]. The EWMA control chart for the fraction of nonconforming can also be constructed as the binomial EWMA. The ARL performance of the binomial EWMA is superior to the standard p chart [13]. Vargas et al. [1] showed that, in practice, the CUSUM control chart cannot help to find the points out-of-control for the levels of variation between ± 1.0 standard deviation. Hence, for these variation levels, the EWMA control chart is more efficient than CUSUM.

A moving average control chart is a type of memory-based control chart. Khoo [13] presents a way to extend the moving average chart to the case of binomial data for the fraction of nonconforming. He stated that due to its simplicity and superiority in detecting shifts, the moving average control chart for the fraction of nonconforming should be given a high priority by quality control practitioners; and since it is more likely for a process to start in-control first, following a warm-up period before the process shifts to an out-of-control state and in cases when a shift occurs, the binomial EWMA may perform poorly.

The main idea in this research is to develop an effective analytic variable limit np control chart to be more sensitive to small deviations. To achieve this goal, a new variable limit np control chart is developed and an initial belief about the process's treatment, based on previous information, is quantified. Then, by using the Bayesian concept and a recursive function, an np control chart with variable control limits is introduced. Also, for verification of the method, the proposed method is compared with binomial EWMA, binomial CUSUM, moving average and Shewhart np control charts.

It is noticeable that the proposed method effectively improves the performance of the attribute control chart in detecting small deviations, compared to other mentioned methods.

DESCRIPTION OF OUR METHODOLOGY

In this section, a methodology for improvement in the performance of attribute control charts will be suggested in two phases. Phase I is dedicated to the establishment of the control chart and in Phase II, this control chart will be implemented.

To simplify, it is assumed that sample size n is constant. At iteration k, $k \geq 1$, of the data gathering process, we define:

$$O_k = (x_1, x_2, \cdots, x_k),$$

as an observations vector where x_i is the number of defective products in iteration i, for $i = 1, 2, \cdots, k$. After observing the number of defects at iteration k, namely x_k, we quantify the belief of being in an out-of-control state by $B(x_k, O_{k-1})$ as follows:

$$B(x_k, O_{k-1}) = \Pr\{OOF \,|x_k, O_{k-1}\}, \qquad (1)$$

where OOF stands for out-of-control.

At iteration k, we want to improve the belief of being in an out-of-control state, based on observations vector, O_{k-1} and a new observation, x_k. We define:

$$B(k-1) = B(x_{k-1}, O_{k-2}),$$

as the prior belief and $B(k) = B(x_k, O_{k-1})$ as the posterior belief of the out-of-control state for $k \geq 1$. It is clear from the prior belief that for $k = 1$ we need to estimate an initial value, $B(0)$, based on our initial information about the process. If there is no initial information about the process, $B(0)$ is set equal to 0.5. Furthermore, we set $0 < B(0) < 1$.

We assume that the observations in iterations are taken independently, then we have:

$$\Pr\{x_k \,|OOF, O_{k-1}\} = \Pr\{x_k \,|OOF\}. \qquad (2)$$

With this feature, by using the Bayesian rule, the posterior belief, $B(k)$, is:

$$B(k) = B(x_k, O_{k-1}) = \Pr\{OOF|x_k, O_{k-1}\}$$

$$= \frac{\Pr\{OOF, x_k \,|O_{k-1}\}}{\Pr\{x_k \,|O_{k-1}\}}$$

$$= \frac{\Pr\{OOF \,|O_{k-1}\}\Pr\{x_k \,|OOF, O_{k-1}\}}{\Pr\{x_k \,|O_{k-1}\}}. \qquad (3)$$

Consequently, we can rewrite Equation 3 as:

$$B(x_k, O_{k-1})$$

$$= \frac{\Pr\{OOF|O_{k-1}\}\Pr\{x_k|OOF\}}{\Pr\{OOF|O_{k-1}\}\Pr\{x_k|OOF\} + \Pr\{IC|O_{k-1}\}\Pr\{x_k|IC\}}$$

$$= \frac{B(O_{k-1})\Pr\{x_k|OOF\}}{B(O_{k-1})\Pr\{x_k|OOF\} + (1 - B(O_{k-1}))\Pr\{x_k|IC\}}, \qquad (4)$$

where IC stands for in-control. If we assume that the quality characteristic of the process has a binomial

distribution with parameter p, the probabilities of being in an out-of-control or an in-control state are:

$$\Pr\{x_k|OOF\} = \binom{n}{x_k} p_1^{x_k}(1-p_1)^{n-x_k}, \qquad (5)$$

and:

$$\Pr\{x_k|IC\} = \binom{n}{x_k} p_0^{x_k}(1-p_0)^{n-x_k}, \qquad (6)$$

where p_0 is an in-control quality level and we are going to detect step shift, p_1, in a fraction of nonconforming and $p_0 \neq p_1$. By substituting Equations 5 and 6 in Equation 4, we have:

