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Research Note

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Quintic Spline Solution of Boundary Value Problems in the Plate Deflection Theory

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Abstract. *In this paper, Quintic spline in off-step points is used for the solution of fourth-order boundary value problems. Spline relations and boundary formulas are developed and the convergence analysis of the given method is investigated. Numerical illustrations are given to show the applicability and efficiency of our method.*

Keywords: *Fourth-order ordinary differential equation; Quintic spline; Off-step points; Convergence analysis; Monotone matrix.*

INTRODUCTION

We consider the fourth-order boundary value problems of the form:

$$u^{(4)}(x)+f(x)u(x)=g(x), a\leq x\leq b, \tag{1}$$

subject to the following boundary conditions:

$$u(a)=A_1, \qquad u(b)=A_2,$$

$$u''(a)=B_1, \qquad u''(b)=B_2, \tag{2}$$

where $f(x)$ and $g(x)$ are continuous on $[a,b]$ and A_i, B_i, $i=1,2$ are real finite constants. Such types of fourth-order boundary value problems arise frequently in the plate deflection theory [1]. The analytical solution of the Problem in Equation 1 for the arbitrary choice of $f(x)$ and $g(x)$ cannot be determined [1-5]. We assume that $u(x)$ is sufficiently differentiable and that a unique solution of the problem in Equation 1 exists [2,3,6].

Further discussions of fourth-order boundary value problems are given in [1-3,7]. Usmani [1] discussed the existence and uniqueness solutions of such problems when subjected to the following boundary conditions:

$$u(a)=A_1, \qquad u(b)=A_2,$$

$$u'(a)=B_1, \qquad u'(b)=B_2.$$

Sixth order methods for solving this problem were used by Usmani [2,3]. Methods of order two and four, based on quintic and sextic splines, were developed by Usmani [4,5]. Later, Usmani [6] used quartic splines for the numerical solution of fourth-order boundary value problems. Rashidinia [8] and Usmani et al. [9] derived the quintic spline and non-polynomial quintic spline methods for the solution of linear fourth-order boundary value problems. But all the derived methods use nodal points; only in [6] a quartic spline with off-step points is used.

In this paper, first a direct method based on the quintic spline for fourth-order boundary value problems (Equation 1) is presented. Our aim is to approximate $u(x)$ satisfying (Equation 1) by using quintic spline functions, $\in C^4[a,b]$. This approach will employ consistency relations at midknot.

Then, the quintic spline formulation is derived for the numerical solution of Equations 1 and 2. Following that to retain the bandwidth of the coefficient matrix of the system as five, we develop the end conditions of $O(h^6)$. Subsequently, convergence analysis is proved so that the matrix associated with the system of linear equations that arises is not assumed to be monotone, as often believed in the post. Finally, the numerical

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evidence is included to demonstrate the efficiency of the presented method.

QUINTIC SPLINE FUNCTION

We consider a uniform mesh, Δ, with nodal points, x_i on $[a,b]$ such that:

$$\Delta : a = x_0 < x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_N = b,$$

where $x_{i-\frac{1}{2}} = a+(i-\frac{1}{2})h$, $i = 1,2,\cdots,N$ and $h = \frac{b-a}{N}$. Also, we denote a function value, $u(x_i)$ by u_i.

Definition

A quintic spline function, $S_i(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as following:

1. In each subinterval, $[x_i, x_{i+1}]$, $S_i(x)$ is a polynomial of, at most, degree five;
2. The first-fourth derivatives of $S_i(x)$ are continuous on $[a,b]$;
3. $S_i(x_i) = u(x_i)$, $i = 0(1)N$.

The spline function, $S_i(x)$, for $x \in [x_i, x_{i+1}]$, is defined by:

$$S_i(x) = \sum_{k=0}^{5} a_i^{(k)}(x-x_i)^k, \qquad i = 0,1,2,\cdots,N, \quad (3)$$

where $a_i^{(k)}$, $k = 0,1,\cdots,5$ are constants to be determined.

