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The Adomian-Tau Method for Solving a System of Non-linear Differential Equations

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Abstract.   *In this paper, we study a new simple method to find a numerical solution for the System of Non-linear Differential Equations (SNDE). The method is based on a matrix form of SNDE and we use computational aspects of special matrices to find the expansion of the unknown function. Finally, some numerical examples illustrate the accuracy of the method.*

Keywords: *Differential equations; Matrix forms; Numerical solutions.*

INTRODUCTION

Many problems of physics and mathematics are reduced to solving non-linear systems of differential equations. Solving these systems analytically is difficult, so many classical methods for the numerical solution of such systems has been developed, such as: different versions of the Runge-Kutta method, multi-step Adam's methods, spectral or pseudo spectral methods, the operational approach of the Tau method and the Adomian decomposition method [1-7]. Most of these methods lead to solving system of non-linear algebraic equations with high computational costs. Our method in this paper, which reduces computational costs, is a simple operational approach along with the Adomian decomposition method, lead to a system of linear algebraic equations that is solved simply. It must be mentioned that the Adomian decomposition method plays an important role in this paper, since by using this method we convert the non-linear terms of SNDE to the polynomials in terms of independent variables, and then use the operational matrices of the Tau method [1] to convert the SNDE to the system of algebraic equations.

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SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS

Consider a non-linear system of differential equations of the form:

$$y_i'(x) = f_i(x, y_1, y_2, \cdots, y_r),$$

$$i = 1, 2, \cdots, r, \qquad x \in [0, a], \tag{1}$$

with the initial conditions:

$$y_i(0) = \alpha_i, \qquad i = 1, 2, \cdots, r, \tag{2}$$

where $f_i(x, y_1, y_2, \cdots, y_r)$, $i = 1, 2, \cdots, r$ denote the non-linear terms of SNDE. Assume that $y_{in}(x)$ is a polynomial approximation of degree n for $y_i(x)$, then, one can write:

$$y_{in}(x) = \sum_{j=0}^{n} a_{ij} x^j = \underline{\mathbf{a}}_i \underline{\underline{\mathbf{X}}}, \tag{3}$$

where $\underline{\mathbf{a}}_i = [a_{i0}, a_{i1}, \cdots, a_{in}, 0, \cdots]$ and $\underline{\underline{\mathbf{X}}} = [1, x, x^2, \cdots]^T$.

CONVERTING SNDE TO THE SYSTEM OF ALGEBRAIC EQUATIONS

The effect of differentiation or shifting on coefficients $\underline{\mathbf{p}}_n = [p_0, p_1, \cdots, p_n, 0, 0, \cdots]$ of polynomial $p_n(x) = \underline{\mathbf{p}}_n \underline{\underline{\mathbf{X}}}$ is the same as that of the post-multiplication of

$\underline{\mathbf{p}}_n$ by either matrix η or matrix μ, defined by:

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & \vdots & \\ & & & 0 & & \\ & & \cdots & & & \ddots \end{bmatrix},$$

and:

$$\eta = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 2 & 0 & & \vdots & \\ 0 & 0 & 3 & 0 & & \\ & & \cdots & & & \ddots \end{bmatrix}.$$

Lemma 1

Let $p_n(x)$ be a polynomial of the form:

$$p_n(x) = \sum_{i=0}^{n} p_i x^i = \underline{\mathbf{p}}_n \underline{\underline{\mathbf{X}}}.$$

Then:

i) $\dfrac{d^k}{dx^k} p_n(x) = \underline{\mathbf{p}}_n \eta^k \underline{\underline{\mathbf{X}}}, \qquad k = 0, 1, 2, \cdots,$

ii) $x^k p_n(x) = \underline{\mathbf{p}}_n \mu^k \underline{\underline{\mathbf{X}}}, \qquad k = 0, 1, 2, \cdots.$

Proof: see [1].

By using Lemma 1, one can write:

$$y_i'(x) = \underline{\mathbf{a}}_i \eta \underline{\underline{\mathbf{X}}}, \qquad i = 1, 2, \cdots, r. \tag{4}$$

We now use the Adomian decomposition (see [2]), to simplify the non-linear term of Equations 1.

By setting $\hat{f}_i(x) = f_i(x, y_1, y_2, \cdots, y_r)$, and substituting $y_i(x) = \sum_{j=0}^{\infty} a_{ij} x^j$, we get:

$$\hat{f}_i(x) = f_i\left(x, \sum_{j=0}^{\infty} a_{1j} x^j, \sum_{j=0}^{\infty} a_{2j} x^j, \cdots, \sum_{j=0}^{\infty} a_{rj} x^j \right)$$

$$= \sum_{j=0}^{\infty} A_j^{f_i} x^j = \underline{\mathbf{A}}^{f_i} \underline{\underline{\mathbf{X}}}, \qquad i = 1, 2, \cdots, r, \tag{5}$$

where $\underline{A}^{f_i} = [A_0^{f_i}, A_1^{f_i}, \cdots]$, with:

$$A_k^{f_i} = \frac{1}{k!} \left\{ \frac{d^k}{dx^k} f_i\left(x, \sum_{j=0}^{\infty} a_{1j} x^j, \sum_{j=0}^{\infty} a_{2j} x^j, \right. \right.$$

$$\left. \left. \cdots, \sum_{j=0}^{\infty} a_{rj} x^j \right) \right\}_{x=0} = \frac{\hat{f}_i^{(k)}(0)}{k!},$$

$i = 1, 2, \cdots, r, \quad k = 0, 1, \cdots$

which depends on:

$$a_{10}, a_{11}, \cdots, a_{1k}, \cdots, a_{r0}, a_{r1}, \cdots, a_{rk},$$

for $k = 0, 1, \cdots$.

From Relations 4 and 5 the matrix form of Equations 1 can be written as:

$$\underline{\mathbf{a}}_i \eta \underline{\underline{\mathbf{X}}} = \underline{\mathbf{A}}^{f_i} \underline{\underline{\mathbf{X}}}, \qquad i = 1, 2, \cdots, r,$$

which yields:

$$\underline{\mathbf{a}}_i \eta = \underline{\mathbf{A}}^{f_i}, \qquad i = 1, 2, \cdots, r, \tag{6}$$

since $\underline{\underline{\mathbf{X}}}$ is a base vector. Consequently the unknown coefficients in Relation 3 can be determined from Conditions 2 and Relation 6. In fact, we use Conditions 2 to write:

$$a_{i0} = \alpha_i, \qquad i = 1, 2, \cdots, r,$$

and determine other coefficients by forward substituting from the following systems:

$$\begin{cases} a_{1j} = \dfrac{A_{j-1}^{f_1}}{j} \\ a_{2j} = \dfrac{A_{j-1}^{f_2}}{j} \\ \vdots \\ a_{rj} = \dfrac{A_{j-1}^{f_r}}{j} \end{cases}$$

for $j = 1, \cdots, n$.

ERROR ESTIMATION

In this section, an error estimator is obtained for the approximate solution of Equations 1 and 2. Let us call $e_{in}(x) = y_i(x) - y_{in}(x)$ as the error function of the approximate solution $y_{in}(x)$ to $y_i(x)$, where $y_i(x)$ is the exact solution of Equations 1 and 2. Hence, $y_{in}(x)$ satisfies in:

$$y_{in}'(x) = f_i(x, y_{1n}, y_{2n}, \cdots, y_{rn}) + H_{in}(x),$$

$$i = 1, 2, \cdots, r, \quad x \in [0, a], \tag{7}$$

with the initial conditions:

$$y_{in}(0) = \alpha_i, \qquad i = 1, 2, \cdots, r. \tag{8}$$

The perturbation term, $H_{in}(x)$, can be obtained by substituting the computed solution, $y_{in}(x)$, into the following equation:

$$H_{in}(x) = y_{in}'(x) - f_i(x, y_{1n}, y_{2n}, \cdots, y_{rn}),$$

$$i = 1, 2, \cdots, r, \qquad x \in [0, a]. \tag{9}$$

Subtracting Equations 7 and 8 from Equations 1 and 2, respectively, and taking two terms of expansion, $f_i(x, y_1, y_2, \cdots, y_r)$ around $y_{1n}, y_{2n}, \cdots, y_{rn}$, the error function, $e_{in}(x)$, satisfies the problem:

$$e'_{in}(x) = \sum_{j=1}^{r} e_{jn}(x) \frac{\partial f_i(x, y_{1n}, y_{2n}, \cdots, y_{rn})}{\partial y_{jn}} - H_{in}(x),$$

$$i = 1, 2, \cdots, r, \quad x \in [0, a], \tag{10}$$

with:

$$e_{in}(0) = 0, \qquad i = 1, 2, \cdots, r. \tag{11}$$

We proceed to find an approximation, $e_{in,N}(x)$, to the error function, $e_{in}(x)$, in the same way as we did before for the solution of problem Equations 1 and 2.

It should be noted that in order to construct the approximation $e_{in,N}(x)$ to $e_{in}(x)$, only the right hand side of system (Relation 6) needs to be recomputed.

NUMERICAL EXAMPLES

In this section, we illustrate the accuracy of the presented method by computing $e_{in,N}(x)$ (estimation) and absolute errors for various numerical examples. Tables 1 to 5 show these errors for Examples 1 to 4 with $n = 5, 10, 15$. Note that all of the presented results are obtained by programming in the maple 8 software environment.

Example 1

$$\begin{cases} y'_1(x) = y_2(x) e^{y_1^2(x) + y_2^2(x) - 1}, \\ y'_2(x) = -\frac{y_1(x) y_2(x)}{\cos x}, \\ y_1(0) = 0, \quad y_2(0) = 1, \quad 0 \le x \le \frac{\pi}{2}. \end{cases}$$

The exact solutions are:

$$y_1(x) = \sin x,$$

$$y_2(x) = \cos x.$$

Example 2

$$\begin{cases} y'_1(x) = \sin^2(x) + y_2^2(x) - 1 - e^x y_1^2(x), \\ y'_2(x) = -e^x \tan x y_1(x) y_2(x), \\ y_1(0) = y_2(0) = 1, \quad 0 \le x \le \frac{\pi}{2}. \end{cases}$$

The exact solutions are:

Table 1. Absolute and estimation errors for Example 1.

n	x	Exacty_1	App.y_1	App.Err.y_1	Est.Err.y_1	Exacty_2	App.y_2	App.Err.y_2	Est.Err.y_2
	0.00	0.000000	0.000000	0	0	1.000000	1.000000	0	0
	0.31	0.309017	0.309017	5.984438e-08	5.992645e-08	0.951057	0.951058	1.332912e-06	1.335263e-06
5	0.63	0.587785	0.587793	7.628678e-06	7.670586e-06	0.809017	0.809102	8.485701e-05	8.545682e-05
	0.94	0.809017	0.809146	1.294553e-04	1.310592e-04	0.587785	0.588743	9.581179e-04	9.734066e-04
	1.26	0.951057	0.952017	9.606070e-04	9.818350e-04	0.309017	0.314335	5.317684e-03	5.469236e-03
	1.57	1.000000	1.004525	4.524856e-03	4.681754e-03	0.000000	0.019969	1.996896e-02	2.086348e-02
	0.00	0.000000	0.000000	0	0	1.000000	1.000000	0	0
	0.31	0.309017	0.309017	7.365770e-14	7.370431e-14	0.951057	0.951057	1.928528e-15	1.929574e-15
10	0.63	0.587785	0.587785	1.505651e-10	1.509464e-10	0.809017	0.809017	7.886421e-12	7.903536e-12
	0.94	0.809017	0.809017	1.298247e-08	1.305650e-08	0.587785	0.587785	1.020469e-09	1.025455e-09
	1.26	0.951057	0.951057	3.060324e-07	3.091383e-07	0.309017	0.309017	3.209384e-08	3.237288e-08
	1.57	1.000000	1.000004	3.542584e-06	3.598843e-06	0.000000	0.000000	4.647660e-07	4.710875e-07
	0.00	0.000000	0.000000	0.000000e-01	0	1.000000	1.000000	0	0
	0.31	0.309017	0.309017	7.949760e-24	7.952054e-24	0.951057	0.951057	4.301682e-22	4.303070e-22
15	0.63	0.587785	0.587785	1.041090e-18	1.042292e-18	0.809017	0.809017	2.816425e-17	2.820060e-17
	0.94	0.809017	0.809017	1.024268e-15	1.026930e-15	0.587785	0.587785	1.846966e-14	1.852330e-14
	1.26	0.951057	0.951057	1.359868e-13	1.366152e-13	0.309017	0.309017	1.838656e-12	1.848154e-12
	1.57	1.000000	1.000000	6.023421e-12	6.066936e-12	0.000000	0.000000	6.513361e-11	6.565963e-11