$$B(x_k, O_{k-1}) =$$

$$\frac{B(O_{k-1})\binom{n}{x_k}p_1^{x_k}(1-p_1)^{n-x_k}}{B(O_{k-1})\binom{n}{x_k}p_1^{x_k}(1-p_1)^{n-x_k}+(1-B(O_{k-1}))\binom{n}{x_k}p_0^{x_k}(1-p_0)^{n-x_k}}, \quad (7)$$

and:

$$1 - B(x_k, O_{k-1}) =$$

$$\frac{(1-B(O_{k-1}))\binom{n}{x_k}p_0^{x_k}(1-p_0)^{n-x_k}}{B(O_{k-1})\binom{n}{x_k}p_1^{x_k}(1-p_1)^{n-x_k}+(1-B(O_{k-1}))\binom{n}{x_k}p_0^{x_k}(1-p_0)^{n-x_k}}. \quad (8)$$

To derive the required recursive functions, assuming $z_k = \frac{B(x_k, O_{k-1})}{1-b(x_k, O_{k-1})}$ we have:

$$z_k = \frac{B(x_k, O_{k-1})}{1 - B(x_k, O_{k-1})}$$

$$= \frac{B(O_{k-1})}{1 - B(O_{k-1})}\left(\frac{p_1}{p_0}\right)^{x_k}\left(\frac{1-p_1}{1-p_0}\right)^{n-x_k}$$

$$= z_{k-1}\left(\frac{p_1}{p_0}\right)^{x_k}\left(\frac{1-p_1}{1-p_0}\right)^{n-x_k}. \qquad (9)$$

Hence:

$$z_k = \left(\frac{p_1}{p_0}\right)^{x_k}\left(\frac{1-p_1}{1-p_0}\right)^{n-x_k} z_{k-1} = \cdots$$

$$= \left(\frac{p_1}{p_0}\right)^{\sum_{i=1}^{k}x_i}\left(\frac{1-p_1}{1-p_0}\right)^{nk-\sum_{i=1}^{k}x_i} z_0, \qquad (10)$$

where:

$$z_0 = \frac{B(0)}{1 - B(0)}. \qquad (11)$$

As mentioned before, $B(0)$ is the probability of an out-of-control state of the process and is estimated based on our initial information. If we take Ln from both sides of Equation 10, we have:

$$Ln(z_k) = \left(Ln\frac{p_1}{p_0}\right)\sum_{i=1}^{k}x_i$$

$$+ \left(Ln\frac{1-p_1}{1-p_0}\right)\left(nk - \sum_{i=1}^{k}x_i\right) + Ln(z_0). \qquad (12)$$

Or:

$$Ln(z_k) = \left[\left(Ln\frac{p_1}{p_0} - Ln\frac{1-p_1}{1-p_0}\right)\right]\sum_{i=1}^{k}x_i$$

$$+ \left[Ln\frac{1-p_1}{1-p_0}\right](nk) + Ln(z_0). \qquad (13)$$

It is obvious that:

$$\sum_{i=1}^{k}x_i \approx \text{Binomial}\,(nk, p_0). \qquad (14)$$

Applying a normal approximation to binomial distributions, we get:

$$nkp_0 - c\sqrt{nkp_0(1-p_0)} \leq \sum_{i=1}^{k}x_i \leq nkp_0$$

$$+ c\sqrt{nkp_0(1-p_0)}, \qquad (15)$$

where parameter c is equal to $z_{\alpha/2}$ in which z is a statistic with standard normal distribution and α is the value of type I error. From the above equation, we determine UCL and LCL for $Ln(z_k)$ as follows:

$$UCL_k =$$

$$\left(Ln\frac{p_1}{p_0} - Ln\frac{1-p_1}{1-p_0}\right)\left(nkp_0 + c\sqrt{nkp_0(1-p_0)}\right)$$

$$+ (nk)\left(Ln\frac{1-p_1}{1-p_0}\right) + Ln(z_0), \qquad (16)$$

where UCL_k and LCL_k indicate upper and lower limits for statistics, $Ln(z_k)$, respectively, when $p_0 < p_1$. If $p_0 > p_1$, Equation 16 is LCL_k and Equation 17 is UCL_k. As observed from Equations 13, 16 and 17, $Ln(z_0)$ is a value that is added to statistics $Ln(z_k)$ and also its lower and upper bounds. To construct our control limits, we suppose $B(0) = 0.5$ and then, $Ln(z_0) = 0$.

In summary, if $Ln(z_k)$ is less than UCL_k and more than LCL_k, then, the process is in-control, and otherwise, the process is in an out-of-control state.