We further require that the values of the first-, second-, third- and fourth-order derivatives are the same for the pair of segments that join at each point (x_i, u_i).

To derive an expression for the coefficients of Equation 3, in terms of $u_{i-\frac{1}{2}}$, $u_{i+\frac{1}{2}}$, $M_{i-\frac{1}{2}}$, $M_{i+\frac{1}{2}}$, $F_{i-\frac{1}{2}}$ and $F_{i+\frac{1}{2}}$, we first denote:

(i) $\quad S_i(x_{i-\frac{1}{2}}) = u_{i-\frac{1}{2}}$,

(ii) $\quad S_i(x_{i+\frac{1}{2}}) = u_{i+\frac{1}{2}}$,

(iii) $\quad S_i''(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}$,

(iv) $\quad S_i''(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}$,

(v) $\quad S_i^{(4)}(x_{i-\frac{1}{2}}) = F_{i-\frac{1}{2}}$,

(vi) $\quad S_i^{(4)}(x_{i+\frac{1}{2}}) = F_{i+\frac{1}{2}}$. $\quad (4)$

From algebraic manipulation, we get the following expression:

$$a_i^{(0)} = \frac{1}{768}\left[5h^4\left(F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}\right) - 48h^2\left(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}\right) + 384\left(u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}\right)\right],$$

$$a_i^{(1)} = \frac{1}{5760h}\left[7h^4\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) + 240h^2\left(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}\right) + 5760\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right)\right],$$

$$a_i^{(2)} = \frac{1}{32}\left[-h^2\left(F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}\right) + 8\left(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}\right)\right],$$

$$a_i^{(3)} = \frac{1}{144h}\left[h^2\left(F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}\right) + 24\left(M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}\right)\right],$$

$$a_i^{(4)} = \frac{1}{48}\left(F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}\right),$$

$$a_i^{(5)} = \frac{1}{120h}\left(F_{i+\frac{1}{2}} + F_{i-\frac{1}{2}}\right),$$

where $i = 0,1,2,\cdots,N$. The continuity of the first derivative implies:

$$M_{i-\frac{3}{2}} + 22M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}} = \frac{h^2}{240}(7F_{i-\frac{3}{2}} - 254F_{i-\frac{1}{2}} + 7F_{i+\frac{1}{2}}) + \frac{24}{h^2}(u_{i-\frac{3}{2}} - 2u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}),$$

$$i = 2(1)N-1, \quad (5)$$

and the continuity of the third derivative yields:

$$M_{i-\frac{3}{2}} - 2M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}} = \frac{h^2}{24}(F_{i-\frac{3}{2}} + 22F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}),$$

$$i = 2(1)N-1. \quad (6)$$

Subtracting Equation 6 from Equation 5 and dividing it by 24, we obtain:

$$M_{i-\frac{1}{2}} = \frac{1}{h^2}(u_{i-\frac{3}{2}} - 2u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}) - \frac{h^2}{1920}(F_{i-\frac{3}{2}} + 158F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}). \tag{7}$$

Elimination of M_i's between Equations 6 and 7 leads to the following useful relation:

$$u_{i-\frac{5}{2}} - 4u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} - 4u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} = \frac{h^4}{1920}(F_{i-\frac{5}{2}} + 236F_{i-\frac{3}{2}} + 1446F_{i-\frac{1}{2}} + 236F_{i+\frac{1}{2}} + F_{i+\frac{3}{2}}),$$
$$i = 3(1)N-2. \tag{8}$$

NUMERICAL METHOD

Now, we consider Equation 1 subject to boundary conditions (Equation 2). We discretize the given system in Equation 1 at the grid points, x_i, $i = 3, 4, \cdots, N-2$, and use the spline relation (Equation 8). We obtain the $(N-4)$ linear algebraic equation in the (N) unknowns, $u_{i-\frac{1}{2}}$, $i = 1, 2, \cdots, N$, as:

$$\left(1 + \frac{1}{1920}h^4 f_{i-\frac{5}{2}}\right)u_{i-\frac{5}{2}} + \left(-4 + \frac{236}{1920}h^4 f_{i-\frac{3}{2}}\right)u_{i-\frac{3}{2}} + \left(6 + \frac{1446}{360}f_{i-\frac{1}{2}}\right)h^4 u_{i-\frac{1}{2}} + \left(-4 + \frac{236}{1920}h^4 f_{i+\frac{1}{2}}\right)u_{i+\frac{1}{2}} + \left(1 + \frac{1}{1920}h^4 f_{i+\frac{3}{2}}\right)u_{i+\frac{3}{2}}$$
$$= \frac{h^4}{1920}(g_{i-\frac{5}{2}} + 236g_{i-\frac{3}{2}} + 1446g_{i-\frac{1}{2}} + 236g_{i+\frac{1}{2}} + g_{i+\frac{3}{2}}), \qquad i = 3(1)N-2, \tag{9}$$

where $f_i = f(x_i)$ and $g_i = g(x_i)$.

To obtain the unique solution of the above systems, we need four more equations. By using a Taylor series and the method of undetermined coefficients, the boundary formulas associated with boundary conditions (Equation 2) can be determined as follows:

$$a_0u_0 + a_1u_{\frac{1}{2}} + a_2u_{\frac{3}{2}} + a_3u_{\frac{5}{2}} + ch^2u_0'' + h^4\left[b_1u^{(4)}_{\frac{1}{2}} + b_2u^{(4)}_{\frac{3}{2}} + b_3u^{(4)}_{\frac{5}{2}}\right] + t_1 = 0, \tag{10}$$

$$a_0'u_0 + a_1'u_{\frac{1}{2}} + a_2'u_{\frac{3}{2}} + a_3'u_{\frac{5}{2}} + a_4'u_{\frac{7}{2}} + c'h^2u_0'' + h^4\left[b_1'u^{(4)}_{\frac{1}{2}} + b_2'u^{(4)}_{\frac{3}{2}} + b_3'u^{(4)}_{\frac{5}{2}} + b_4'u^{(4)}_{\frac{7}{2}}\right] + t_2 = 0, \tag{11}$$

and:

$$a_0^{'*}u_N + a_1^{'*}u_{N-\frac{1}{2}} + a_2^{'*}u_{N-\frac{3}{2}} + a_3^{'*}u_{N-\frac{5}{2}} + a_4^{'*}u_{N-\frac{7}{2}} + c^{'*}h^2u_N'' + h^4\left[b_1^{'*}u^{(4)}_{N-\frac{1}{2}} + b_2^{'*}u^{(4)}_{N-\frac{3}{2}} + b_3^{'*}u^{(4)}_{N-\frac{5}{2}} + b_4^{'*}u^{(4)}_{N-\frac{7}{2}}\right] + t_{N-1} = 0, \tag{12}$$

$$a_0^*u_N + a_1^*u_{N-\frac{1}{2}} + a_2^*u_{N-\frac{3}{2}} + a_3^*u_{N-\frac{5}{2}} + c^*h^2u_N'' + h^4\left[b_1^*u^{(4)}_{N-\frac{1}{2}} + b_2^*u^{(4)}_{N-\frac{3}{2}} + b_3^*u^{(4)}_{N-\frac{5}{2}}\right] + t_N = 0, \tag{13}$$

In order that t_1, t_2, t_{N-1} and t_N are $O(h^6)$, we find that:

$$(a_0, a_1, a_2, a_3, c) = (a_0^*, a_1^*, a_2^*, a_3^*, c^*) = \left(-6, 10, -5, 1, \frac{5}{4}\right),$$

$$(b_1, b_2, b_3) = (b_1^*, b_2^*, b_3^*) = -\left(\frac{383}{960}, \frac{383}{1920}, \frac{1}{1920}\right),$$

$$(a_0', a_1', a_2', a_3', a_4', c') = (a_0^{'*}, a_1^{'*}, a_2^{'*}, a_3^{'*}, a_4^{'*}, c^{'*}) = \left(2, -5, 6, -4, 1, \frac{1}{4}\right),$$

$$(b_1', b_2', b_3', b_4') = (b_1^{'*}, b_2^{'*}, b_3^{'*}, b_4^{'*}) = -\left(\frac{383}{1920}, \frac{113}{192}, \frac{33}{160}, \frac{1}{1920}\right).$$

From the above relations, we obtain the following equations:

$$\left(10 + \frac{383}{960}h^4 f_{\frac{1}{2}}\right)u_{\frac{1}{2}} + \left(-5 + \frac{383}{360}h^4 f_{\frac{3}{2}}\right)u_{\frac{3}{2}} + \left(1 + \frac{1}{1920}h^4 f_{\frac{5}{2}}\right)u_{\frac{5}{2}} = 6u_0 - \frac{5}{4}h^2u_0'' + \frac{h^4}{1920}\left[766g_{\frac{1}{2}} + 383g_{\frac{3}{2}} + g_{\frac{5}{2}}\right] + \frac{181}{11520}h^6u^{(6)}(\xi_1) + O(h^7), \qquad i = 1, \tag{14}$$

$$\left(-5+\frac{383}{1920}h^4f_{\frac{1}{2}}\right)u_{\frac{1}{2}}+\left(6+\frac{113}{192}h^4f_{\frac{3}{2}}\right)u_{\frac{3}{2}}$$

$$+\left(-4+\frac{33}{160}h^4f_{\frac{5}{2}}\right)u_{\frac{5}{2}}+\left(1+\frac{1}{1920}h^4f_{\frac{7}{2}}\right)u_{\frac{7}{2}}$$

$$=-2u_0-\frac{1}{4}h^2u_0''+\frac{h^4}{1920}\Big[383g_{\frac{1}{2}}+1130g_{\frac{3}{2}}$$

$$+396g_{\frac{5}{2}}+g_{\frac{7}{2}}\Big]+\frac{497}{11520}h^6u^{(6)}(\xi_1)+O(h^7),$$

$$i=2, \tag{15}$$

and:

$$\left(1+\frac{1}{1920}h^4f_{N-\frac{7}{2}}\right)u_{N-\frac{7}{2}}$$

$$+\left(-4+\frac{33}{160}h^4f_{N-\frac{5}{2}}\right)u_{N-\frac{5}{2}}$$

$$+\left(6+\frac{113}{192}h^4f_{N-\frac{3}{2}}\right)u_{N-\frac{3}{2}}$$

$$+\left(-5+\frac{383}{1920}h^4f_{N-\frac{1}{2}}\right)u_{N-\frac{1}{2}}=-2u_N-\frac{1}{4}h^2u_N''$$

$$+\frac{h^4}{1920}\Big[g_{N-\frac{7}{2}}+396g_{N-\frac{5}{2}}+1130g_{N\frac{3}{2}}+383g_{N-\frac{1}{2}}\Big]$$

$$+\frac{497}{11520}h^6u^{(6)}(\xi_{N-1})+O(h^7), \qquad i=N-1, \tag{16}$$

$$\left(1+\frac{1}{1920}h^4f_{N-\frac{5}{2}}\right)u_{N-\frac{5}{2}}$$

$$+\left(-5+\frac{38}{1920}h^4f_{N-\frac{3}{2}}\right)u_{N-\frac{3}{2}}$$

$$+\left(10+\frac{383}{960}h^4f_{N-\frac{1}{2}}\right)u_{N-\frac{1}{2}}=6u_N-\frac{5}{4}h^2u_N''$$

$$+\frac{h^4}{1920}\Big[g_{N-\frac{5}{2}}+383g_{N-\frac{3}{2}}+766g_{N-\frac{1}{2}}\Big]$$

$$+\frac{181}{11520}h^6u^{(6)}(\xi_N)+O(h^7), \qquad i=N. \tag{17}$$

The scheme of Equation 9 along with boundary formulae (Equations 14, 15, 16 and 17) yields the five diagonal linear system of order $N\times N$ and may be written in matrix form as:

$$AU=C+T, \tag{18}$$

$$A\overline{U}=C, \tag{19}$$

$$AE=T, \tag{20}$$

where $U=(u_{i-\frac{1}{2}})$, $\overline{U}=\left(\overline{u}_{i-\frac{1}{2}}\right)$, $T=\left(t_{i-\frac{1}{2}}\right)$ and $E=\left(e_{i-\frac{1}{2}}\right)=\left(u_{i-\frac{1}{2}}-\overline{u}_{i-\frac{1}{2}}\right)$ for $i=1(1)N$ are N-dimensional column vectors.

Matrix A can be denoted by $A=A_0+h^4BF$, where:

$$A_0=$$

$$\begin{pmatrix}
10 & -5 & 1 & & & & & & \\
-5 & 6 & -4 & 1 & & & & & \\
1 & -4 & 6 & -4 & 1 & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & & 1 & -4 & 6 & -5 \\
 & & & & & & 1 & -5 & 10
\end{pmatrix}$$

$$=P^2,$$

P is a monotone three diagonal matrix defined by:

$$p_{ij}=\begin{cases}3 & i=j=1,N,\\ 2 & i=j=2,3,\cdots,N-1,\\ -1 & |i-j|=1,\\ 0 & \text{otherwise.}\end{cases} \tag{21}$$

Matrix B is defined by:

$$B=\frac{1}{1920}$$

$$\begin{pmatrix}
766 & 383 & 1 & & & & & & \\
383 & 1130 & 396 & 1 & & & & & \\
1 & 236 & 1446 & 236 & 1 & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & 1 & 236 & 1446 & 236 & 1 \\
 & & & & & 1 & 56 & 246 & 383 \\
 & & & & & & 1 & 56 & 245
\end{pmatrix}$$

$C = [c_{\frac{1}{2}}, c_{\frac{3}{2}}, \cdots, c_{N-\frac{1}{2}}]^T$ given by:

$$c_{\frac{1}{2}} = 6A_1 - \frac{5}{4}h^2B_1 + \frac{h^4}{1920}(766g_{\frac{1}{2}} + 383g_{\frac{3}{2}} + g_{\frac{5}{2}}),$$

$$c_{\frac{3}{2}} = -2A_1 - \frac{1}{4}h^2B_1 + \frac{h^4}{1920}(383g_{\frac{1}{2}} + 1130g_{\frac{3}{2}} + 396g_{\frac{5}{2}} + g_{\frac{7}{2}}),$$

$$c_{i-\frac{1}{2}} = 0, \qquad i = 3(1)N-2,$$

$$c_{N-\frac{3}{2}} = -2A_2 - \frac{1}{4}h^2B_2 + \frac{h^4}{1920}(g_{N-\frac{7}{2}} + 396g_{N-\frac{5}{2}} + 1130g_{N-\frac{3}{2}} + 383g_{N-\frac{1}{2}}),$$

$$c_{N-\frac{1}{2}} = 6A_2 - \frac{5}{4}h^2B_2 + \frac{h^4}{1920}(g_{N-\frac{5}{2}} + 383g_{N-\frac{3}{2}} + 766g_{N-\frac{1}{2}}).$$

The above system in Equation 18 can be solved by any direct or iterative methods.

CONVERGENCE ANALYSIS

Here, we investigate the convergence analysis of the given method. Here, e_i is the discretization error and t_i is the local truncation error defined by:

$$t_i = \begin{cases} -\frac{181}{11520}h^6u^{(6)}(\xi_1), & i = 1 \\ -\frac{497}{11520}h^6u^{(6)}(\xi_2), & i = 2, \\ -\frac{1}{24}h^6u^{(6)}(\xi_i), & i = 3(1)N-2, \\ -\frac{497}{11520}h^6u^{(8)}(\xi_{N-1}), & i = N-1, \\ -\frac{181}{11520}h^6u^{(8)}(\xi_N), & i = N, \end{cases}$$

$$\begin{array}{l} x_0 < \xi_1 < x_{\frac{1}{2}}, \\ x_0 < \xi_2 < x_{\frac{1}{2}}, \\ x_{i-\frac{1}{2}} < \xi_i < x_{i+\frac{1}{2}}, \\ x_{N-\frac{1}{2}} < \xi_{N-1} < x_N, \\ x_{N-\frac{1}{2}} < \xi_N < x_N. \end{array} \tag{22}$$

Theorem 1

Let $u(x)$ be the exact solution of the boundary value problem in Equation 1 and $\overline{u}_{i-\frac{1}{2}}$, $i = 1, 2, \cdots, N$ be the numerical solution obtained by the difference scheme (Equation 19). Then:

$$\|E\|_\infty = O(h^2),$$

provided $h^4|f(x)| < 1$.

Proof

We can write error Equation 20 in the following form:

$$E = A^{-1}T = [A_0 + h^4BF]^{-1}T = [I + h^4A_0^{-1}BF]^{-1}A_0^{-1}T,$$

$$\|E\|_\infty \le \|A^{-1}\|_\infty\|T\|_\infty \le \|[I + h^4A_0^{-1}BF]^{-1}\|_\infty\|A_0^{-1}\|_\infty\|T\|_\infty,$$

$$\|E\|_\infty \le \frac{\|A_0^{-1}\|_\infty\|T\|_\infty}{1 - h^4\|A\|_\infty\|B\|_\infty\|F\|_\infty}, \tag{23}$$

provided that $h^4\|A\|_\infty\|B\|_\infty\|F\|_\infty < 1$.

For our numerical procedure based on Equations 9, 14 to 17 we have $B = I_N$, a unit matrix, thus $\|B\|_\infty = 1$.

Following [6], we have:

$$\|A_0^{-1}\|_\infty \le \frac{5(b-a)^4 + 10(b-a)^2h^2 + 9h^4}{384h^4}. \tag{24}$$

Also, we have:

$$\|T\|_\infty \le \frac{497}{11520}h^6M_6, \tag{25}$$

where $M_6 = max|u^{(6)}(\xi)|$, $a \le \xi \le b$.

Substituting $\|A_0^{-1}\|_\infty$, $\|B\|_\infty$ and $\|T\|_\infty$ from Rations 23 and simplifying we obtain:

$$\|E\|_\infty \le \frac{497\phi M_6h^2}{11520(384 - \phi|f(x)|)} = O(h^2), \tag{26}$$

where $\phi = [5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]$, provided that:

$$|f(x)| < \frac{384}{(b-a)^4(5 + \frac{10}{N^2} + \frac{9}{N^4})}.$$

Consequently, it follows that the prescribed numerical method is a second-order convergent process. This completes the proof of Theorem 1.□

NUMERICAL ILLUSTRATIONS

To illustrate the applicability and effectiveness of our method and also to compare our results with existing methods, we consider the following fourth-order boundary value problems. These problems have been solved by the presented method with step lengths $h = 2^{-m}$, $m = 2, 3, \cdots, 8$, and the maximum absolute errors in numerical solutions are listed in Tables 1 to 3. The computed results verified that, by reducing the step size from h to $h/2$, the observed errors are approximately reduced by a factor $(\frac{1}{2})^2$ verifying the theoretical order of the presented method. We also compared our results with the second-order methods in references [5,6,8-10].

Table 1. The maximum absolute errors in the solution of Problem 1.

h	Our Method	Order
$\frac{1}{4}$	4.23(−4)	
$\frac{1}{8}$	3.89(−5)	3.44
$\frac{1}{16}$	2.02(−5)	
$\frac{1}{32}$	5.74(−6)	1.81
$\frac{1}{64}$	1.47(−6)	
$\frac{1}{128}$	3.71(−7)	1.98

Problem 1

Consider the linear BVP in [7]:

$$u^4(x) - u(x) = 4e^x, \qquad 0 < x < 1,$$

$$u(0) = 1, \qquad u''(0) = 3,$$

$$u(1) = 2e, \qquad u''(1) = 4e.$$

The theoretical solution for this problem is:

$$u(x) = (1 + x)e^x.$$

This problem has been solved with different values of $h = \frac{1}{4}, \cdots, \frac{1}{128}$ and the maximum absolute errors in the solutions are tabulated in Table 1.

