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A New Category of Relations: Combinationally Constrained Relations

S.M.T. Rohani Rankoohi[1] and S.H. Mirian Hosseinabadi[2,*]

Abstract. *The normalization theory in relational database design is a classical subject investigated in different papers. The results of these research works are the stronger normal forms such as 5NF, DKNF and 6NF. In these normal forms, there are less anomalies and redundancies, but it does not mean that these stronger normal forms are free of anomalies and redundancies. Each normal form discussion is based on a particular constraint. In this paper, we introduce relations which contain a new kind of constraint called "combinational constraint". We distinguish two important kinds of this constraints, namely Strong and Weak. Also we classify the Combinationally Constrained Relations as Single and Multiple. We introduce all kinds of such relations and specify them using their quantitative properties, formally. It can be shown that these relations are in 5NF or 6NF and still they contain redundancies and have some anomalies.*

Keywords: *Relation; Constrained attribute; Free attribute; Combinational constraint; Combinationally constrained relation; Weak combinational constraint; Strong combinational constraint.*

INTRODUCTION

The subject of normalization in the process of relational database design and even in the process of object-oriented database design is a classical and well-known subject. In everyday-life applications and, in particular, when we have many relations and a large amount of data, normalization of relations is still one of the important phases in the designing of application systems. However, there are situations and applications in which non-normal relations are more efficient.

The main objective of the normalization process is to reduce the redundancy and anomalies as much as possible. Different normal forms are introduced, among which the strongest form is 6NF [1,2] (leave aside Domain Key Normal Form (DKNF) which is not reachable in practice [3]). Based on the main objective of the normalization process [4], a relation which is in 5NF should have less redundancies and be free of 4NF relation anomalies. This is the same in 6NF which is the latest and strongest normal form.

As we will see later in the related works, the research on the normalization theory was done until the late 80s, but less in the 90s. In particular, there is not much theoretical research on the subject of 5NF and 6NF relation schemas. Each normal form discussion is based on a particular constraint, for example in the 4NF relations, there exists a special semantic constraint, i.e. the so called "Cyclic Nature Constraint [4]".

In this paper, we introduce another kind of constraint in relations, which we call the "combinational constraint". First, we introduce two types of this constraint, namely Strong and Weak. Then, we distinguish two kinds of relation, namely Single and Multiple. And finally, based on these two types, we introduce various kinds of Combinationally Constrained Relations and investigate some of their quantitative properties. In our future paper, we will show that these relations are in 5NF or 6NF and despite their high level of normality, they can have a lot of redundancies and some anomalies.

From now on, we use the basic symbols given in Table 1. The complete set of symbols that are

1. *Faculty of Electrical and Computer Engineering, Shahid Beheshti University, Tehran, P.O. Box 1983963113, Iran.*
2. *Department of Computer Engineering, Sharif University of Technology, Tehran, Iran.*
*. *Corresponding author. E-mail: hmirian@sharif.ac.ir*

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Table 1. Basic Symbols

Symbol	Definition
$\mathcal{C}$	Combinational constraint
$\mathcal{R}$	Combinationally Constrained Relation
r	Body of $\mathcal{R}$
c	The number of $\mathcal{C}$ in a r
$\#r$	Cardinality of $\mathcal{R}$ (the number of tuples in r)
n	Degree of relation
$[A]_t$	Attribute(s) A of tuple t
$\mathbf{[}A\mathbf{]}_t$	Value of attribute(s) A in tuple t
$\mathbb{P}$	Power set
$\emptyset$	Empty set

presented and used in this paper will be given in the nomenclature at the end of paper. To present the formal definitions, we use the First Order Logic and Predicate Calculus and follow the notation suggested by Spivey in the Z notation manual [5].

The rest of the paper is organized as follows: First, an overview of related works is given. Then, basic definitions and concepts are introduced. Next, the main categories of $\mathcal{R}$, namely Strong, Weak, Single and Multiple $\mathcal{R}$, are discussed. This is followed by other kinds of $\mathcal{R}$, and their properties. Subsequently, a case study is given to show that there are everyday used relations having this type of semantic constraint. Finally, the conclusion and future works are discussed.