EXPERIMENTAL RESULTS

Visual basic 7 is used to compare our method with the standard np chart, Binomial EWMA, Binomial CUSUM and moving average chart. Through simulation, after establishing the control chart, the obtained control limits from Equations 16 and 17 are divergent and very close to each other at the beginning. It means that our interval is very tight at first. When the process is in-control, this causes some initial data to fall to out-of-control zones immediately, and some of them to fall to out-of-control zones later. Consequently, we are faced by a very large variation of run length (VRL_0). To overcome this problem, a constant value, l, will be added to the parameter, k, as follows:

$$UCL_{k+l} = \left(Ln\frac{p_1}{p_0} - Ln\frac{1-p_1}{1-p_0} \right) \left(n(k+l)p_0 \right.$$
$$\left. + c\sqrt{n(k+l)p_0(1-p_0)} \right)$$
$$+ (n(k+l))\left(Ln\frac{1-p_1}{1-p_0} \right), \qquad (17)$$

and:

$$LCL_{k+l} = \left(Ln\frac{p_1}{p_0} - Ln\frac{1-p_1}{1-p_0} \right) \left(n(k+l)p_0 \right.$$
$$\left. - c\sqrt{n(k+l)p_0(1-p_0)} \right)$$
$$+ (n(k+l))\left(Ln\frac{1-p_1}{1-p_0} \right). \qquad (18)$$

By this modification, the values of VRL_0 become smaller and the appropriate ARL_0 will be reached. In other words, by using this parameter, we start at point l of the control limits instead of point zero. In the next stage, we should estimate a value for statistics, $Ln(z_{k+l})$, when $k = 0$, the initial value of the statistics, or the value of $Ln(z_l)$. As mentioned before, if we do not have any information about the process, we put the initial value of $B(l)$ equal to 0.5; l is a new value instead of zero. Furthermore, under this condition, we assume $Ln(z_l)$ at the middle of the interval (LCL_l, UCL_l). It means that $Ln(z_l)$ is equal to:

$$0.5 \times (LCL_l + UCL_l).$$

Therefore, we have:

$$Ln(z_l) = B(l) \times (LCL_l + UCL_l).$$

It is noticeable that if we guess that the fraction of nonconforming is greater than (less than) p_0, then we set a value $B(l) \geq 0.5$ $(B(l) \leq 0.5)$. Parameter l is one of the parameters of the discussed model, and

it is defined in such a way that in combination with the defined value of c, the desired ARL_0 and ARL_1 is reached. Afterwards, we compare the presented method with some other methods developed in the literature.

The selected methods for comparison are binomial EWMA for $\lambda = 0.02$, 0.05 and 0.08 and various values of parameter A, binomial CUSUM, moving average for $w = 2$, 3 and 4, and the standard np control chart. Comparison of the values for parameters of the binomial EWMA and the moving average, are selected as those in Khoo [13]. The values of l and c are selected, based on the acquisition of good results for ARL_0 and standard deviation of RL_0 ($SDRL_0$).

To check the validity of our method, we generate two independent binomial distributions with parameters $p_0 = 0.1$, $p_0 = 0.2$ and $n = 200$. In the next step, using Equation 10, we update belief z_k. When z_k is greater than UCL_{k+l} in Equation 18, or less than LCL_{k+l} in Equation 19, then, an out-of-control signal is observed.

For the comparison study, we calculate the stock-ticker ARL_1 values for all considered methods by 10000 independent replications, i.e. $M = 10000$, for different values of p_1, which are indicated in the first column of Tables 1 and 2.

The simulation results given in Tables 1 and 2 and also Figures 1 and 2, clearly show that the proposed method is better than binomial EWMA, binomial CUSUM, the moving average and standard np control chart for shifts of small magnitude from the target value, p_0, but for shifts of large magnitude from the target value, our method is not as good as the others.

According to the results represented in Tables 1 and 2, the following are observed:
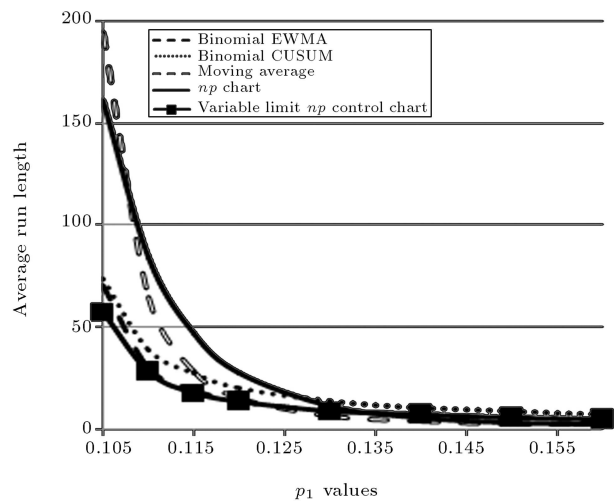

Figure 1. Comparison between binomial EWMA, binomial CUSUM, moving average, np chart and variable limit np control chart for $p_0 = 0.1$ (results of EWMA and MA are the best results shown in Table 1).

Table 1. ARL profiles for binomial EWMA, binomial CUSUM, moving average, np chart and variable limit np control chart based on $p_0 = 0.10$ and $n = 200$, $M=10000$ computed by means of a simulation.

p_1	Binomial EWMA						Binomial CUSUM		Moving Average						np Chart		Variable Limit np Control Chart	
	$\lambda = 0.02$ $A = 2.1257$		$\lambda = 0.05$ $A = 2.652$		$\lambda = 0.08$ $A = 2.7650$		$U = 0$ $K = 0.1$ $H = 0.39$		$W = 2$		$W = 3$		$W = 4$		$C = 3$		$l = 120$ $C = 1$ $B(l) = 0.5$	
	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.040	4.4	0.5	3.7	0.6	3.2	0.5	-	-	1.7	0.7	1.6	0.6	1.6	0.6	2.21	1.6	4.5	0.6
0.050	5.3	0.8	4.4	0.8	3.8	0.7	-	-	2.5	1.5	2.3	1	2.2	0.9	4.7	4.2	5.3	0.8
0.060	6.6	1.1	5.6	1.1	4.9	1.1	-	-	5	4	3.7	2.3	3.3	1.7	12.1	11.5	6.5	1.1
0.070	8.8	1.9	7.6	2	6.8	2	-	-	13.8	12.5	8.3	6.7	6.4	4.5	36.2	34.7	8.7	1.9
0.080	13.5	3.9	12.4	4.3	11.5	4.7	-	-	54.8	55	29	27.3	19.5	17.6	122.8	122.3	13.2	3.7
0.085	18.7	6.5	18.1	7.9	17.6	9.1	-	-	123.3	121.1	67.3	66.7	43.1	40.9	226.7	224.3	17.6	5.8
0.090	30.1	13.3	32.7	18.6	34.3	22.9	-	-	284	271.6	178.2	175.5	116.6	112.9	381.2	381.3	26.9	11.3
0.095	71.8	46.3	102.7	82.8	120.2	108.3	-	-	568	574.6	467.6	461	345.2	344.3	441.2	435.3	59.7	37.4
0.105	70.7	47.1	93.3	76	101.9	92.3	73.3	35.1	213.4	213.4	208	207.2	194.7	186.6	161.6	160.8	57.7	38.7
0.110	30.1	14.5	32	19.6	32.7	22.5	38.8	13.9	92.3	89.6	77.1	74.7	64.3	62.3	84.1	82.9	27.9	12.9
0.115	18.9	7.5	18.1	8.8	17.2	9.6	26.9	7.5	43	40.7	34.9	33	28.4	26	46.9	45.8	17.9	6.7
0.120	13.7	4.7	12.4	5.1	11.5	5.4	20	5	23.1	22.1	17.9	16.7	14.9	13	27.7	26.9	13.2	4.3
0.130	8.9	2.5	7.8	2.6	6.9	2.6	13.9	3	8.3	7.4	6.7	5.6	6	4.6	11.2	10.4	8.8	2.4
0.140	6.6	1.7	5.7	1.6	5	1.6	10.5	1.9	4.3	3.5	3.7	2.6	3.5	2.3	5.6	5.1	6.6	1.6
0.150	5.3	1.2	4.5	1.2	3.9	1.1	8.3	1.4	2.6	1.9	2.5	1.5	2.4	1.3	3.3	2.7	5.4	1.1
0.160	4.5	1	3.8	1	3.3	0.9	7.2	1.24	1.9	1.1	1.9	1	1.8	0.9	2.2	1.6	4.6	0.9
$p_0 = 0.1$	390.1	392	585.3	625.9	537.7	553.6	416	342.6	496.6	495.7	549.7	564.4	556.7	535	302	302	860.5	2391.9
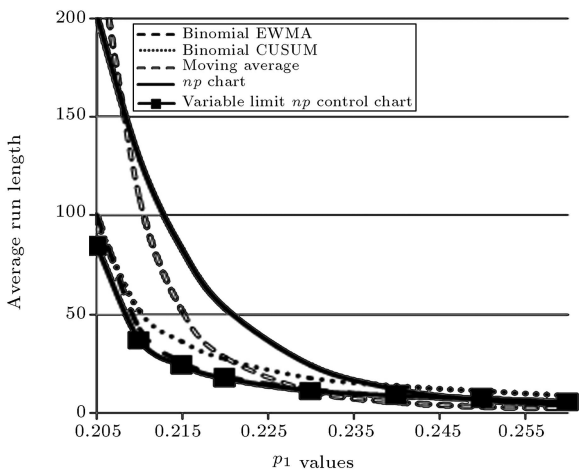
Figure 2. Comparison between binomial EWMA, binomial CUSUM, moving average, np chart and variable limit np control chart for $p_0 = 0.2$ (results of EWMA and MA are the best results shown in Table 2).

1. The initial value of z_l is assumed to be $0.5 \times$ (LCL$_l$ + UCL$_l$). It means that the process is in an out-of-control state with the probability of $B(l) = 0.5$. We define this initial value based on our recognition about the process, and it can have any other values in the (LCL$_l$, UCL$_l$) interval.

2. We define parameters c and l to obtain the desired ARL$_0$ and ARL$_1$,

3. The standard deviation of RL$_0$ (SDRL$_0$) in our approach is large in comparison with other methods; but the ARL$_0$ value is pretty large. Also, as shown in Tables 3 and 4, the initial 10 percent of RL$_0$ in the new method is pretty equivalent to other methods. Therefore, overall, the ARL$_0$ in the new method is better than the others.

4. It is noticeable that the ARL$_0$ in the new method is high and that the ARL$_1$ is low for small process

Table 2. ARL profiles for binomial EWMA, binomial CUSUM, moving average, np chart and variable limit np control chart based on $p_0 = 0.20$ and $n = 200$, $M = 10000$ computed by means of a simulation.

p_1	Binomial EWMA						Binomial CUSUM		Moving Average						np Chart		Variable Limit np Control Chart	
	$\lambda = 0.02$ $A = 2.1257$		$\lambda = 0.05$ $A = 2.6150$		$\lambda = 0.08$ $A = 2.7650$		$U = 0$ $K = 0.1$ $H = 0.52$		$W = 2$		$W = 3$		$W = 4$		$C = 3$		$l = 120$ $C = 1$ $B(l) = 0.5$	
	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
0.140	5.9	1	4.9	1	4.3	1	-	-	3.2	2.3	2.8	1.5	2.6	1.3	5.5	5	5.9	1
0.150	7	1.4	5.9	1.4	5.3	1.4	-	-	5.5	4.4	4.1	2.8	3.7	2.2	10.7	10.3	7	1.4
0.160	8.8	2.1	7.6	2.1	6.8	2.1	-	-	11	9.7	7.4	5.9	6	4.3	21.7	21.2	8.7	1.9
0.170	11.9	3.3	10.6	3.5	9.9	3.8	-	-	27.3	26.3	16.2	14.7	12.4	10.3	48.9	48.1	11.7	3
0.180	18.7	6.8	17.7	7.9	17.5	9.2	-	-	80.8	81.1	48.5	47.1	35	33	121.8	121	17.5	5.7
0.185	26.1	11.3	26.5	14.8	27.6	18.2	-	-	154.2	157.1	94.9	92.3	70.2	68.3	187.7	196.8	23.6	9.6
0.190	43.1	24.2	49.6	34.5	56.1	43	-	-	289.6	284.6	206.1	202.7	158.1	156.1	272.4	279.7	37.9	20.5
0.195	102.6	77.6	144.8	129.8	172.7	167.6	-	-	490.4	509.9	408	412.2	331.9	335.4	326.5	328.4	79.4	61.9
0.205	100.8	74.6	133.4	122.2	154.8	144.5	96.7	52.4	297.4	316.1	257.3	267.1	239.4	246.1	201.8	202	84.5	63.8
0.210	42.8	24.2	48.2	34.9	53.5	43.2	51.2	19.8	158.6	167.7	127.6	133	108.8	112.2	129	127.7	37.3	19.6
0.215	26.2	12	26.4	15.1	27	17.6	36.1	10.1	84	85.4	63.2	63.4	51.2	51.9	82.6	83.1	24.2	10.4
0.220	18.8	7.4	17.7	8.4	17.1	9.5	27.4	7.3	47.9	46.6	34.8	33.6	28.4	26.8	53.5	53.2	17.6	6.6
0.230	12	3.9	10.6	4	9.9	4.3	18.2	4.2	18.3	17.5	13.2	12	11	9.4	24.4	24.4	11.7	3.5
0.240	8.9	2.4	7.6	2.4	6.9	2.5	13.8	2.8	8.8	8	6.7	5.5	5.9	4.4	12.5	11.9	8.8	2.3
0.250	7.1	1.7	6	1.7	5.3	1.6	11.1	2	4.9	4.1	4	3	3.7	2.5	6.9	4.4	7	1.7
0.260	5.9	1.3	4.9	1.25	4.4	1.2	9.2	1.62	3.2	2.4	2.8	1.8	2.7	1.5	4.4	3.8	5.9	1.3
p_0=0.2	359	348.2	496.8	515.9	490.6	498.8	400.8	329.7	475.2	481.3	460.3	459	418.1	428.7	284.2	282.8	698.9	1747.9

Table 3. RL$_0$ for initial 10 percent of data ($p_0 = 0.1$, $n = 200$, $M = 10000$).

