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Eigensolution for Adjacency and Laplacian Matrices of Large Repetitive Structural Models

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Abstract. *Many structural models such as grids, barrel vaults, trusses and frames with repetitive units, known as regular structures, have structural matrices in the form of* $M = F(B, A, B^T)$. *In this paper, a simple and efficient method is presented for calculating the eigenvalues of the adjacency and Laplacian matrices of regular structures. These eigenvalues can be used in studying the combinatorial properties of these structures. Examples are included to show the accuracy of the presented approach.*

Keywords: *Canonical forms; Tri-diagonal; Eigensolution; Adjacency matrices; Laplacian matrices; Repetitive structural models.*

INTRODUCTION

In order to calculate the eigenvalues of a matrix, the characteristic equation of the matrix should be formed and the corresponding equation of order n should be solved. The solution of this equation for a large n is not only difficult but is also often accompanied by some errors.

In the past decade, canonical forms have been developed and used for the eigensolution of bi-lateral symmetric structures [1,2]. Other canonical forms consist of block tri-diagonal and block penta-digonal matrices arising from more general symmetries and regular structures [3,4]. For tri-diagonal cases, the corresponding matrices are often in the form $M = F(B, A, B)$, and the eigensolution of these problems can be simplified using special decomposition methods [5,6]. An excellent review of symmetry can be found in the work of Kangwai et al. [7]. Regular structures are those obtained by the graph products [8]. Definitions and concepts of product graphs may be found in the work of Imrich and Klavzar [9].

The adjacency, Laplacian matrices and the stiffness and mass matrices of cyclic repetitive structures also have block tri-diagonal forms with additional blocks at the far ends of their cross diagonal. The presences of these blocks, simplify the process of finding their eigenvalues [10,11] by decomposition approaches. For general cases, like repetitive structures or those obtained by different graph products, these matrices have the form $M = F(B, A, B^T)$.

One method for the eigensolution of this canonical form is adding some members to matrix M to convert B^T to B in order to obtain the form $M = F(B, A, B)$ [6] for easy approximate decomposition of M. There are also classic methods for solving $M = F(B, A, B^T)$ based on LU decomposition, preconditioning, divide and counter algorithms and other approximate methods [12-15].

In this paper, considering the properties of the matrices of the form $M = F(B, A, B^T)$, a special method is developed to simplify the calculations. This can be used in combinatorial optimization problems such as in the ordering and partitioning of graph models using a Fiedler vector [7,16]; it can also be employed in stability and dynamic analyses of repetitive space structures and finite element models.

BASIC DEFINITIONS OF GRAPH THEORY

Definitions from Graph Theory

A *graph*, $S(N, E)$, consists of a set of elements, $N(S)$, called *nodes* and a set of elements, $E(S)$, called

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members (*edges*), together with a relation of incidence which associates two distinct nodes with each member, known as its *ends*. Two nodes of a graph are called *adjacent* if these nodes are the end nodes of a member. A member is called *incident* with a node if it is an end node of the member [17]. The *degree* of a node is the number of edges incident with the node.

Matrices Associated with a Graph

Let S be a graph with N nodes. The adjacency matrix A is an N^*N matrix in which the entry in row I and column j is 1, if node n_i is adjacent to n_j, and is zero otherwise. This matrix is symmetric and the row sums of A are the degrees of nodes of S.

The Laplacian matrix, L, of graph S is defined as:

$$L = D - A, \tag{1}$$

where D is known as the degree matrix; it is also a diagonal matrix in which the ith diagonal entry is equal to the degree of node i.

The adjacency and Laplacian matrices are important matrices in the theory of graphs and their eigenvalues and eigenvectors form the foundation of a branch of mathematics known as the algebraic graph theory [18-20].

BLOCK TRI-DIAGONAL MATRICES IN STRUCTURAL MECHANICS

Many structural matrices are highly sparse and, with an appropriate nodal ordering, one can transform these matrices into banded forms. These matrices can then be partitioned to produce block tri-diagonal matrices [14]. The procedure is schematically illustrated in Figure 1.

In many regular and symmetric structures and especially in repetitive structures, by appropriate partitioning and nodal numbering, blocks can be produced as shown in Figure 2.