Table 2. Absolute and estimation errors for Example 2.

n	x	Exacty_1	App.y_1	App.Err.y_1	Est.Err.y_1	Exacty_2	App.y_2	App.Err.y_2	Est.Err.y_2
	0.00	1.000000	1.000000	0	0	1.000000	1.000000	0	0
	0.31	0.730403	0.730401	1.277610e-06	1.335263e-06	0.951057	0.951058	1.332912e-06	1.335263e-06
5	0.63	0.533488	0.533410	7.834912e-05	8.545682e-05	0.809017	0.809102	8.485701e-05	8.545682e-05
	0.94	0.389661	0.388805	8.563109e-04	9.734066e-04	0.587785	0.588743	9.581179e-04	9.734066e-04
	1.26	0.284610	0.279987	4.622520e-03	5.469236e-03	0.309017	0.314335	5.317684e-03	5.469236e-03
	1.57	0.207880	0.190917	1.696257e-02	2.086348e-02	0.000000	0.019969	1.996896e-02	2.086348e-02
	0.00	1.000000	1.000000	0	0 1	1.000000	1.000000	0	0
	0.31	0.730403	0.730403	7.182034e-14	7.370431e-14	0.951057	0.951057	1.928528e-15	1.929574e-15
10	0.63	0.533488	0.533488	1.434084e-10	1.509464e-10	0.809017	0.809017	7.886421e-12	7.903536e-12
	0.94	0.389661	0.389661	1.210068e-08	1.305650e-08	0.587785	0.587785	1.020469e-09	1.025455e-09
	1.26	0.284610	0.284610	2.796356e-07	3.091383e-07	0.309017	0.309017	3.209384e-08	3.237288e-08
	1.57	0.207880	0.207883	3.178900e-06	3.598843e-06	0.000000	0.000000	4.647660e-07	4.710875e-07
	0.00	1.000000	1.000000	0	0	1.000000	1.000000	0	0
	0.31	0.730403	0.730403	4.224914e-22	4.303070e-22	0.951057	0.951057	4.301682e-22	4.303070e-22
15	0.63	0.533488	0.533488	2.719352e-17	2.820060e-17	0.809017	0.809017	2.816425e-17	2.820060e-17
	0.94	0.389661	0.389661	1.754760e-14	1.852330e-14	0.587785	0.587785	1.846966e-14	1.852330e-14
	1.26	0.284610	0.284610	1.720483e-12	1.848154e-12	0.309017	0.309017	1.838656e-12	1.848154e-12
	1.57	0.207880	0.207880	6.008156e-11	6.565963e-11	0.000000	0.000000	6.513361e-11	6.565963e-11

$$y_1(x) = e^{-x}, \qquad y_2(x) = \cos x.$$

Example 3 [3,4]

$$\begin{cases} y_1'(x) = -20y_1(x) - 0.25y_2(x) - 19.75y_3(x), \\ y_2'(x) = 20y_1(x) - 20.25y_2(x) + 0.25y_3(x), \\ y_3'(x) = 20y_1(x) - 19.75y_2(x) - 0.25y_3(x), \\ y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = -1. \end{cases}$$

The exact solutions are

$$\begin{cases} y_1(x) = \frac{1}{2}\left[e^{-\frac{1}{2}x} + e^{-20x}(\cos(20x) + \sin(20x))\right], \\ y_2(x) = \frac{1}{2}\left[e^{-\frac{1}{2}x} - e^{-20x}(\cos(20x) - \sin(20x))\right], \\ y_3(x) = -\frac{1}{2}\left[e^{-\frac{1}{2}x} + e^{-20x}(\cos(20x) - \sin(20x))\right]. \end{cases}$$

Example 4 [3,4]

$$\begin{cases} y_1'(x) = -y_1(x), \\ y_2'(x) = y_1(x) - y_2^2(x), \\ y_3'(x) = y_2^2(x), \\ y_1(0) = 1, \quad y_2(0) = y_3(0) = 0. \end{cases}$$

The exact solutions are:

$$\begin{cases} y_1(x) = e^{-x} \\ y_2(x) = x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{5}{4!}x^4 + \frac{3}{5!}x^5 - \frac{71}{6!}x^6 \\ \qquad + \frac{95}{7!}x^7 - \frac{1469}{8!}x^8 - \frac{6909}{9!}x^9 - \frac{35311}{10!}x^{10} \\ \qquad + \frac{440023}{11!}x^{11} + \frac{384933}{12!}x^{12} - \frac{30642437}{13!}x^{13} \\ \qquad + \frac{99642009}{14}x^{14} + \frac{2280282799}{15!}x^{15} \\ \qquad - \frac{21150331603}{16!}x^{16} + \cdots \\ y_3(x) = \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{2}{5!}x^5 + \frac{70}{6!}x^6 - \frac{94}{7!}x^7 \\ \qquad - \frac{1470}{8!}x^8 + \frac{6910}{9!}x^9 - \frac{35310}{10!}x^{10} \\ \qquad - \frac{440022}{11!}x^{11} + \frac{384934}{12!}x^{12} + \frac{30642438}{13!}x^{13} \\ \qquad - \frac{99642010}{14}x^{14} - \frac{2280282798}{15!}x^{15} \\ \qquad + \frac{21150331602}{16!}x^{16} + \cdots \end{cases}$$

For $n = 5, 10, 15$, we achieved the following approximate solutions:

Table 3. Absolute and estimation errors for Example 3.

		$n = 5$			$n = 10$		
x	$\mathbf{Exact}y_1$	$\mathbf{App.}y_1$	$\mathbf{App.Err.}y_1$	$\mathbf{Est.Err.}y_1$	$\mathbf{App.}y_1$	$\mathbf{App.Err.}y_1$	$\mathbf{Est.Err.}y_1$
0.000	1.000000	1.000000	0	0	1.000000	0	0
0.002	0.998721	0.998721	2.249679e-11	2.275556e-11	0.998721	3.351229e-22	3.362437e-22
0.004	0.995968	0.995968	1.423400e-09	1.456356e-09	0.995968	6.840366e-19	6.886271e-19
0.006	0.991861	0.991861	1.602857e-08	1.658880e-08	0.991861	5.896901e-17	5.956456e-17
0.008	0.986515	0.986515	8.903105e-08	9.320676e-08	0.986515	1.391509e-15	1.410308e-15
0.010	0.980040	0.980040	3.357456e-07	3.555556e-07	0.980040	1.614463e-14	1.641815e-14
x	$\mathbf{Exact}y_2$	$\mathbf{App.}y_2$	$\mathbf{App.Err.}y_2$	$\mathbf{Est.Err.}y_2$	$\mathbf{App.}y_2$	$\mathbf{App.Err.}y_2$	$\mathbf{Est.Err.}y_2$
0.000	0.000000	0.000000	0	0	0.000000	0	0
0.002	0.038700	0.038700	2.275427e-11	2.275556e-11	0.038700	1.113935e-24	1.252605e-41
0.004	0.075804	0.075804	1.456029e-09	1.456356e-09	0.075804	4.534668e-21	2.565336e-38
0.006	0.111319	0.111319	1.658049e-08	1.658880e-08	0.111319	5.847433e-19	2.218953e-36
0.008	0.145255	0.145255	9.312452e-08	9.320676e-08	0.145255	1.834653e-17	5.253808e-35
0.010	0.177629	0.177629	3.550697e-07	3.555556e-07	0.177629	2.653365e-16	6.116237e-34
x	$\mathbf{Exact}y_3$	$\mathbf{App.}y_3$	$\mathbf{App.Err.}y_3$	$\mathbf{Est.Err.}y_3$	$\mathbf{App.}y_3$	$\mathbf{App.Err.}y_3$	$\mathbf{Est.Err.}y_3$
0.000	-1.000000	-1.000000	0	0	-1.000000	0	0
0.002	-0.960300	-0.960300	2.275427e-11	2.275556e-11	-0.960300	1.113935e-24	1.252605e-41
0.004	-0.922198	-0.922198	1.456029e-09	1.456356e-09	-0.922198	4.534668e-21	2.565336e-38
0.006	-0.885686	-0.885686	1.658049e-08	1.658880e-08	-0.885686	5.847433e-19	2.218953e-36
0.008	-0.850753	-0.850753	9.312452e-08	9.320676e-08	-0.850753	1.834653e-17	5.253808e-35
0.010	-0.817383	-0.817384	3.550697e-07	3.555556e-07	-0.817383	2.653365e-16	6.116237e-34

		$n = 15$		
x	$\mathbf{Exact}y_1$	$\mathbf{App.}y_1$	$\mathbf{App.Err.}y_1$	$\mathbf{Est.Err.}y_1$
0.000	1.000000	1.000000	0	0
0.002	0.998721	0.998721	2.627518e-34	2.627545e-34
0.004	0.995968	0.995968	1.721917e-29	1.721988e-29
0.006	0.991861	0.991861	1.130967e-26	1.131072e-26
0.008	0.986515	0.986515	1.128337e-24	1.128522e-24
0.010	0.980040	0.980040	4.008290e-23	4.009316e-23
x	$\mathbf{Exact}y_2$	$\mathbf{App.}y_2$	$\mathbf{App.Err.}y_2$	$\mathbf{Est.Err.}y_2$
0.000	0.000000	0.000000	0	0
0.002	0.038700	0.038700	2.615208e-34	2.627545e-34
0.004	0.075804	0.075804	1.705853e-29	1.721988e-29
0.006	0.111319	0.111319	1.115211e-26	1.131072e-26
0.008	0.145255	0.145255	1.107468e-24	1.128522e-24
0.010	0.177629	0.177629	3.916027e-23	4.009316e-23
x	$\mathbf{Exact}y_3$	$\mathbf{App.}y_3$	$\mathbf{App.Err.}y_3$	$\mathbf{Est.Err.}y_3$
0.000	-1.000000	-1.000000	0	0
0.002	-0.960300	-0.960300	2.615207e-34	2.627545e-34
0.004	-0.922198	-0.922198	1.705853e-29	1.721988e-29
0.006	-0.885686	-0.885686	1.115211e-26	1.131072e-26
0.008	-0.850753	-0.850753	1.107468e-24	1.128522e-24
0.010	-0.817383	-0.817383	3.916027e-23	4.009316e-23

Table 4. Estimation error for Example 4.

n	x	Est.Err.y_1	Est.Err.y_2	Est.Err.y_3
	0.00	0	0	0
	0.02	8.888889e-14	6.311111e-12	6.222222e-12
5	0.04	5.688889e-12	4.039111e-10	3.982222e-10
	0.06	6.480000e-11	4.600800e-09	4.536000e-09
	0.08	3.640889e-10	2.585031e-08	2.548622e-08
	0.10	1.388889e-09	9.861111e-08	9.722222e-08
	0.00	0	0	0
	0.02	5.130672e-27	2.257614e-21	2.257608e-21
10	0.04	1.050762e-23	4.623593e-18	4.623582e-18
	0.06	9.088831e-22	3.999295e-16	3.999286e-16
	0.08	2.151960e-20	9.469118e-15	9.469096e-15
	0.10	2.505211e-19	1.102350e-13	1.102348e-13
	0.00	0	0	0
	0.02	3.132278e-41	6.624872e-31	6.624872e-31
15	0.04	2.052770e-36	4.341676e-26	4.341676e-26
	0.06	1.348343e-33	2.851790e-23	2.851790e-23
	0.08	1.345303e-31	2.845361e-21	2.845361e-21
	0.10	4.779477e-30	1.010875e-19	1.010875e-19

Table 5. Estimation error for Example 4.

n	x	Est.Err.y_1	Est.Err.y_2	Est.Err.y_3
	0.00	0	0	0
	0.20	8.888889e-08	6.311111e-06	6.222222e-06
5	0.40	5.688889e-06	4.039111e-04	3.982222e-04
	0.60	6.480000e-05	4.600800e-03	4.536000e-03
	0.80	3.640889e-04	2.585031e-02	2.548622e-02
	1.00	1.388889e-03	9.861111e-02	9.722222e-02
	0.00	0	0	0
	0.20	5.130672e-16	2.257614e-10	2.257608e-10
10	0.40	1.050762e-12	4.623593e-07	4.623582e-07
	0.60	9.088831e-11	3.999295e-05	3.999286e-05
	0.80	2.151960e-09	9.469118e-04	9.469096e-04
	1.00	2.505211e-08	1.102350e-02	1.102348e-02
	0.00	0	0	0
	0.20	3.132278e-25	6.624872e-15	6.624872e-15
15	0.40	2.052770e-20	4.341676e-10	4.341676e-10
	0.60	1.348343e-17	2.851790e-07	2.851790e-07
	0.80	1.345303e-15	2.845361e-05	2.845361e-05
	1.00	4.779477e-14	1.010875e-03	1.010875e-03

$$
\begin{cases}
n = 5 \\
y_{1,5}(x) = 1 - x + \frac{x^2}{!2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \\
y_{2,5}(x) = x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{5}{4!}x^4 + \frac{3}{5!}x^5 \\
y_{3,5}(x) = \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{2}{5!}x^5
\end{cases}
$$

$$
\begin{cases}
n = 10 \\
y_{1,10}(x) = 1 - x + \frac{x^2}{!2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \\
\qquad - \frac{x^7}{7!} + \frac{x^8}{8!} - \frac{x^9}{9!} + \frac{x^{10}}{10!} \\
y_{2,10}(x) = x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{5}{4!}x^4 + \frac{3}{5!}x^5 - \frac{71}{6!}x^6 \\
\qquad + \frac{95}{7!}x^7 - \frac{1469}{8!}x^8 - \frac{6909}{9!}x^9 - \frac{35311}{10!}x^{10} \\
y_{3,10}(x) = \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{2}{5!}x^5 + \frac{70}{6!}x^6 - \frac{94}{7!}x^7 \\
\qquad - \frac{1470}{8!}x^8 + \frac{6910}{9!}x^9 - \frac{35310}{10!}x^{10}
\end{cases}
$$

$$
\begin{cases}
n = 15 \\
y_{1,15}(x) = 1 - x + \frac{x^2}{!2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \\
\qquad - \frac{x^7}{7!} + \frac{x^8}{8!} - \frac{x^9}{9!} + \frac{x^{10}}{10!} - \frac{x^{11}}{11!} + \frac{x^{12}}{12!} \\
\qquad - \frac{x^{13}}{13!} + \frac{x^{14}}{14!} - \frac{x^{15}}{15!} \\
y_{2,15}(x) = x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{5}{4!}x^4 + \frac{3}{5!}x^5 - \frac{71}{6!}x^6 \\
\qquad + \frac{95}{7!}x^7 - \frac{1469}{8!}x^8 - \frac{6909}{9!}x^9 - \frac{35311}{10!}x^{10} \\
\qquad + \frac{440023}{11!}x^{11} + \frac{384933}{12!}x^{12} - \frac{30642437}{13!}^{13} \\
\qquad + \frac{99642009}{14!}x^{14} + \frac{2280282799}{15!}x^{15} \\
y_{3,15}(x) = \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{2}{5!}x^5 + \frac{70}{6!}x^6 - \frac{94}{7!}x^7 \\
\qquad - \frac{1470}{8!}x^8 + \frac{6910}{9!}x^9 - \frac{35310}{10!}x^{10} \\
\qquad - \frac{440022}{11!}x^{11} + \frac{384934}{12!}x^{12} + \frac{30642438}{13!}x^{13} \\
\qquad - \frac{99642010}{14!}x^{14} - \frac{2280282798}{15!}x^{15}
\end{cases}
$$

For comparing the numerical results of the presented method with other methods, see Tables 4 and 5.

Remark 1

Note that in Tables 1–4, the notations Exacty_i, App.y_i, Abs.Err.y_i and Est.Err.y_i ($i = 1, 2$ or 3) have been used for the exact solution, approximate solution, absolute and estimate errors respectively.

CONCLUSION

In this paper, we have solved a problem in a general form, which is important in practice. We have also designed a remarkably simple method which has high accuracy for solving these types of problem, and clarified accuracy by solving numerical examples (see Tables 1 to 5).

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BIOGRAPHIES

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