	Binomial EWMA			Binomial CUSUM	Moving Average			np Chart	Variable Limit np Control Chart
	$\lambda = 0.02$ $A = 2.127$	$\lambda = 0.05$ $A = 2.652$	$\lambda = 0.08$ $A = 2.765$	$U = 0$ $K = 0.2$ $H = 0.52$	$C = 3$ $W = 2$	$C = 3$ $W = 3$	$C = 3$ $W = 4$	$C = 3$	$l = 120$ $C = 1$ $B(l) = 0.5$
Min	18	17	7	43	1	1	1	1	9
Max	67	78	73	123	56	63	66	33	47
Average	43.63	48.3	35.4	89.2	29.4	34.8	37.1	16.5	33.9
Variance	193.6	533.8	288.3	506	316.9	373.9	407.5	92.7	99.6

deviations, compared to other considered methods in this research.

5. The standard deviation of RL$_1$ (SDRL$_1$) in the variable limit control chart is usually less than its value in other methods.

6. The proposed method is effective for the control of production processes, in which the recognition of small deviations from p_0 is important, such as in high tech processes.

7. In accordance with the gained control limits from Equations 17 and 18, it is clear that they are divergent, therefore, in the long run, our interval

Table 4. RL_0 for initial 10 percent of data ($p_0 = 0.2$, $n = 200$, $M = 10000$).

	Binomial EWMA			Binomial CUSUM	Moving Average			np Chart	Variable Limit np Control Chart
	$\lambda = 0.02$ $A = 2.127$	$\lambda = 0.05$ $A = 2.615$	$\lambda = 0.08$ $A = 2.765$	$U = 0$ $K = 0.2$ $H = 0.52$	$C = 3$ $W = 2$	$C = 3$ $W = 3$	$C = 3$ $W = 4$	$C = 3$	$l = 120$ $C = 1$ $B(l) = 0.5$
Min	12	10	9	35	1	1	1	1	6
Max	66	54	51	104	44	53	39	29	43
Average	42	37.9	35.4	76	21.2	22	22	14.2	29.1
Variance	223.5	101.9	101.4	338	185.9	231	139.4	61.7	98.5

Table 5. ARL sensitivity analysis for parameter $B(l)$ ($p_0 = 0.2$, $n = 200$, $M = 10000$).

p_1	$B(l) = 0.5$ $C = 1$; $l = 120$	$B(l) = 0.55$ $C = 1$; $l = 120$	$B(l) = 0.6$ $C = 1$; $l = 120$	$B(l) = 0.65$ $C = 1$; $l = 120$
0.14	5.9	6.4	6.9	7.4
0.15	7	7.7	8.2	8.9
0.16	8.7	9.6	10.3	11.1
0.17	11.7	12.7	13.8	14.8
0.18	17.5	19.2	20.7	22.3
0.185	23.6	26.5	28.1	30.3
0.19	37.9	42.3	46	49.1
0.195	79.4	85.7	94.4	101.2
0.205	84.5	75.9	66.5	59.9
0.21	37.3	34	31.4	27.2
0.215	24.2	22.1	19.8	17.6
0.22	17.6	16.2	14.5	12.6
0.23	11.7	10.5	9.3	8.5
0.24	8.8	8	7.2	6.4
0.25	7	6.5	5.7	5.1
0.26	5.9	5.4	4.8	4.4
$p_0 = 0.2$	698.9	558.5	565.5	504.5

will be wide and the ARL_1 of the control chart is increased. Consequently, it is applicable for evaluation of the initial setup of a process.

In general, using the new method will results a better ARL_0 despite showing a larger $SDRL_0$. Also, for small deviations from the target, the ARL_1 and $SDRL_1$ of this approach are better than other methods. Hence, the proposed method offers a better performance under certain conditions.

In the next stage, a sensitivity analysis is performed for the parameters of our model. There are six parameters in our model, namely, n, c, l, p_0, p_1 and

$B(l)$. Based on simulation results, sensitive parameters in our model are $B(l)$, c and l. The variations of other parameters are not effective. The sensitivity analysis for parameter $B(l)$ is given in Table 5. Appropriate values of c and l depend on process conditions and must be evaluated properly.

Table 5 demonstrates that when the value of $B(l)$ is increased, our method is more capable of recognizing an upward shift in the quantity of nonconforming ($p_1 > p_0$), and vice-versa. Also, for fair comparison, we used other good values for parameters of Binomial EWMA and Binomial CUSUM, the results of which are shown in Tables 6 and 7.

Table 6. Other simulation results for binomial EWMA and binomial CUSUM ($p_0 = 0.1$, $n = 200$, $M = 10000$).

p_1	Binomial EWMA					Binomial CUSUM			
	$\lambda = 0.02$ $A = 2.1275$	$\lambda = 0.05$ $A = 2.6520$	$\lambda = 0.08$ $A = 2.7650$	$\lambda = 0.1$ $A = 2.8155$	$\lambda = 0.2$ $A = 2.9$	$U = 0$ $K = 0.1$ $H = 0.39$	$U = 0.05$ $K = 0.1$ $H = 0.39$	$U = 0$ $K = 0.101$ $H = 0.3$	$U = 0.05$ $K = 0.101$ $H = 0.3$
	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
0.105	70.7	93.3	101.9	111	119.7	73.3	68.6	66.8	62.3
0.110	30.1	32	32.7	33.7	38.7	38.8	36.1	33.5	29.4
0.115	18.9	18.1	17.2	17.3	18.6	26.9	23.9	22	18.9
0.120	13.7	12.4	11.5	11.3	11.3	20	18.1	16.4	14.1
0.130	8.9	7.8	6.9	6.5	5.9	13.9	12.2	11.1	9.4
0.140	6.6	5.7	5	4.7	4	10.5	9.2	8.3	7
0.150	5.3	4.5	3.9	3.7	3.1	8.3	7.5	7.4	5.8
0.160	4.5	3.8	3.3	3.1	2.5	7.2	6.3	5.7	4.9
$p_0 = 0.1$	390.1	585.3	537.7	531	443	416	395.5	415	402

Table 7. Other simulation results for binomial EWMA and binomial CUSUM ($p_0 = 0.2$, $n = 200$, $M = 10000$).

p_1	Binomial EWMA					Binomial CUSUM			
	$\lambda = 0.02$ $A = 2.1275$	$\lambda = 0.05$ $A = 2.6150$	$\lambda = 0.08$ $A = 2.7650$	$\lambda = 0.1$ $A = 2.84$	$\lambda = 0.2$ $A = 2.9$	$U = 0$ $K = 0.2$ $H = 0.52$	$U = 0.05$ $K = 0.2$ $H = 0.52$	$U = 0$ $K = 0.201$ $H = 0.45$	$U = 0.05$ $K = 0.201$ $H = 0.45$
	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
0.205	100.8	133.4	154.8	170.4	175.4	96.7	90.9	95.8	91.5
0.210	42.8	48.2	53.5	58.9	67.4	51.2	49.2	49	45.4
0.215	26.2	26.4	27	28.5	31.6	36.1	33.1	32.6	29.9
0.220	18.8	17.7	17.1	17.6	18.1	27.4	25	24.4	22.2
0.230	12	10.6	9.9	9.5	8.9	18.2	16.8	16.4	14.6
0.240	8.9	7.6	6.9	6.6	5.8	13.8	12.5	12.3	11
0.250	7.1	6	5.3	5.1	4.4	11.1	10.2	9.9	8.8
0.260	5.9	4.9	4.4	4.2	3.4	9.2	8.5	8.3	7.5
$p_0 = 0.2$	359	496.8	490.6	508	391	400.8	385.6	430.4	427.3

Also, for low values of p_1 that are related to high tech processes, Table 8 demonstrates that our method is superior to others.

As clear from Tables 6 to 8, the results of ARL_1 related to the proposed control chart are better than the best results of Binomial EWMA and Binomial CUSUM.

NUMERICAL EXAMPLE

In this section, we describe our methodology step by step as follows.

1. Input l; $B(l)$; c; n; p_0 and p_1,
2. For $k = 0 : m$,
3. Calculate LCL_{k+l} and UCL_{k+l},
4. End,
5. Set $Ln(z_l) = B(l)(LCL_l + UCL_l)$,
6. For $k = 1 : m$,
7. Determine X_k and $Ln(z_{k+l})$,
8. If $LCL_{k+l} \leq Ln(z_{k+l}) \leq UCL_{k+l}$,
9. The process is in-control,
10. Else,
11. The process is out-of-control and makes an alarm,
12. Check the process and perform corrective action,
13. Go to step 3,
14. End,
15. End.