RELATED WORKS

Each normal form in the normalization theory is based on a specific constraint. Beside Codd classical normal forms (1NF, 2NF and 3NF), in stronger normal forms these constraints are:

- In BCNF: Each determinant must be a candidate key;
- In 4NF: "Multivalued dependencies" must not exist;
- In 5NF: "Join dependencies" not implied by the candidate keys must not exist;
- In 6NF: Join dependencies must not exist, either implied by the candidate keys or by non-key attributes.

But, the constraint we introduce in this paper, namely combinational constraint, to the best of our knowledge, has not been considered before. It can be shown that, although some of the Combinationally Constrained Relations are in 5NF and even some in 6NF, these relations still contain a lot of redundancies and, as a result, they have anomalies.

Many of the advanced normal forms are due to Fagin. Specifically the discussion on 4NF, 5NF (PJNF) and DKNF can be found in [6-8]. Since their introduction, many database texts have discussed them [3,4,9-11], but mostly through a simple example. During the 1980s, there were a number of research works on relational database design, particularly on decomposition and join dependency. The decomposability phenomenon for relations with a degree of n, where $n > 2$, was first noted in [12]. The work of Gyssens can also be mentioned in this respect [8,13,14]. The justification for 4NF can be found in [15]. Other work by Vincent on normal forms [16-19] are also notable. Since 2000, there have been a number of works done on the decomposition of relations and in [20], a justification for inclusion dependency can be found. In [21], partitioned normal form relations are discussed. An information theory based approach for normal forms (4NF and 5NF) is proposed in [22]. The 6NF is solely introduced in [1,2].

BASIC DEFINITIONS AND CONCEPTS

In this section, the concept of $\mathcal{R}$ as well as other preliminary concepts are introduced.

Definition 1

Combinationally Constrained Relation

Let $R(A_1, A_2, \cdots, A_n)$ be a relation schema and n and k two integers such that $n > 2$ and $2 \leq k \leq n-1$. R is combinationally constrained if, and only if, R satisfies the following conditions:

1. Attributes $A_1, A_2, \cdots, A_n$ take their values from the disjoint domains $D_1, D_2, \cdots, D_n$, respectively.
2. The body of R satisfies the following two time-independent constraints:
   (a) The cardinality of $R(\#r)$ is greater than or equal to $C_k^n = \frac{n!}{(n-k)! * k!}$, i.e. the number of combinations of k from n.
   (b) The values of each attribute in C_{k-1}^{n-1} tuples of r are equal. We call such attributes, in each tuple, "combinationally constrained attributes" and the set of such attributes for each tuple, t, is denoted by $[\mathcal{A}^{\mathcal{C}}]_t$.

As said before, we call such a relation a Combinationally Constrained Relation, which is denoted by $\mathcal{R}$.□

It is worthy to mention that there are many subsets of $D_1 \times D_2 \times \cdots \times D_n$ which satisfy the above constraints and they are not rare in practice. We will show some samples of these relations in our case study.

Definition 2

Constraint Degree

The constraint degree of $\mathcal{R}$ is defined as the number of attributes under the $\mathcal{C}$ in r and we denote it by $\#\mathcal{A}^{\mathcal{C}}$ cardinality of $\mathcal{A}^{\mathcal{C}}$).□

Definition 3

Free Attributes

In each tuple of r there exists the same number of attributes that are not combinationally constrained, which we call Free. The set of "free attributes" of a $\mathcal{R}$ in each tuple t is denoted by $[\mathcal{A}^{\mathcal{F}}]_t$, and the number of such attributes is shown by $\#\mathcal{A}^{\mathcal{F}}$ (cardinality of $\mathcal{A}^{\mathcal{F}}$).□

Motivating Example

Here, we discuss a real example to show the effect of such a constraint on a relation. We will continue this example in the case study. Consider a university as a micro real world. Assume the following entities in this environment: Professor, Course, Department, Book, