

# Eigenproblems of Symmetric Planar Frames

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**Abstract.** *In this paper, the graph models of planar frame structures with different symmetries are decomposed and appropriate processes are designed for their healing in order to form the corresponding factors. The eigenvalues and eigenvectors of the entire structure are then obtained by evaluating those of its factors. The methods developed in this article simplify calculation of the natural frequencies and natural modes of the planar frames with different types of symmetry.*

**Keywords:** *Symmetry; Planar frames; Decomposition; Graph factors; Eigenvalues; Natural frequencies.*

## INTRODUCTION

Symmetry has been widely studied in science and engineering [1-5]. Large eigenvalue problems arise in many scientific and engineering problems [6-8]. While the basic mathematical ideas are independent of the size of matrices, the numerical determination of eigenvalues and eigenvectors becomes more complicated as the dimensions and sparsity of the matrices increase. General methods were developed for the solution of large-scale problems in electrical networks as early as 1957 by Kron [9] and further applied to structures by Prezemieniecki [10]. The substructuring methods have also been extended to the eigensolution of structures [11]. Though these methods are powerful tools, special methods are needed for the efficient solution of those structural models which have different kinds of symmetry.

Methods are developed for decomposing the symmetric graph models of structures in order to calculate the eigenvalues of matrices with special patterns [12-15]. The application of these methods is extended to the free vibration of mass-spring systems [16] and the free and forced vibration of frame structures [17,18]. Symmetry is also employed in the stability analysis of frames [19,20].

In this paper, graph models are associated with frame structures. Decomposition approaches are em-

ployed to form matrices of special patterns for such systems. The problem of finding eigenvalues and eigenvectors of symmetric frames is transferred into calculating those of their factors. The factors of a symmetric model are obtained by a decomposition followed by a new healing process. This results in efficient methods for evaluating the natural frequencies and natural modes of symmetric frames. Different forms of the symmetry are accompanied by simple illustrative examples.

The application of the present method becomes more apparent when the dynamic behavior of large-scale structures is studied. The concepts and methods introduced in this paper can also be applied to eigenproblems involved in civil and other branches of engineering.

## PRELIMINARY DEFINITIONS

### Definitions from Theory of Graphs

A graph,  $S$ , consists of a set of elements,  $N(S)$ , called *nodes* and a set of elements,  $M(S)$ , called *members*, together with a relation of incidence that associates two distinct nodes with each member known as its *ends*. If the end nodes of a member coincide, then the member is called a *loop*, and if there exists more than one member between two nodes, it is called a *multiple member*. Two nodes of a graph are called *adjacent* if these nodes are the end nodes of a member. A member is considered incident with a node if it is an end node of the member. The *degree* of a node is the number of members incident with that node. Two

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graphs are called *isomorphic*; if there is a one-to-one correspondence between their nodes, the adjacency is preserved. A *subgraph*,  $S_i$ , of a graph,  $S$ , is a graph for which  $N(S_i) \subseteq N(S)$  and  $M(S_i) \subseteq M(S)$  and each member of  $S_i$  has the same ends as in  $S$ . For detailed explanations and applications, the reader may refer to [14].

The adjacency matrix,  $A = [a_{ij}]_{n \times n}$ , of a labeled graph,  $S$ , containing  $n$  nodes is defined as:

$$a_{ij} = \begin{cases} 1 & \text{if node } n_i \text{ is adjacent to } n_j \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The degree matrix,  $D = [d_{ij}]_{n \times n}$ , is a diagonal matrix of node degrees.  $d_{ij}$  is equal to the degree of the  $i$ th node.

The Laplacian matrix,  $L = [l_{ij}]_{n \times n}$ , is defined as:

$$L = D - A. \quad (2)$$

Therefore, the entries of  $L$  are as follows:

$$l_{ij} = \begin{cases} -1 & \text{if node } n_i \text{ is adjacent to } n_j \\ \deg(n_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

### Decomposition of Matrices of Form II and Form III

Consider an  $N \times N$  symmetric matrix,  $M$ , with all entries being real. For two symmetry forms, the eigenvalues of  $M$  are obtained using the properties of its submatrices.

#### Canonical Form II

For this case, matrix  $M$  can be decomposed into the following form:

$$[M] = \begin{bmatrix} [A]_{n \times n} & [B]_{n \times n} \\ [B]_{n \times n} & [A]_{n \times n} \end{bmatrix}_{N \times N}. \quad (4)$$

The eigenvalues of matrix  $M$  can be calculated as:

$$\{\lambda M\} = \{\lambda C\} \bar{\cup} \{\lambda D\}, \quad (5)$$

where  $\bar{\cup}$  is the sign for the collection of the eigenvalues of the submatrices.

It can be proved that  $\det M = \det C \times \det D$  [14]. Here,  $C$  and  $D$  are called condensed submatrices of  $M$ ,

$$C = A + B, \quad D = A - B. \quad (6)$$

#### Canonical Form III

This form has a Form II submatrix augmented by  $k$  rows and columns as shown in the following:

$$[M] = \begin{bmatrix} [A] & [B] \\ [B] & [A] \\ \hline C(2n+1, 1) & \cdots & C(2n+1, 2n) \\ \vdots & \ddots & \vdots \\ Z(2n+k, 1) & \cdots & Z(2n+k, 2n) \\ \hline L_{11} & \cdots & L_{1k} \\ L_{21} & \cdots & L_{2k} \\ L_{n1} & \cdots & L_{nk} \\ L_{11} & \cdots & L_{1k} \\ L_{21} & \cdots & L_{2k} \\ L_{n1} & \cdots & L_{nk} \\ C(2n+1, 2n+1) & \cdots & C(2n+1, 2n+k) \\ \vdots & \ddots & \vdots \\ Z(2n+k, 2n+1) & \cdots & Z(2n+k, 2n+k) \end{bmatrix}, \quad (7)$$

where  $M$  is a  $(N+k) \times (N+k)$  matrix with a  $N \times N$  submatrix with the pattern of Form II ( $N = 2n$ ) and  $k$  augmented columns and rows. The entries of the augmented columns are the same in each column and all the entries of  $M$  are real numbers.  $C(i, j)$  and  $Z(i, j)$  are arbitrary real numbers.

The set of eigenvalues for  $M$  is obtained as:

$$\{\lambda M\} = \{\lambda D\} \bar{\cup} \{\lambda E\}, \quad (8)$$

where  $D$  and  $E$  are constructed as follows:

$$D = A - B, \quad (9)$$

$$[E] = \begin{bmatrix} [A+B] \\ \hline C(2n+1, 1) + C(2n+1, n+1) & \cdots & C(2n+1, 2n+k) \\ \vdots & \ddots & \vdots \\ Z(2n+k, 1) + Z(2n+k, n+1) & \cdots & Z(2n+k, 2n+k) \\ \hline L_{11} & \cdots & L_{1k} \\ L_{21} & \cdots & L_{2k} \\ L_{n1} & \cdots & L_{nk} \\ C(2n+1, 2n+1) & \cdots & C(2n+1, 2n+k) \\ \vdots & \ddots & \vdots \\ Z(2n+k, 2n+1) & \cdots & Z(2n+k, 2n+k) \end{bmatrix}, \quad (10)$$

and:

$$\det M = \det D \times \det E. \quad (11)$$

A special case of the Form III canonical matrix can be considered as:

$$[M] = \begin{bmatrix} A_{\frac{N}{2} \times \frac{N}{2}} & B_{\frac{N}{2} \times \frac{N}{2}} & S_{\frac{N}{2} \times k} \\ B_{\frac{N}{2} \times \frac{N}{2}} & A_{\frac{N}{2} \times \frac{N}{2}} & S_{\frac{N}{2} \times k} \\ S_{k \times \frac{N}{2}}^T & S_{k \times \frac{N}{2}}^T & X_{k \times k} \end{bmatrix}, \quad (12)$$

resulting in:

$$D = A - B,$$

and:

$$E = \begin{bmatrix} A + B & S_{\frac{N}{2} \times k} \\ 2S_{k \times \frac{N}{2}}^T & X_{k \times k} \end{bmatrix}, \quad (13)$$

where  $A$ ,  $B$ ,  $S$  and  $X$  have the dimensions as shown. The matrix of this form will frequently be used in the analysis of symmetric structures.

### Symmetry of Form II and III

When the matrices of the previous section are considered as the Laplacian matrices of the graph models of structures, the canonical forms can be interpreted as follows.

#### Form II Symmetry

The axis of symmetry passes through members, and graph  $S$  has an even number of nodes. The members cut by the axis of symmetry are called link members, and their end nodes are taken as linked nodes. Link members connect the two isomorphic subgraphs,  $S_1$  and  $S_2$ , to each other. For this case, two different types of connection are considered. The first one is a direct connection and the second is called a cross-connection.

In a direct connection, a typical node  $i$  in  $S_1$  is connected to the node labeled as  $i + N/2$  in  $S_2$  by a link member. In cross-connection, a typical pair of nodes  $i$  and  $j$  in  $S_1$  is connected to  $j + N/2$  and  $i + N/2$ , respectively.

#### Form III Symmetry

In this case, the axis of symmetry passes through nodes, while the conditions of direct or cross-connections are not fulfilled. The nodes on the axis of symmetry are called central nodes.

Once the types of symmetry are identified, the isomorphic subgraphs,  $S_1$  and  $S_2$ , are modified such that the union of eigenvalues of the structural matrices of the two modified subgraphs becomes the same as the eigenvalues of the entire graph,  $S$ . The process of the modifications made to the subgraphs is called

the healing of the subgraphs and the entire process may be considered as the factorization of a graph. The subgraphs obtained for graph  $S$  after healing are called the factors of  $S$ .

### DECOMPOSITION OF PLANAR SYMMETRIC FRAMES WITH ODD NUMBER OF SPANS

The algorithm presented in this section decomposes a symmetric structure with an odd number of spans into two factors,  $C$  and  $D$  or  $E$  and  $D$ . By obtaining the dynamic properties of each factor and considering their union, the dynamic properties of entire frames are obtained. Here, for simplicity, the axial deformations are not included and the corresponding Degrees Of Freedom (DOFs) are taken as zero.

In the following, first Algorithm A is presented, and then the necessary new terms are introduced.

#### Algorithm A

The algorithm for the decomposition of planar frames with an odd number of spans, with or without sway, is designed as:

- Step 1: Delete all the beams crossing the axis of symmetry.
- Step 2: The columns corresponding to the left part, which are connected to the eliminated beams, are doubled by  $tc$  columns. This half, for the case of non-sway frames, forms the factor  $C$ , and in the case of sway frames, together with the translation DOFs, forms the factor  $E$ .
- Step 3: The columns of the right half, which were connected to the eliminated beams, are doubled by  $cc$  columns. This half, for the cases of sway and non-sway frames, forms the factor  $D$ , and in the case of sway frames, together with the translation DOFs, is deleted.

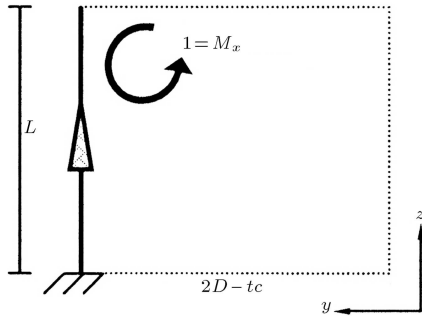
### Definitions

#### The New Element $tc$ for Planar Case

The elements defined in the following are used in the above algorithm for doubling some columns in place of deleting the beams crossing the axis of symmetry. The new column is denoted by  $tc$ , as shown in Figure 1, and is defined as follows.

Let the properties of the deleted beam crossing the axis of symmetry be denoted by  $L_b$ ,  $m_b$  and  $EI_b$ , then the stiffness and mass matrices of the new column,  $tc$ , will be:

$$K_{tc} = \frac{EI_b}{L_b} \times [6],$$



**Figure 1.** The new column  $tc$ , fixed in  $F_y$  direction.

$$M_{tc} = \frac{m_b L_b^3}{420} \times [1], \quad (14)$$

where  $E$  is the elastic modulus,  $I$  is the moment of inertia of the cross section,  $m$  is the mass of unit length and  $L_b$  is the length of the new column.

#### The New Element $cc$ for Planar Case

Again, considering the properties of the deleted beam as  $L_b$ ,  $m_b$  and  $EI_b$ , the stiffness and mass matrices of the new column,  $cc$ , will be:

$$K_{cc} = \frac{EI_b}{L_b} \times [2], \quad M_{cc} = \frac{m_b L_b^3}{420} \times [7]. \quad (15)$$

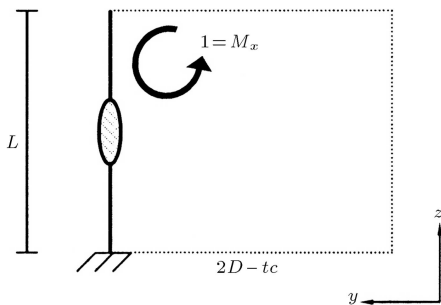
This column is illustrated in Figure 2. The way these new elements,  $tc$  and  $cc$ , are defined, is explained in the subsequent section.

#### Potential of the Algorithm A

In this section, it is shown how Algorithm A decomposes a symmetric planar frame with an odd number of spans into two substructures, leading to correct eigenvalues.

#### Algorithm for Numbering the DOFs of a Planar Frame

The following algorithms are used for numbering the DOFs of a frame, such that the corresponding stiffness and mass matrices become canonical forms explained in the previous section.



**Figure 2.** The new column  $cc$ .

#### Numbering for Frames with Odd Number of Spans

##### Algorithm

Consider a frame with  $N$  total DOFs and  $p$  lateral translation DOFs.

1. First, we specify the symmetry line and the number of the rotation DOFs for the left part, starting from an arbitrary node and assigning  $1, 2, 3, \dots, N/2$ .
2. The rotation DOFs for the right part are numbered, such that the difference between the numbers for the symmetric DOFs is equal to  $N/2$ .
3. The translation DOFs are numbered, starting from an arbitrary translation DOF in an arbitrary manner assigning  $N, N+1, N+2, \dots, N+p$ .

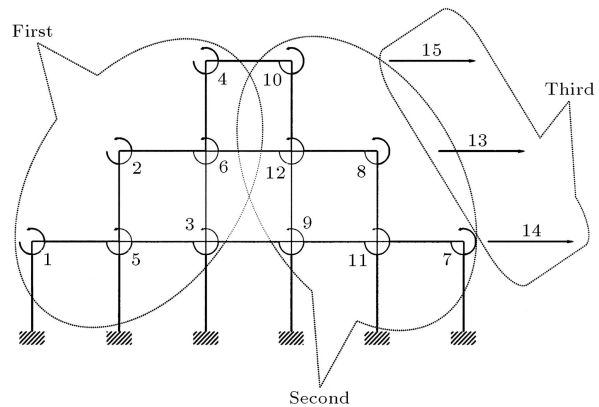
With this numbering for planar frames with an odd number of spans, the stiffness and mass matrices for the non-sway case will have Form II symmetry and for a sway case will have Form III symmetry. Figure 3 shows an example of such numbering.

#### Numbering for Frames with Even Number of Spans

##### Algorithm

Consider a frame with  $N$  rotation DOFs in two symmetric halves,  $r$  rotation DOFs in central nodes existing in the axis of symmetry, and  $p$  lateral translation DOFs.

1. First, the axis of symmetry is specified and then the rotation DOFs for the left part are numbered, starting from an arbitrary node and assigning  $1, 2, 3, \dots, N/2$ .
2. The rotation DOFs for the right part are numbered, such that the difference between the numbers for the symmetric DOFs is equal to  $N/2$ .
3. The rotation DOFs of central nodes coinciding with the axis of symmetry and all the translation DOFs



**Figure 3.** Numbering of the DOFs in a symmetric frame with an odd number of spans.

are numbered starting from an arbitrary DOF in an arbitrary manner and assigning  $N, N+1, N+2, \dots, N+p+r$ . Figure 4 illustrates such a numbering.

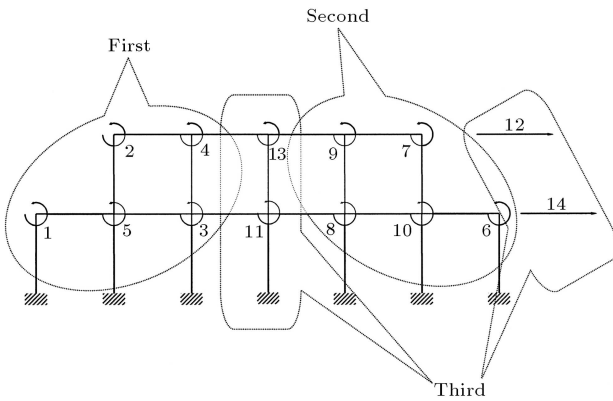
Performing this numbering, there will be a linear relationship between the stiffness and mass matrices and we will have a factorization, resulting in two substructures.

On the other hand, if the numbering of the DOFs in the main structure is performed in the manner presented, the corresponding stiffness and mass matrices become canonical forms. In the other words, in the case of non-sway frames with an odd number of spans, the problem is solved by constructing the submatrices,  $M_C, K_C$  and  $M_D, K_D$ , corresponding to the Form II symmetry presented in the previous section. Furthermore, in the sway case of all types of frame, with either an odd or even number of spans, and in the case of non-sway frames with an even one, the problem is solved by forming submatrices,  $M_D, K_D$  and  $f(M_E, m), f(K_E, m)$ , corresponding to Form III symmetry, as presented in the previous section. For a structure with an even number of spans,  $m$  is the total number of the translation DOFs (if they exist) and rotation DOFs of central nodes, i.e.  $m = r + p$ , where  $r$  and  $p$  are previously defined.

If the numbering of the DOFs in the main structure is performed in the manner presented, then the corresponding stiffness and mass matrices will have the following forms:

$$K, M = \begin{bmatrix} A_{\frac{N}{2} \times \frac{N}{2}} & B_{\frac{N}{2} \times \frac{N}{2}} & S_{\frac{N}{2} \times m} \\ B_{\frac{N}{2} \times \frac{N}{2}} & A_{\frac{N}{2} \times \frac{N}{2}} & S_{\frac{N}{2} \times m} \\ S_{m \times \frac{N}{2}}^T & S_{m \times \frac{N}{2}}^T & X_{m \times m} \end{bmatrix}$$

$$\Rightarrow \underbrace{K_D, M_D}_{\text{real}} = A - B,$$



**Figure 4.** Numbering of the DOFs in a symmetric frame with an even number of spans.

and:

$$\underbrace{K_E, M_E}_{\text{real}} = \begin{bmatrix} A + B & S_{\frac{N}{2} \times m} \\ 2S_{m \times \frac{N}{2}}^T & X_{m \times m} \end{bmatrix}, \quad (16)$$

which are the same as Equation 12.

However, the stiffness and mass matrices of substructure  $E$  in Algorithm A differing to submatrix  $E$  in Equation 14 are obtained as:

$$\underbrace{K_D, M_D}_{\text{algorithm}} = A - B,$$

and:

$$\underbrace{K_E, M_E}_{\text{algorithm}} = \begin{bmatrix} A + B & S \\ S^T & X/2 \end{bmatrix}. \quad (17)$$

In this algorithm, the stiffness and mass matrices of factor  $E$  are not the same as those obtained from the partitioning of the corresponding matrices of the original structure, as presented in Equations 6 or 9 and 10. However, the responses consisting of the determinant and eigenvalues are identical as proved in the following.

#### **The Function $f(A, m)$ and Proof of a Theorem**

Function  $f(A, m)$

Consider  $A$  as a matrix. The function  $f(A, m)$  multiplies the last  $m$  rows of  $A$  by 2.

An example is provided in the following:

$$A = \begin{bmatrix} 3 & 3 & 6 & 9 & 5 \\ 7 & 3 & 5 & 1 & 9 \\ 4 & 6 & 2 & 8 & 6 \\ 5 & 4 & 1 & 7 & 7 \\ 6 & 5 & 9 & 2 & 5 \end{bmatrix},$$

$$f(A, 2) = \begin{bmatrix} 3 & 3 & 6 & 9 & 5 \\ 7 & 3 & 5 & 1 & 9 \\ 4 & 6 & 2 & 8 & 6 \\ 10 & 8 & 2 & 14 & 14 \\ 12 & 10 & 18 & 4 & 10 \end{bmatrix},$$

$$f(A, 3) = \begin{bmatrix} 3 & 3 & 6 & 9 & 5 \\ 7 & 3 & 5 & 1 & 9 \\ 8 & 12 & 4 & 16 & 12 \\ 10 & 8 & 2 & 14 & 14 \\ 12 & 10 & 18 & 4 & 10 \end{bmatrix}.$$

#### **Theorem**

If  $m$  is a number of rows of a matrix and  $K$  and  $M$  are arbitrary matrices, then the eigenvalues of  $M^{-1} \times K$  and  $[f(M, m)]^{-1} \times f(K, m)$  are identical.

*Proof*

When a row or column of a matrix is multiplied by a constant number, then the determinant is multiplied by the same number. Thus, employing function  $f$ , we have:

$$\begin{aligned}\det(f(K, m)) &= 2^m \det K, \\ \det(f(M, m)) &= 2^m \det M.\end{aligned}\quad (18)$$

If  $\lambda$  and  $\varphi$  are the eigenvalues and eigenvectors of  $M^{-1} \times K$ :

$$M^{-1} \times K\varphi = \lambda\varphi. \quad (19)$$

Pre-multiplying by  $M$  leads to:

$$M \times M^{-1}K\varphi = M\lambda\varphi. \quad (20)$$

Thus:

$$K\varphi = \lambda M\varphi, \quad (21)$$

or:

$$(K - \lambda M) \times \varphi = 0. \quad (22)$$

Hence:

$$\det(K - \lambda M) = 0. \quad (23)$$

On the other hand, we have:

$$\det(f(K, m) - \lambda f(M, m)) = 2^m \det(K - \lambda M) = 0. \quad (24)$$

Therefore:

$$\text{eig}([f(M, m)]^{-1} \times f(K, m)) = \lambda = \text{eig}(M^{-1} \times K). \quad (25)$$

In dynamic analysis,  $\lambda$  can be considered as  $\omega^2$  resulting in:

$$[K - \omega^2 M] \varphi = 0, \quad (26)$$

where  $\omega$  are natural frequencies of a structure. Thus, the theorem can be restated as follows.

If  $m$  is an integer number defined previously and  $K$  and  $M$  are the stiffness and mass matrices of a structure, then the natural frequencies obtained from matrices  $M$  and  $K$  and those obtained from matrices  $f(K, m)$  and  $f(M, m)$  are identical.

Therefore, we can conclude that the eigenvalues obtained from the stiffness and mass matrices of each substructure, such as  $E$  in the above relationship (Equation 17), are equal to those obtained from the main structure (Equation 16), i.e:

$$\underbrace{K_D}_{\text{real}} = f(\underbrace{K_D, m}_{\text{algorithm}}),$$

and:

$$\underbrace{M_D}_{\text{real}} = f(\underbrace{M_D, m}_{\text{algorithm}}), \quad (27)$$

$$\underbrace{K_E}_{\text{real}} = f(\underbrace{K_E, m}_{\text{algorithm}}),$$

and:

$$\underbrace{M_E}_{\text{real}} = f(\underbrace{M_E, m}_{\text{algorithm}}). \quad (28)$$

It is important to note that we have obtained substructures for which the stiffness and mass matrices are symmetric and in order to obtain their eigenvalues, one does not need to utilize function  $f$ . At the same time, the factorization can be performed in a simple manner.

#### *Properties of the Columns $tc$ and $cc$*

The properties of the new columns are obtained considering the inter-relation of the DOFs of the members. For the frames with an odd number of spans, where the axis of symmetry passes through beams, the effect of the deleted beams should be included in the decomposed subgraphs. Adding the new columns serves as a means for transferring the properties of the main structure into the decomposed substructures. These operations are healings which change the subgraphs into the factors. The new columns introduced here are more effective than using springs, masses, and other devices as have been used in [17].

If we can construct substructures with stiffness and mass matrices corresponding to the Form II canonical form, then we can form the factors.

If the numbering of DOFs is performed corresponding to canonical Form II, then submatrix B will represent the relation between the DOFs of the right and left side of the frame and submatrix A represents the relation between the DOFs of each half of the structure.  $A$ ,  $B$ ,  $C$  and  $D$  are shown in Equations 4 and 6.

In general, for a beam-column with one rotational DOF per node, we have:

$$K = \frac{EI}{L} \times \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \quad M = \frac{mL^3}{420} \times \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}. \quad (29)$$

Considering the relationship between the DOFs of the connecting beams, it becomes obvious that entries (1,1) and (1,2) in the mass and stiffness matrices of substructures C and D should be added and subtracted,

respectively:

$$\begin{aligned}
 C: K &= \frac{EI}{L} \times [4 + 2] = \frac{EI}{L} \times [6], \\
 M &= \frac{mL^3}{420} \times [4 + (-3)] = \frac{mL^3}{420} \times [1], \\
 D: K &= \frac{EI}{L} \times [4 - 2] = \frac{EI}{L} \times [2], \\
 M &= \frac{mL^3}{420} \times [4 - (-3)] = \frac{mL^3}{420} \times [7].
 \end{aligned} \quad (30)$$

It is obvious that the length and the elastic properties in these relationships correspond to the connecting beams, which are supposed to be deleted:

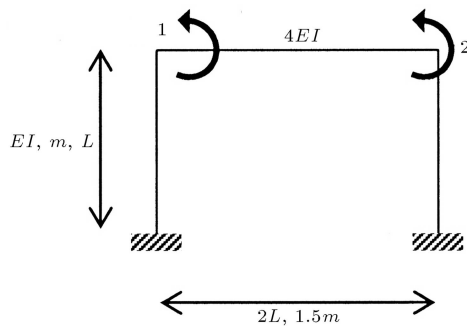
$$\begin{aligned}
 K_{tc} &= \frac{EI_b}{L_b} \times [6], \\
 M_{tc} &= \frac{m_b L_b^3}{420} \times [1], \\
 K_{cc} &= \frac{EI_b}{L_b} \times [2], \\
 M_{cc} &= \frac{m_b L_b^3}{420} \times [7].
 \end{aligned} \quad (31)$$

In this way, the properties of the new columns are obtained.

Now, if we want to use this algorithm, submatrices  $D$  and  $E$  can be obtained much easier than the previous methods wherein we had to add masses and springs [17]. Another advantage of this algorithm is that, unlike previous methods, the substructures in both sway and non-sway cases are similar, and the only difference is in a non-sway case for the left half substructure of the frame where we have translation DOFs.

#### Example 1

The symmetric frame shown in Figure 5 is considered. This is constrained against sway and has only 2



**Figure 5.** A symmetric frame with two DOFs.

rotation DOFs as shown in the figure.

The distribution of mass in the link beam that crosses the axis of symmetry should also be symmetric.

According to Algorithm A, the decomposition of the frame is obtained in a step by step manner, whereas, in previously developed methods, the factors were obtained by adding springs and masses [17].

The properties of the added columns (Figure 6) are as follows:

$$\begin{aligned}
 K_{tc} &= \frac{EI_b}{L_b} \times [6] = \left[ \frac{12EI}{L} \right], \\
 M_{tc} &= \frac{m_b L_b^3}{420} \times [1] = \left[ \frac{6mL^3}{210} \right], \\
 K_{cc} &= \frac{EI_b}{L_b} \times [2] = \left[ \frac{4EI}{L} \right], \\
 M_{cc} &= \frac{m_b L_b^3}{420} \times [7] = \left[ \frac{42mL^3}{210} \right].
 \end{aligned}$$

Now, the stiffness and mass matrices of factors  $C$  and  $D$  are formed as:

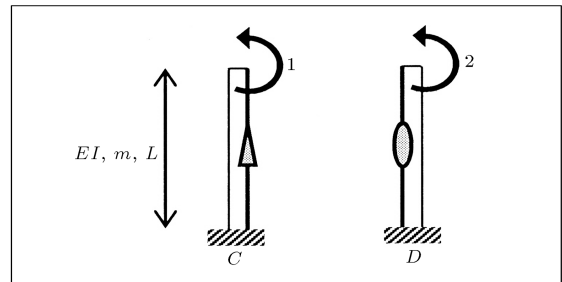
$$\begin{aligned}
 K_C &= \left[ \frac{4EI}{L} + \frac{12EI}{L} \right] = \left[ \frac{16EI}{L} \right], \\
 M_C &= \left[ \frac{mL}{420} \times 4L^2 + \frac{6mL^3}{210} \right] = \left[ \frac{8mL^3}{210} \right],
 \end{aligned}$$

$$\omega^2 = X,$$

$$\Rightarrow \omega_1 = \sqrt{\frac{420EI}{mL^4}},$$

and:

$$\begin{aligned}
 K_D &= \left[ \frac{4EI}{L} + \frac{4EI}{L} \right] = \left[ \frac{8EI}{L} \right], \\
 M_D &= \left[ \frac{mL}{420} \times 4L^2 + \frac{42mL^3}{210} \right] = \left[ \frac{44mL^3}{210} \right],
 \end{aligned}$$



**Figure 6.** Factors of the frame of Figure 3.

$$\omega^2 = X$$

$$\Rightarrow \omega_2 = \sqrt{\frac{420EI}{11mL^4}},$$

and the natural frequencies are easily obtained.

#### Example 2

The frame shown in Figure 7 has 10 DOFs and has Form II symmetry. The factors are constructed as shown in Figure 8.

The stiffness and mass matrices of the added columns are as follows:

$$tc_1 : \Rightarrow K_{tc_1} = \frac{EI_b}{L_b} \times [6] = [2EI],$$

$$M_{tc_1} = \frac{m_b L_b^3}{420} \times [1] = \left[ \frac{9m}{140} \right],$$

$$cc_1 : \Rightarrow K_{cc_1} = \frac{EI_b}{L_b} \times [2] = \left[ \frac{2EI}{3} \right],$$

$$M_{cc_1} = \frac{m_b L_b^3}{420} \times [7] = \left[ \frac{9m}{20} \right],$$

$$tc_2 : \Rightarrow K_{tc_2} = \frac{EI_b}{L_b} \times [6] = \left[ \frac{2EI}{3} \right],$$

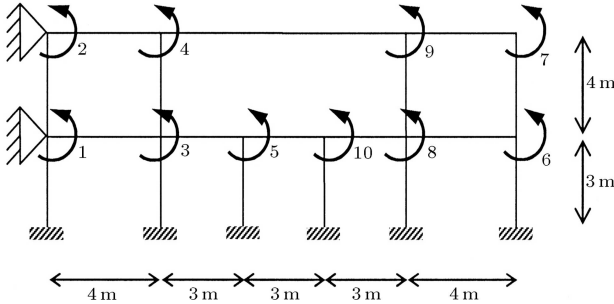


Figure 7. A symmetric frame with 10 DOFs.

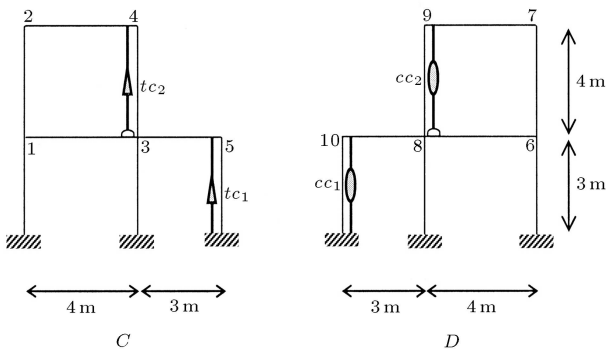


Figure 8. Factors of the frame of Figure 7.

$$M_{tc_2} = \frac{m_b L_b^3}{420} \times [1] = \left[ \frac{243m}{70} \right],$$

$$cc_2 : \Rightarrow K_{cc_2} = \frac{EI_b}{L_b} \times [2] = \left[ \frac{2EI}{9} \right],$$

$$M_{cc_2} = \frac{m_b L_b^3}{420} \times [7] = \left[ \frac{243m}{10} \right].$$

The stiffness and mass matrices of factors  $C$  and  $D$  are constructed as:

$$K_C = EI$$

$$\times \begin{bmatrix} \frac{4}{3} + \frac{4}{4} + \frac{4}{4} & \frac{2}{4} & \frac{2}{4} \\ \frac{2}{4} & \frac{4}{4} + \frac{4}{4} & 0 \\ \frac{2}{4} & 0 & \frac{4}{3} + \frac{4}{4} + \frac{4}{3} + \frac{4}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 \\ \frac{2}{4} & 0 & 0 \\ \frac{2}{4} & \frac{2}{3} & 0 \\ \frac{4}{4} + \frac{4}{4} + \frac{2}{3} & 0 & 0 \\ 0 & \frac{4}{3} + \frac{4}{3} + 2 & 0 \end{bmatrix}$$

$$= 2EI \begin{bmatrix} 5/3 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1 & 0 & 1/4 & 0 \\ 1/4 & 0 & 7/3 & 1/4 & 1/3 \\ 0 & 1/4 & 1/4 & 4/3 & 0 \\ 0 & 0 & 1/3 & 0 & 7/3 \end{bmatrix},$$

$$M_C = \frac{m}{420} \begin{bmatrix} 620 & -192 & -192 & 0 & 0 \\ -192 & 512 & 0 & -192 & 0 \\ -192 & 0 & 728 & -192 & -81 \\ 0 & -192 & -192 & 1241 & 0 \\ 0 & 0 & -81 & 0 & 243 \end{bmatrix},$$

$$K_D = EI$$

$$\times \begin{bmatrix} \frac{4}{3} + \frac{4}{4} + \frac{4}{4} & \frac{2}{4} & \frac{2}{4} \\ \frac{2}{4} & \frac{4}{4} + \frac{4}{4} & 0 \\ \frac{2}{4} & 0 & \frac{4}{3} + \frac{4}{4} + \frac{4}{3} + \frac{4}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 \\ \frac{2}{4} & 0 & 0 \\ \frac{2}{4} & \frac{2}{3} & 0 \\ \frac{4}{4} + \frac{4}{4} + \frac{2}{9} & 0 & 0 \\ 0 & \frac{4}{3} + \frac{4}{3} + \frac{2}{3} \end{bmatrix}$$

$$= 2EI \begin{bmatrix} 5/3 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1 & 0 & 1/4 & 0 \\ 1/4 & 0 & 7/3 & 1/4 & 1/3 \\ 0 & 1/4 & 1/4 & 10/9 & 0 \\ 0 & 0 & 1/3 & 0 & 5/3 \end{bmatrix},$$

$$M_D = \frac{m}{420} \begin{bmatrix} 620 & -192 & -192 & 0 & 0 \\ -192 & 512 & 0 & -192 & 0 \\ -192 & 0 & 728 & -192 & -81 \\ 0 & -192 & -192 & 5615 & 0 \\ 0 & 0 & -81 & 0 & 405 \end{bmatrix}.$$

In this way, the natural frequencies and natural modes of this frame with 10 DOFs are obtained using the equation of motion of two factors each having 5 DOFs, as:

$$\det[K_C - \omega^2 M_C]_{5 \times 5} = 0,$$

leading to:

$$\omega_1 = \sqrt{\frac{0.6EI}{m}},$$

$$\omega_2 = \sqrt{\frac{1.42EI}{m}},$$

$$\omega_3 = \sqrt{\frac{2.15EI}{m}},$$

$$\omega_4 = \sqrt{\frac{5.24EI}{m}},$$

$$\omega_5 = \sqrt{\frac{9.56EI}{m}},$$

and:

$$\det[K_D - \omega^2 M_D]_{5 \times 5} = 0,$$

leading to:

$$\omega_6 = \sqrt{\frac{0.15EI}{m}},$$

$$\omega_7 = \sqrt{\frac{1.1EI}{m}},$$

$$\omega_8 = \sqrt{\frac{2EI}{m}},$$

$$\omega_9 = \sqrt{\frac{3.52EI}{m}},$$

$$\omega_{10} = \sqrt{\frac{5.57EI}{m}}.$$

*Example 3*

Consider the sway frame shown in Figure 9, having 12 DOFs. The factors are shown in Figure 10.

The natural frequencies are similar to those of Example 2. Here, there is no need to solve equation  $\det[K_D - \omega^2 M_D]_{5 \times 5} = 0$  for finding the eigenvalues. The formation of factor  $D$  can be avoided:

$$cc_1 : \Rightarrow K_{cc_1} = \frac{EI_b}{L_b} \times [2] = \left[ \frac{2EI}{3} \right],$$

$$M_{cc_1} = \frac{m_b L_b^3}{420} \times [7] = \left[ \frac{9m}{20} \right],$$

$$cc_2 : \Rightarrow K_{cc_2} = \frac{EI_b}{L_b} \times [2] = \left[ \frac{2EI}{9} \right],$$

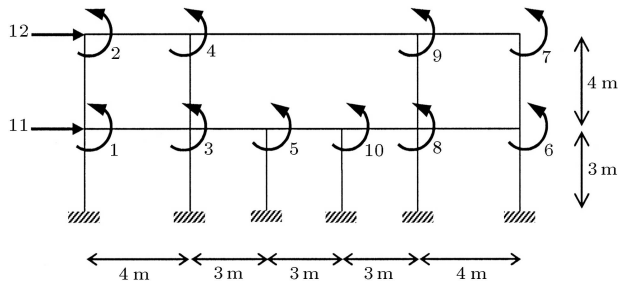
$$M_{cc_2} = \frac{m_b L_b^3}{420} \times [7] = \left[ \frac{243m}{10} \right].$$

The stiffness and the mass matrices of factor  $E$  are as:

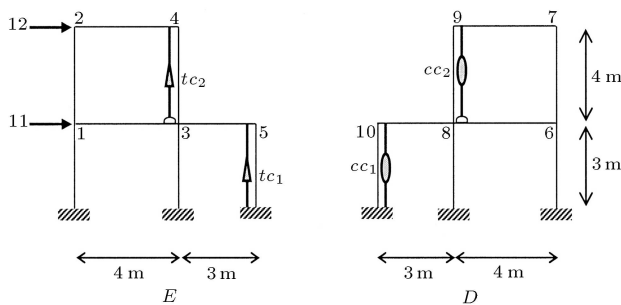
$$K_E = EI \times$$

$$\begin{bmatrix} \frac{4}{3} + \frac{4}{4} + \frac{4}{4} & \frac{2}{4} & \frac{2}{4} & 0 & 0 \\ \frac{2}{4} & \frac{4}{4} + \frac{4}{4} & 0 & \frac{2}{4} & 0 \\ \frac{2}{4} & 0 & \frac{4}{3} + \frac{4}{4} + \frac{4}{3} + \frac{4}{4} & \frac{2}{4} & \frac{2}{3} \\ 0 & \frac{2}{4} & \frac{2}{4} & \frac{4}{4} + \frac{4}{4} + \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{4}{3} + \frac{4}{3} + 2 \\ \frac{6}{3^2} - \frac{6}{4^2} & \frac{-6}{4^2} & \frac{6}{3^2} - \frac{6}{4^2} & \frac{-6}{4^2} & \frac{6}{3^2} \\ \frac{6}{4^2} & \frac{6}{4^2} & \frac{6}{4^2} & \frac{6}{4^2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{3^2} - \frac{6}{4^2} & \frac{6}{4^2} \\ \frac{-6}{4^2} & \frac{6}{4^2} \\ \frac{6}{3^2} - \frac{6}{4^2} & \frac{6}{4^2} \\ \frac{-6}{4^2} & \frac{6}{4^2} \\ \frac{6}{3^2} & 0 \\ 3 \times \frac{12}{3^3} + 2 \times \frac{12}{4^3} & -2 \times \frac{12}{4^3} \\ -2 \times \frac{12}{4^3} & 2 \times \frac{12}{4^3} \end{bmatrix},$$



**Figure 9.** A sway frame with 12 DOFs.



**Figure 10.** Factors  $D$  and  $E$  of the sway frame of Figure 9.

$$M_E = \frac{m}{420}$$

$$\begin{bmatrix} 620 & -192 & -192 & 0 \\ -192 & 512 & 0 & -192 \\ -192 & 0 & 728 & -192 \\ 0 & -192 & -192 & 1241 \\ 0 & 0 & -81 & 0 \\ -22 \times 3^2 + 22 \times 4^2 & -13 \times 4^2 & -22 \times 3^2 + 22 \times 4^2 & -13 \times 4^2 \\ 13 \times 4^2 & -22 \times 4^2 & 13 \times 4^2 & -22 \times 4^2 \\ 0 & -22 \times 3^2 + 22 \times 4^2 & 13 \times 4^2 & -22 \times 4^2 \\ 0 & -13 \times 4^2 & -22 \times 4^2 & 13 \times 4^2 \\ -81 & -22 \times 3^2 + 22 \times 4^2 & 13 \times 4^2 & -22 \times 4^2 \\ 0 & -13 \times 4^2 & -22 \times 4^2 & 0 \\ 243 & -22 \times 3^2 & 0 & 0 \\ -22 \times 3^2 & 3 \times 156 \times 3 + 2 \times 156 \times 4 & 2 \times 54 \times 4 & 2 \times 156 \times 4 \\ 0 & 2 \times 54 \times 4 & 2 \times 156 \times 4 & 0 \end{bmatrix}.$$

In this way, the natural frequencies and natural modes of this frame with 12 DOFs are obtained using the equation of motion of two factors having 5 and 7 DOFs.

The first 5 frequencies are obtained as:

$$\omega_1 = \sqrt{\frac{0.15EI}{m}},$$

$$\omega_2 = \sqrt{\frac{1.1EI}{m}},$$

$$\omega_3 = \sqrt{\frac{2EI}{m}},$$

$$\omega_4 = \sqrt{\frac{3.52EI}{m}},$$

$$\omega_5 = \sqrt{\frac{5.57EI}{m}}.$$

The remaining 7 frequencies are calculated from factor  $E$  as:

$$\det[K_E - \omega^2 M_E]_{7 \times 7} = 0,$$

leading to:

$$\omega_6 = \sqrt{\frac{0.022EI}{m}},$$

$$\omega_7 = \sqrt{\frac{0.25EI}{m}},$$

$$\omega_8 = \sqrt{\frac{0.61EI}{m}},$$

$$\omega_9 = \sqrt{\frac{1.81EI}{m}},$$

$$\omega_{10} = \sqrt{\frac{2.97EI}{m}},$$

$$\omega_{11} = \sqrt{\frac{5.67EI}{m}},$$

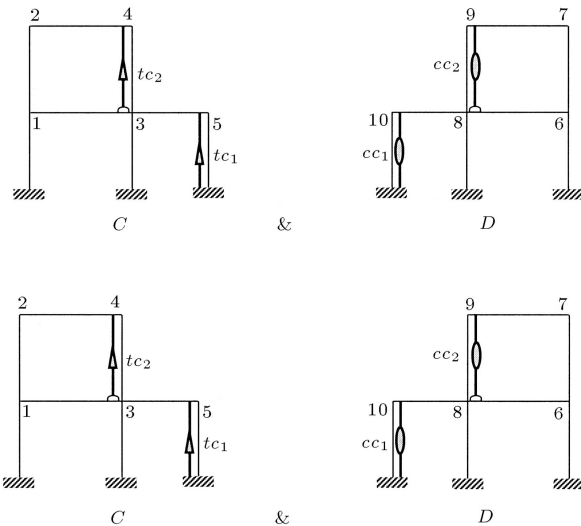
$$\omega_{12} = \sqrt{\frac{10.27EI}{m}}.$$

The factors of the main frame in the case of sway and non-sway are identical (Figure 11).

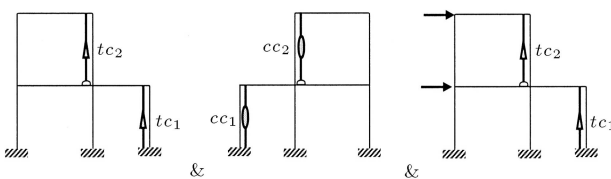
Only factor  $E$  has the additional translation DOF. Thus, for calculating the responses of frames in sway and non-sway cases, instead of solving a problem with  $n \times n$  and  $(n+m) \times (n+m)$  matrices, we need to solve three problems corresponding to  $n/2 \times n/2$ ,  $n/2 \times n/2$  and  $(n/2+m) \times (n/2+m)$  matrices (Figure 12).

### DECOMPOSITION OF SYMMETRIC PLANAR FRAMES WITH EVEN NUMBER OF SPANS

According to the algorithm, each symmetric structure with an even number of spans is decomposed into two factors without introducing a new element. By obtaining the dynamic properties of each factor and considering the union of the results, one can obtain the dynamic properties of the entire structure.



**Figure 11.** Factors of the frame in non-sway and sway cases.



**Figure 12.** Three factors to be considered for the solution.

### Definitions

A central element is defined as a column which coincides with the axis of symmetry. Central nodes are taken as the nodes that coincide with the axis of symmetry.

### Algorithm B

This algorithm is simple and consists of the following steps.

- Step 1: Divide the frame into two halves from the axis of symmetry, such that the moment of inertia for the central column and the mass of their unit length,  $m$ , are reduced to half.
- Step 2: Fix the central nodes in the left half. This half is factor  $D$  and the right half forms factor  $E$ .

All aspects proved previously are applicable to this algorithm. Thus, the results consisting of the determinant and eigenvalues of free vibration from decomposition due to this algorithm are identical to those of the main frame, as was desired. Therefore, we solve the main eigenproblem by constructing submatrices,  $K_D$ ,  $M_D$  and  $K_E$ ,  $M_E$ . In fact, factors  $D$  and  $E$  obtained by this algorithm have the properties of the entire structure.

### Example 4

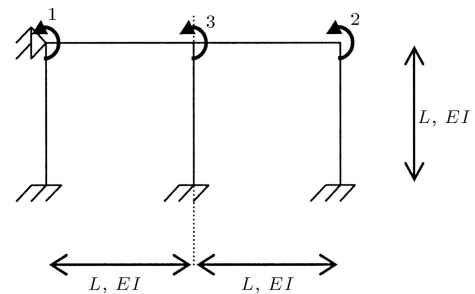
Consider the frame shown in Figure 13, which is constrained against sway. This frame has 3 DOFs. It is assumed that the frame has symmetric elastic properties with respect to the two planes of symmetry.

Factors  $D$  and  $E$  are obtained using Algorithm B, step by step, as shown in Figure 14. These factors can be considered, as shown in Figure 15.

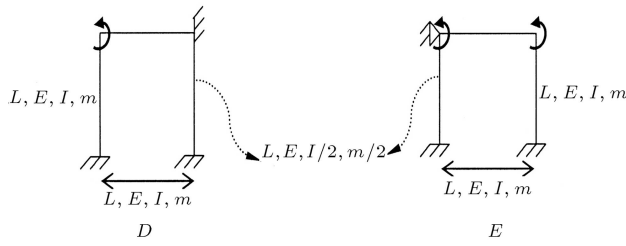
The submatrices corresponding to these two factors are obtained and their characteristic equations lead to the eigen-frequencies required as:

$$K_D = \left[ \frac{4EI}{L} + \frac{4EI}{L} \right] = \left[ \frac{8EI}{L} \right],$$

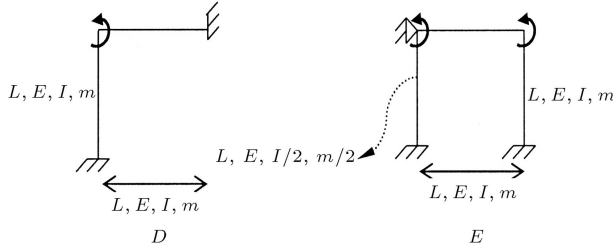
$$M_D = \left[ \frac{4mL^3}{420} + \frac{4mL^3}{420} \right] = \left[ \frac{8mL^3}{420} \right],$$



**Figure 13.** A frame with 3 DOFs.



**Figure 14.** The factors of the frame of Figure 11.



**Figure 15.** Alternative illustration of the factors of the frame of Figure 11.

$$\det[K_D - \omega^2 M_D] = 0 \quad \Rightarrow \quad \omega_1 = \sqrt{\frac{420EI}{mL^4}},$$

$$K_E = \begin{bmatrix} \frac{4EI}{L} + \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} + \frac{4(EI/2)}{L} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4EI}{L} + \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} + \frac{4(EI/2)}{L} \end{bmatrix},$$

$$M_E = \begin{bmatrix} \frac{4mL^3}{420} + \frac{4mL^3}{420} & \frac{-3mL^3}{420} \\ \frac{-3mL^3}{420} & \frac{4mL^3}{420} + \frac{4(m/2)L^3}{420} \end{bmatrix},$$

$$\det[K_E - \omega^2 M_E] = 0,$$

$$\omega_2 = \sqrt{\frac{525EI}{mL^4}},$$

and:

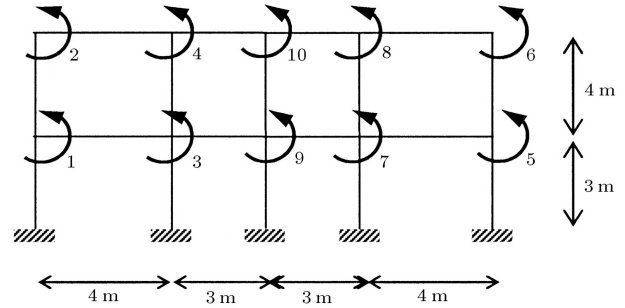
$$\omega_3 = \sqrt{\frac{378EI}{mL^4}}.$$

### Example 5

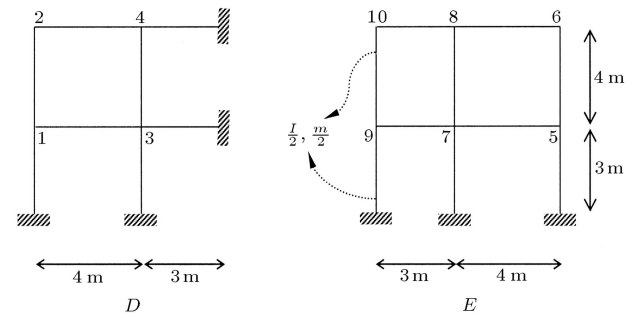
Consider the frame with an even number of spans, as shown in Figure 16, where the frame has 10 DOFs without side sway and 12 DOFs with side sway:

In the case of non-sway, factors  $D$  and  $E$  are obtained, as shown in Figure 17.

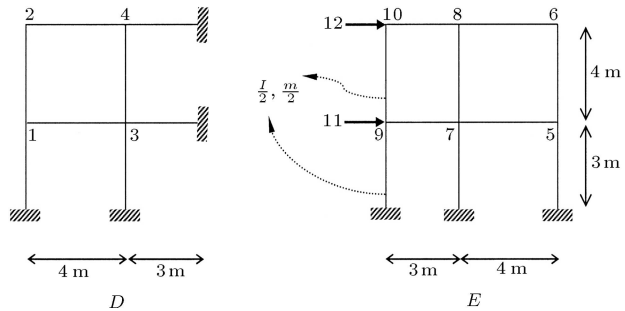
In this case, the eigensolution of a  $10 \times 10$  matrix is transformed into the eigensolution of two  $4 \times 4$  and  $6 \times 6$  matrices.



**Figure 16.** A frame with 4 spans.



**Figure 17.** Factors  $D$  and  $E$  for the non-sway frame.



**Figure 18.** Factors  $D$  and  $E$  for the sway frame.

In the sway case, factors  $D$  and  $E$  are obtained, as shown in Figure 18.

The factors of the main frame in the case of sway and non-sway are identical. Only factor  $E$  has the translation DOF.

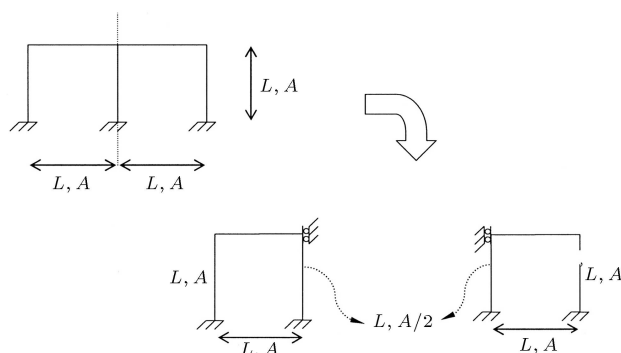
### Decomposition for Static and Dynamic Analysis in the Case of Symmetric Loading

In this section, a symmetric frame is decomposed into two factors. For static analysis, the decomposition of the frame is shown in Figure 19.

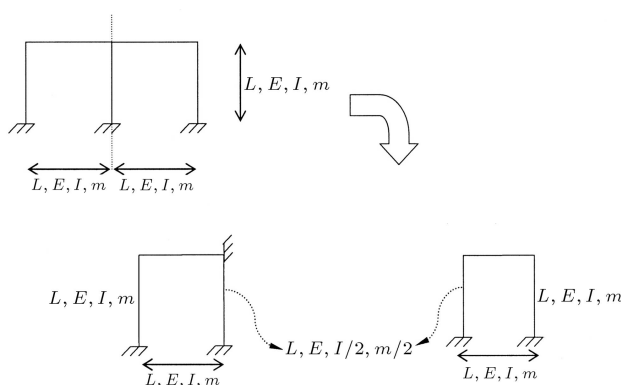
For dynamic analysis, the decomposition is illustrated in Figure 20.

### CONCLUDING REMARKS

The decomposition and healing processes presented in this article reduce the dimensions of the matrices for



**Figure 19.** Decomposition for the static analysis.



**Figure 20.** Decomposition for dynamic analysis.

dynamic analysis of the symmetric frames. Therefore, for large-scale problems, the accuracy of calculation increases and the cost of computation decreases.

It can be observed that for symmetric frames, one of the factors is common in both sway and non-sway cases. Therefore, if a frame has  $n$  symmetric DOFs, then, in both sway and non-sway cases, we will have common results. As an example, for a ten-storey frame with Form II symmetry, the natural frequencies can be obtained by three matrices of dimensions  $45 \times 45$ ,  $45 \times 45$  and  $55 \times 55$  in place of two matrices of dimensions  $100 \times 100$  and  $90 \times 90$ . This results in a considerable saving in computational time. The saving becomes more obvious when large-scale structures are considered.

In this paper, only eigenvalues are calculated, since the eigenvectors for the presented canonical forms can easily be obtained using the simple methods developed in [15].

The present method can also be applied to similar eigensolution problems such as stability analysis of symmetric frames for calculating the critical loads of frames.

## NOMENCLATURE

$K$	stiffness matrix of a member or a structure
$M$	mass matrix of a member or a structure

$m$	mass for one meter of the member
$L$	length of the member
$E$	elastic modulus
$I$	moment inertia of the member
$\omega$	natural frequencies
$\varphi$	vibration modes

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