Problem 2

Consider the following linear BVP from [5,6,8,9]:

$$u^4(x) + xu(x) = -(8 + 7x + x^3)e^x, \qquad 0 < x < 1,$$

$$u(0) = 0, \qquad u''(0) = 0,$$

$$u(1) = 0, \qquad u''(1) = -4e.$$

The theoretical solution for this problem is:

$$u(x) = x(1 - x)e^x.$$

We applied our method and compared the results with those obtained in the quintic spline method at grid points [5,8], the quartic spline method at off-step points [6] and the finite difference method [9]. The results in Table 2 show that our method is giving better accuracy.

Problem 3

Consider the following linear BVP from [5,6,8,10]:

$$u^4(x) - xu(x) = -(11 + 9x + x^2 - x^3)e^x,$$

$$-1 < x < 1,$$

$$u(-1) = 0, \qquad u''(-1) = \frac{2}{e},$$

$$u(1) = 0, \qquad u''(1) = -6e.$$

Table 2. The maximum absolute errors in the solution of Problem 2.

h	Our Method	In [5]	In [6]	In [8]	In [9]
$\frac{1}{4}$	1.54(−3)	3.51(−3)	1.61(−3)	—	7.16(−3)
$\frac{1}{8}$	1.89(−4)	8.67(−4)	4.24(−4)	1.74(−3)	1.74(−3)
$\frac{1}{16}$	9.17(−5)	2.16(−4)	1.08(−4)	4.15(−4)	4.33(−4)
$\frac{1}{32}$	2.59(−5)	5.40(−5)	2.70(−5)	1.07(−4)	1.08(−4)
$\frac{1}{64}$	6.68(−6)	1.35(−5)	6.75(−6)	—	2.70(−5)
$\frac{1}{128}$	1.68(−6)	—	1.69(−6)	—	6.75(−6)
$\frac{1}{256}$	4.23(−7)	—	4.13(−7)	—	—

Table 3. The maximum absolute errors in the solution of Problem 3.

h	Our Method	In [5]	In [6]	In [8]	In [10]
$\frac{1}{4}$	6.56(−2)	3.78(−2)	1.83(−2)	—	—
$\frac{1}{8}$	4.60(−3)	9.38(−3)	4.65(−3)	4.90(−2)	7.50(−2)
$\frac{1}{16}$	1.08(−3)	2.35(−3)	1.17(−3)	1.50(−2)	1.90(−2)
$\frac{1}{32}$	2.11(−4)	5.86(−4)	2.94(−4)	4.30(−3)	4.70(−3)
$\frac{1}{64}$	1.19(−4)	1.47(−4)	7.34(−5)	—	—
$\frac{1}{128}$	1.31(−5)	—	1.84(−5)	—	—
$\frac{1}{256}$	1.83(−5)	—	—	—	—

The theoretical solution for this problem is:

$$u(x) = (1 - x^2)e^x.$$

We solved this problem by our method and compared our results with the quintic spline method at grid points [5,8], the quartic spline method at off-step points [6] and the finite difference method [10]. The maximum absolute error in the solution of problem 3 is tabulated in Table 3, showing that the error in the solution of our method is less than in the methods in [5,6,8,10].

CONCLUSION

As we expected, the numerical results do confirm the second order of approximation. The maximum absolute errors in the solution of the fourth-order two-point boundary value problems given by our method are smaller than the errors in the methods in [5,6,8-10]. Moreover, we found that the developed quintic spline, using the off-step point, gives more accurate results in comparison with the quintic spline used in grid points.

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