EIGENSOLUTION OF GENERAL BLOCK TRI-DIAGONAL MATRICES

Consider the following block diagonal canonical form:

$$M_{(n*m)(n*m)} = F(B_{(m*m)},\ A_{(m*m)},\ B^T_{(m*m)}), \tag{2}$$

where we have n blocks on the diagonal as shown in Equation 3. We assume n to be a large number.

$$M = \begin{bmatrix} A_{m*m} & B^T_{m*m} & & & & & \\ B_{m*m} & A_{m*m} & B^T_{m*m} & & & & \\ & & \cdots & & & & \\ & & B_{m*m} & A_{m*m} & B^T_{m*m} & & \\ & & & \cdots & & & \\ & & & & B_{m*m} & & \\ & & & & & A_{m*m} & B^T_{m*m} \\ & & & & & B_{m*m} & A_{m*m} \end{bmatrix}_{(n*m)(n*m)}. \tag{3}$$

For calculating the eigenvalues of M, this matrix should be written in the following standard form:

$$M\varphi = \lambda\varphi, \tag{4}$$

and, in the developed form, we have:

$$\begin{bmatrix} A-\lambda I & B^T & & & & & \\ B & A-\lambda I & B^T & & & & \\ & & \cdots & & & & \\ & & B & A-\lambda I & B^T & & \\ & & & \cdots & & & \\ & & & & B & & \\ & & & & & A-\lambda I & B^T \\ & & & & & B & A-\lambda I \end{bmatrix} * \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_i \\ \cdots \\ \varphi_{n-1} \\ \varphi_n \end{bmatrix} = 0. \tag{5}$$

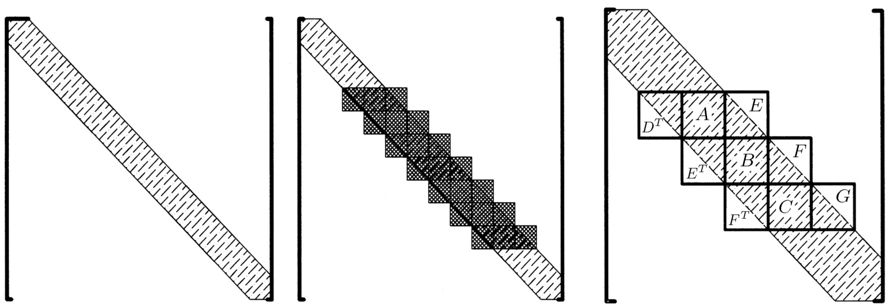

Figure 1. The process of transforming a banded matrix into a block tri-diagonal matrix.

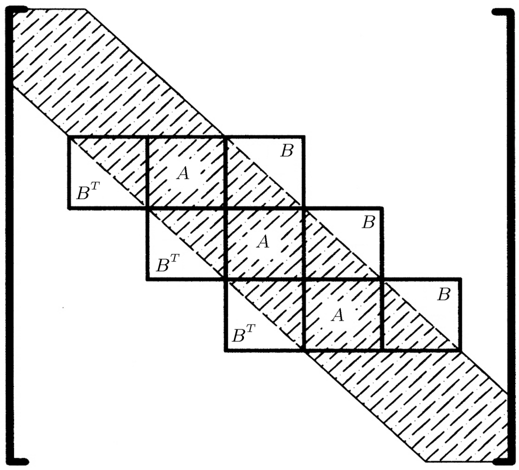

Figure 2. Block triangular diagonal matrix of a repetitive structure.

Expansion of the above matrix leads to:

$$\begin{cases} \cdots \\ \cdots \\ B\varphi_{i-2} + (A - \lambda I)\varphi_{i-1} + B^T\varphi_i = 0 \quad \rightarrow i-1 \\ B\varphi_{i-1} + (A - \lambda I)\varphi_i + B^T\varphi_{i+1} = 0 \quad \rightarrow i \\ \cdots \\ \cdots \end{cases} \tag{6}$$

Using the above set of equations, we have the following:

From the $(i-1)$th row of Equation 6, we consider:

$$\varphi_{i-2} \cong \alpha\varphi_{i-1},$$

and:

$$\varphi_{i-1} \cong \beta\varphi_i. \tag{7}$$

And, from the ith row of this matrix, we define:

$$\varphi_{i-1} \cong \gamma\varphi_i,$$

and:

$$\varphi_i \cong \eta\varphi_{i+1}. \tag{8}$$

The values of α, β, γ and η can easily be found, and since the matrix is considered to have a high dimension, therefore, one can accept $\alpha \cong \gamma \cong \beta \cong \eta$. Taking:

$$\alpha \cong \gamma \cong \beta \cong \eta \cong e^{i\theta}, \tag{9}$$

and considering Equations 8 and 9, we have:

$$\varphi_{i-1} \cong e^{i\theta}\varphi_i,$$

and:

$$\varphi_{i+1} \cong e^{-i\theta}\varphi_i. \tag{10}$$

Substituting Equation 10 in the ith row of Equation 6 leads to:

$$e^{i\theta}B\varphi_i + (A - \lambda I)\varphi_i + e^{-i\theta}B^T\varphi_i = 0, \tag{11}$$

$$(e^{i\theta}B + A + e^{-i\theta}B^T)\varphi_i = \lambda\varphi_i. \tag{12}$$

Equation 12 shows that the eigenvalues of matrix M can be obtained from the eigenvalues of $(e^{i\theta}B + A + e^{-i\theta}B^T)$. Using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ simplifies Equation 12 as:

$$\mathrm{eig}(M) = \bigcup_{j=1}^{n} \mathrm{eig}(A + \cos(\theta) * (B + B^T) + i * \sin(\theta) * (B - B^T)),$$

$$\theta = \frac{j * \pi}{n+1}. \tag{13}$$

Employing the above relationship, the eigensolution of large matrices becomes feasible using the properties of its constituting small blocks.

ADJACENCY AND LAPLACIAN MATRICES OF LARGE REPETITIVE SYSTEMS

For tri-diagonal matrices whose cores are identical, the corresponding adjacency matrices have the following form:

$$A = \begin{bmatrix} A_{m*m} & B^T_{m*m} & & & & & \\ B_{m*m} & A_{m*m} & B^T_{m*m} & & & & \\ & & \cdots & & & & \\ & & B_{m*m} & A_{m*m} & B^T_{m*m} & & \\ & & & \cdots & & & \\ & & & & B_{m*m} & & \\ & & & & & A_{m*m} & B^T_{m*m} \\ & & & & & B_{m*m} & A_{m*m} \end{bmatrix}_{(n*m)(n*m)}. \tag{14}$$

However, the Laplacian matrix does not have this pattern. This matrix and its eigenvalues are in the following form, where the diagonal blocks are not identical:

$$L=\begin{bmatrix} C_{m*m} & B^T_{m*m} & & & & & \\ B_{m*m} & E_{m*m} & B^T_{m*m} & & & & \\ & & \cdots & & & & \\ & & B_{m*m} & E_{m*m} & B^T_{m*m} & & \\ & & & \cdots & & & \\ & & & & B_{m*m} & E_{m*m} & B^T_{m*m} \\ & & & & & B_{m*m} & C_{m*m} \end{bmatrix}_{(n*m)(n*m)}$$

$$\text{eig}(L)=\begin{bmatrix} 0 \\ \lambda_{2L} \\ \lambda_{3L} \\ \cdots \\ \lambda_{iL} \\ \cdots \\ \lambda_{nL} \end{bmatrix}. \tag{15}$$

For this matrix, when the dimension of the matrix increases, its eigenvalues approach to the eigenvalues of the following matrix, M, with some differences:

$$M=\begin{bmatrix} E_{m*m} & B^T_{m*m} & & & & & \\ B_{m*m} & E_{m*m} & B^T_{m*m} & & & & \\ & & \cdots & & & & \\ & & B_{m*m} & E_{m*m} & B^T_{m*m} & & \\ & & & \cdots & & & \\ & & & & B_{m*m} & E_{m*m} & B^T_{m*m} \\ & & & & & B_{m*m} & E_{m*m} \end{bmatrix}_{(n*m)(n*m)}$$

$$\text{eig}(M)=\begin{bmatrix} \lambda_{1M} \\ \lambda_{2M} \\ \lambda_{3M} \\ \cdots \\ \lambda_{iM} \\ \cdots \\ \lambda_{nM} \end{bmatrix}, \tag{16}$$

i.e. when the dimension of the matrix increases, the effect of the difference of C_{m*m} and E_{m*m} in eigenvalues of the Laplacian matrix can be neglected. An approximate relation between the eigenvalues of the L and the M matrices can be expressed as:

$$\begin{bmatrix} \lambda_{1M} \\ \lambda_{2M} \\ \lambda_{3M} \\ \cdots \\ \lambda_{iM} \\ \cdots \end{bmatrix} \cong \begin{bmatrix} \lambda_{2L} \\ \lambda_{3L} \\ \lambda_{4L} \\ \cdots \\ \lambda_{(i+1)L} \\ \cdots \end{bmatrix}. \tag{17}$$

NUMERICAL EXAMPLES

Example 1

As the first example, we consider the adjacency and Laplacian matrices of the graph model of a planar truss, S, as shown in Figure 3.

The pattern of the adjacency matrix of S can be shown as:

$$A(S)=\begin{bmatrix} A_{6*6} & B_{6*6} & 0_{6*6} \\ B^T_{6*6} & A_{6*6} & B_{6*6} \\ 0_{6*6} & B^T_{6*6} & A_{6*6} \end{bmatrix}_{18*18}, \tag{18}$$

in which the submatrices A and B are as follows:

$$A_{6*6}=\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$B_{6*6}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

The exact and approximate eigenvalues for the adjacency matrix of this graph are calculated and compared in Figure 4. In this paper, Matlab is used for the eigensolution of the matrices and the results are referred to as exact solutions. Here, three blocks are considered on the diagonal of the matrix. In all the diagrams for comparison, the x-axis is (i) and the y-axis contains eig (i).

It should be noted that the above matrix can also be partitioned, as follows, with six blocks on the

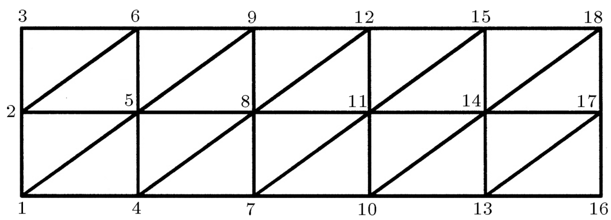

Figure 3. A graph model S of a planar truss.

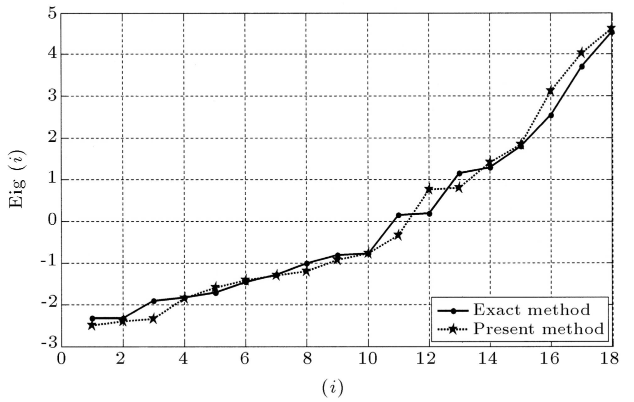

Figure 4. Comparison of the eigenvalues of the adjacency matrix of S.

diagonal of the matrix:

$$A(S)=\begin{bmatrix} A_{3*3} & B_{3*3} & & \\ B^T_{3*3} & A_{3*3} & B_{3*3} & \\ & \cdots & & B_{3*3} \\ & & B^T_{3*3} & A_{3*3} \end{bmatrix}_{18*18},$$

and:

$$A_{3*3}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_{3*3}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

For this case, the eigenvalues are obtained and compared in Figure 5.

Now we study the eigenvalues of the Laplacian matrix of graph S:

$$L(S)=\begin{bmatrix} C_{3*3} & B_{3*3} & & \\ B^T_{3*3} & E_{3*3} & B_{3*3} & \\ & \cdots & & B_{3*3} \\ & & B^T_{3*3} & C_{3*3} \end{bmatrix}_{18*18},$$

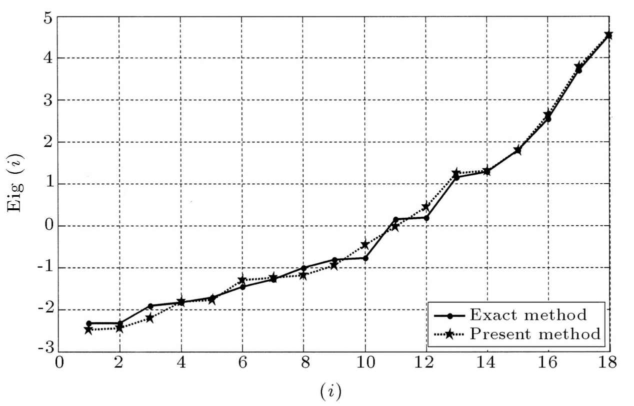

Figure 5. Comparison of the eigenvalues of the adjacency matrix using the present method with six blocks on the diagonal and the exact approach.

$$E_{3*3}=\begin{bmatrix} 4 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 4 \end{bmatrix},$$

$$B_{3*3}=\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$C_{3*3}=\begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{21}$$

It should be noted that for the Laplacian matrix, first and end blocks of the main diagonal C are different from other middle blocks, however, for large Laplacian matrices, considering $C = E$ leads to accurate results and the ith eigenvalue shifts one, i.e. the ith eigenvalues of the main Laplacian matrix is equal to the $(i-1)$th approximated one, when $C = E$. A comparison of the exact and approximate values is illustrated in Figure 6.

Example 2

Now, consider a graph with more nodes and members than that of the previous example, as shown in Figure 7.

The Laplacian and adjacency matrices of S can be written as:

$$A(S)=\begin{bmatrix} A_{3*3} & B_{3*3} & & \\ B^T_{3*3} & A_{3*3} & B_{3*3} & \\ & \cdots & & B_{3*3} \\ & & B^T_{3*3} & A_{3*3} \end{bmatrix}_{48*48},$$

and:

$$A_{3*3}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_{3*3}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

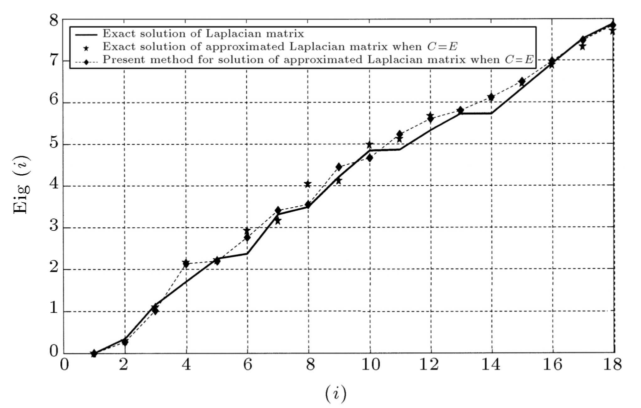

Figure 6. Comparison of the eigenvalues of the Laplacian matrix using the present method with six blocks on the diagonal and the exact approach.

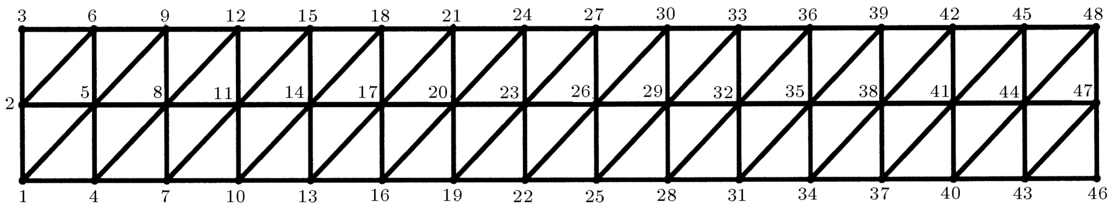

Figure 7. A graph model, S.

$$L(S)=\begin{bmatrix} C_{3*3} & B_{3*3} & & \\ B^T_{3*3} & E_{3*3} & B_{3*3} & \\ & \cdots & & B_{3*3} \\ & & B^T_{3*3} & C_{3*3} \end{bmatrix}_{48*48},$$

$$E_{3*3}=\begin{bmatrix} 4 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 4 \end{bmatrix},$$

$$B_{3*3}=\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$C_{3*3}=\begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{23}$$

The eigenvalues for adjacency and Laplacian matrices are calculated and compared in Figures 8 and 9, respectively.

Example 3

Consider a space truss, S, as shown in Figure 10. The pattern of the adjacency matrix of S is depicted in Figure 11.

For this example, the number of repeated patterns or cores of the truss is $n = 19$, and the matrices A and B are as follows:

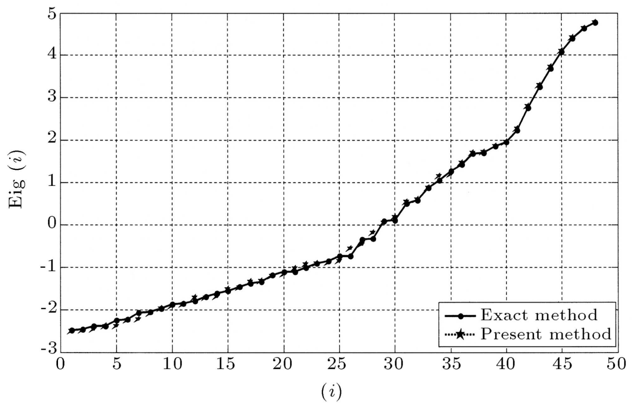

Figure 8. Comparison of the eigenvalues of the adjacency matrix using the present method with 16 blocks on the diagonal and the exact approach.

$$A=\begin{bmatrix}
0&1&&&&&&&&&1&&&&&&& \\
1&0&1&&&&&&&&1&&&&&&& \\
&1&0&1&&&&&&&1&1&1&&&&& \\
&&1&0&1&&&&&&&&1&&&&& \\
&&&1&0&1&&&&&&&1&1&1&&& \\
&&&&1&0&1&&&&&&&&1&&& \\
&&&&&1&0&1&&&&&&&1&1&1& \\
&&&&&&1&0&1&&&&&&&&1& \\
&&&&&&&1&0&1&&&&&&&1&1 \\
&&&&&&&&1&0&1&&&&&&&1 \\
1&1&1&&&&&&&1&0&1&&&&&& \\
&&1&&&&&&&&1&0&1&&&&& \\
&&1&1&1&&&&&&&1&0&1&&&& \\
&&&&1&&&&&&&&1&0&1&&& \\
&&&&1&1&1&&&&&&&1&0&1&& \\
&&&&&&1&&&&&&&&1&0&1& \\
&&&&&&1&1&1&&&&&&&1&0&1 \\
&&&&&&&&1&&&&&&&&1&0
\end{bmatrix}_{18*18},$$

$$B=\begin{bmatrix}
0&&&&&&&&&&&&&&&&& \\
&1&&&&&&&&&1&&&&&&& \\
&&1&&&&&&&&&1&&&&&& \\
&&&1&&&&&&&&&1&&&&& \\
&&&&1&&&&&&&&&1&&&& \\
&&&&&1&&&&&&&&&1&&& \\
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&&&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&&&&1
\end{bmatrix}_{18*18}. \tag{24}$$

It should be mentioned that B is in general form and is not symmetric and the corresponding tri-diagonal matrix is in the form $A(S) = F(B, A, B^T)$. Thus, we can use the proposed method for the eigensolution of

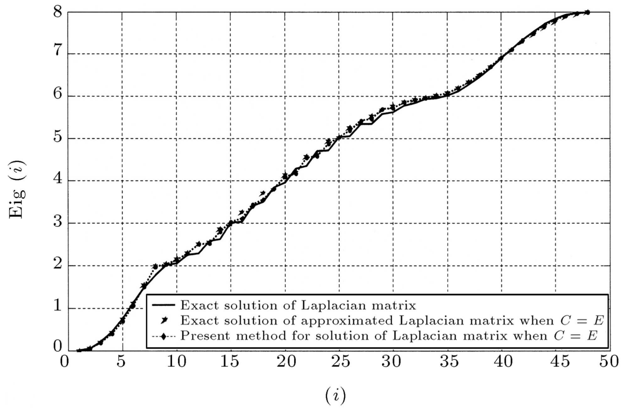

Figure 9. Comparison of the eigenvalues of the Laplacian matrix using the present method with 16 blocks on the diagonal and the exact approach.

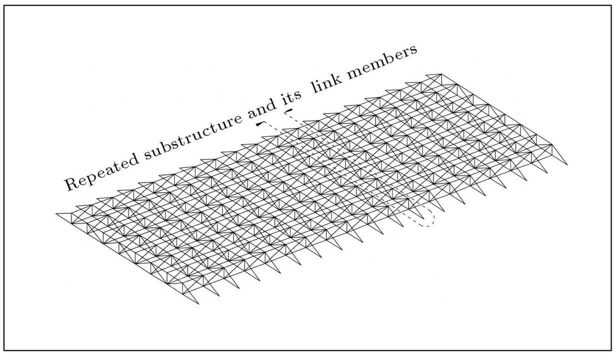

Figure 10. The model of a space repetitive truss, S.

this truss model. Then:

$$\text{eig}(A(S)) = \bigcup_{j=1}^{18} \text{eig}(A + \cos(\theta) * (B + B^T) + i * \sin(\theta) * (B - B^T)),$$

$$\theta = \frac{j * \pi}{20},$$

and:

$$j = 1, 2, 3, \cdots, 19. \tag{25}$$

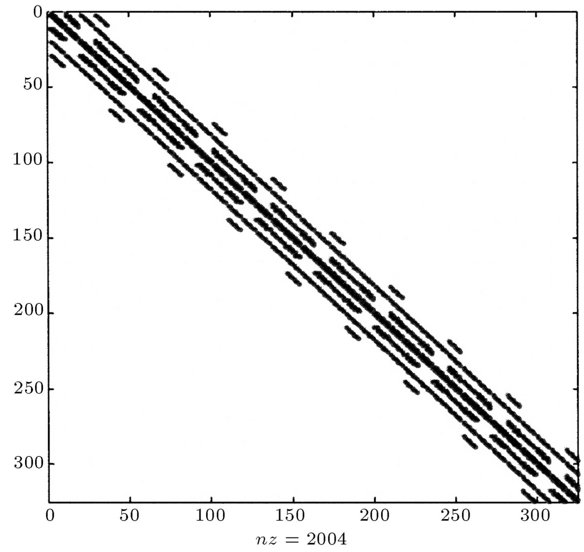

Figure 11. Pattern of the adjacency matrix for S.

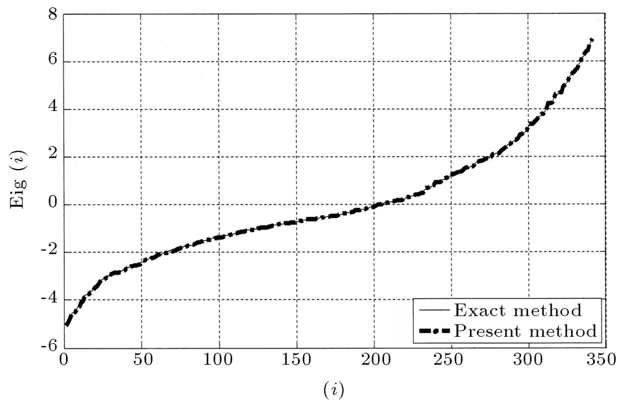

Figure 12. Comparison of eigenvalues of adjacency matrix of presented truss model.

The exact and approximately calculated eigenvalues of the adjacency matrix are compared in Figure 12 and Table 1.

For comparative study, the first five and the last five eigenvalues of the adjacency matrix of the truss model are compared in Table 1.

It can be seen from Table 1 that the results of the present method for eigensolution of the adjacency

Table 1. Comparison of the eigenvalues of the adjacency matrix of Example 3.

First Five Eigenvalues		Last Five Eigenvalues	
Exact Method	**Present Method**	**Exact Method**	**Present Method**
-5.03963	-5.0412	6.410592	6.411105
-4.96772	-4.97384	6.439727	6.446794
-4.84988	-4.86301	6.658422	6.662591
-4.68908	-4.71087	6.818087	6.820003
-4.54520	-4.54703	6.915284	6.915773

matrix are quite close to the eigenvalues of the exact method.

Now, consider the Laplacian matrix of space truss S of Figure 10. The submatrices A and B for the Laplacian matrix of this example are as follows:

$$E = \left[\begin{array}{cccccccccccccccccc}
2 & -1 & & & & & & & & & -1 & & & & & & & \\
-1 & 6 & -1 & & & & & & & & -1 & & & & & & & \\
& -1 & 8 & -1 & & & & & & & -1 & -1 & -1 & & & & & \\
& & -1 & 6 & -1 & & & & & & & & -1 & & & & & \\
& & & -1 & 8 & -1 & & & & & & & -1 & -1 & -1 & & & \\
& & & & -1 & 6 & -1 & & & & & & & & -1 & & & \\
& & & & & -1 & 8 & -1 & & & & & & & -1 & -1 & -1 & \\
& & & & & & -1 & 6 & -1 & & & & & & & & -1 & \\
& & & & & & & -1 & 7 & -1 & & & & & & & -1 & -1 \\
& & & & & & & & -1 & 2 & -1 & & & & & & & -1 \\
-1 & -1 & -1 & & & & & & & -1 & 7 & -1 & & & & & & \\
& & -1 & & & & & & & & -1 & 6 & -1 & & & & & \\
& & -1 & -1 & -1 & & & & & & & -1 & 8 & -1 & & & & \\
& & & & -1 & & & & & & & & -1 & 6 & -1 & & & \\
& & & & -1 & -1 & -1 & & & & & & & -1 & 8 & -1 & & \\
& & & & & & -1 & & & & & & & & -1 & 6 & -1 & \\
& & & & & & -1 & -1 & -1 & & & & & & & -1 & 8 & -1 \\
& & & & & & & & -1 & & & & & & & & -1 & 6
\end{array}\right],$$

.

$$B = -\left[\begin{array}{cccccccccccccccccc}
0 & & & & & & & & & & & & & & & & & \\
& 1 & & & & & & & & & 1 & & & & & & & \\
& & 1 & & & & & & & & & 1 & & & & & & \\
& & & 1 & & & & & & & & & 1 & & & & & \\
& & & & 1 & & & & & & & & & 1 & & & & \\
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& & & & & & & & & & & 1 & & & & & & \\
& & & & & & & & & & & & 1 & & & & & \\
& & & & & & & & & & & & & 1 & & & & \\
& & & & & & & & & & & & & & 1 & & & \\
& & & & & & & & & & & & & & & 1 & & \\
& & & & & & & & & & & & & & & & 1 & \\
& & & & & & & & & & & & & & & & & 1
\end{array}\right]_{18*18} \tag{26}$$

Here, the number of the repeated substructure is $n = 19$ and the eigenvalues of the Laplacian matrix for this example, for the case $C = E$, are compared to the results of the present method in Figure 13.

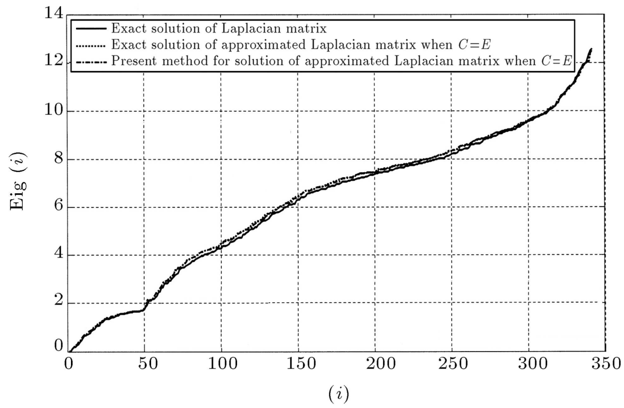

Figure 13. Comparison of the eigenvalues of the Laplacian matrix of S.

For this case, the first five and the last five eigenvalues of the Laplacian matrix of S are compared in Table 2.

CONCLUDING REMARKS

In this paper, a simple method is presented for calculating the eigenvalues of large adjacency and Laplacian matrices of structural models having the canonical form:

$$M = F(B, A, B^T).$$

Examples studied here show that the results obtained by the present method are comparable to those of the exact solution. The calculated eigenvalues are very close to the exact values, and can efficiently be used for solution of the models whose structural matrices are or can be transformed into the block tri-diagonal form, $M = F(B, A, B^T)$.

The present method can be used in combinatorial optimization problems such as the ordering and partitioning of structural models. This method can also be extended to the eigensolutions corresponding to the calculation of the eigen-frequencies and eigen-modes of the repetitive space structures or finite element

Table 2. Comparison of the eigenvalues of the Laplacian matrix of S.

First Five Eigenvalues			Last Five Eigenvalues		
Exact Method	**Approximated Laplacian $C = E$**		**Exact Method**	**Approximated Laplacian $C = E$**	
	Exact Method	**Present Method**		**Exact Method**	**Present Method**
0.00000	0.00000	0.00000	12.17195	11.99790	12.01787
0.03081	0.02919	0.02973	12.36177	12.21275	12.22654
0.12536	0.11518	0.11342	12.50104	12.38541	12.39362
0.17899	0.20413	0.20374	12.58609	12.51176	12.51556
0.20636	0.25293	0.24926	12.58609	12.58881	12.58978

models. The method can also be extended to finding the buckling load of the structures.

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