Table 8. Simulation results for binomial EWMA, binomial CUSUM and variable limit np control chart ($p_0 = 0.01$, $n = 500$, $M = 10000$).

p_1	Binomial EWMA			Variable Limit np Control Chart		Binomial CUSUM					
	$\lambda = 0.02$ $A = 2.1275$	$\lambda = 0.005$ $A = 1.4800$	$\lambda = 0.001$ $A = 0.800$	$c = 0.7$ $l = 350$ $B(l) = 0.5$	$c = 0.7$ $l = 400$ $B(l) = 0.65$	$U = 0$ $K = 0.01$ $H = 0.085$	$U = 0.02$ $K = 0.01$ $H = 0.085$	$U = 0$ $K = 0.0105$ $H = 0.048$	$U = 0.02$ $K = 0.0105$ $H = 0.048$	$U = 0$ $K = 0.0108$ $H = 0.038$	$U = 0.015$ $K = 0.0108$ $H = 0.038$
	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
0.0105	160.6	158.3	167.9	127.4	92.6	140.8	124.3	134.8	113.1	142.3	126.4
0.0110	75.1	78	85.7	63.4	47	79.5	67.1	68.5	52.2	68.8	56.5
0.0115	44.8	51	56.6	43.4	30	56.1	45.3	43.7	30.2	42	32.2
0.0120	31.9	37.4	42.4	31	26.3	42.8	34.4	30.8	20.9	30	20.9
0.0125	24.5	29.6	34	25.8	19.8	34.7	27.6	24.3	15.7	21.8	15.5
0.0130	19.9	24.5	28.3	21.1	15.3	28.9	23.1	20	12.8	17.5	12.1
0.0135	16.8	20.7	24.4	16.5	14.7	25.1	19.8	16.8	11	14.9	10
0.0140	14.5	18.2	21.3	15	11.8	22	17.3	14.6	9.2	12.6	8.4
$p_0=0.01$	406	399	397.6	411	407.8	415.4	394	418.3	394	410.4	391.4

Table 9. Obtained values for LCL_{k+l}, UCL_{k+l}, X_k and $Ln(z_{k+l})$ for mentioned case study when process is in-control.

$l = 120$; $B(l) = 0.5$; $c = 1$; $n = 200$; $p_0 = 0.1$; $p_1 = 0.12$										
k	0	1	2	3	4	5	6	7	8	9
LCL_{k+l}	-15.0	-15.1	-15.3	-15.4	-15.5	-15.7	-15.8	-15.9	-16.0	-16.2
UCL_{k+l}	-6.0	-6.1	-6.1	-6.2	-6.3	-6.4	-6.5	-6.6	-6.6	-6.7
X_k	-	20	16	15	22	15	23	18	20	15
$Ln(z_{k+l})$	-10.5	-10.6	-11.1	-11.8	-11.7	-12.3	-12.1	-12.4	-12.5	-13.1

In the above algorithm, m is the number of subgroups that will be observed from the process. As mentioned before, the best values for l and c are 120 and 1, respectively. For example, assume that $l = 120$, $c = 1$, $n = 200$, $p_0 = 0.1$ and $p_1 = 0.12$. Also, assume that the number of subgroups, m, is 10. Obtained values for LCL_{k+l}, UCL_{k+l}, X_k and $Ln(z_{k+l})$ are shown in Table 9.

The drawing of control limits for the above information is shown in Figure 3. As observed from Table 9 and Figure 3, at first the control limits are divergent and then they have a negative gradient.

CONCLUSIONS

In this research, we applied an initial belief to detect the out-of-control state in the np attribute control charts, and since this approach analyzes data sequentially, it has demonstrated a very good performance. It has been found that, in general, the variable limit control np charts are able to improve the effectiveness of detecting shifts in p_0 to a substantial degree, especially for the small shifts in p_0, without increasing the false alarm rate. Although the cost of running variable limit control np charts is relatively high, the
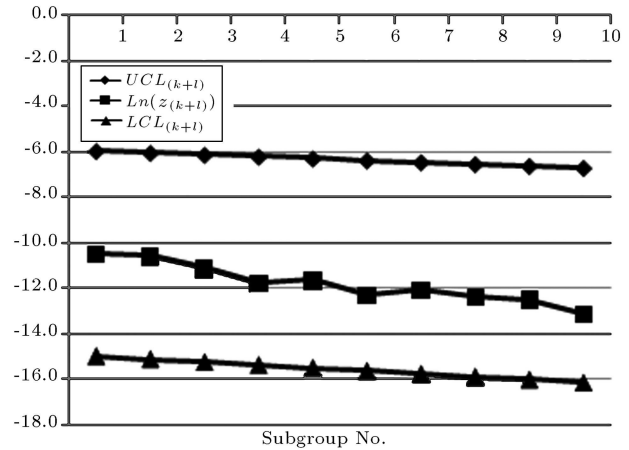

Figure 3. Control limits for $Ln(z_{k+l})$ based on information shown in Table 9.

use of these charts can be justified by the significant improvement in their performance. In general, the proposed method yields a significant improvement in ARL_0 and, for small deviations of the process, it improves ARL_1.

For future research, we propose considering other functions to define the beliefs and economic design of parameters using this approach.

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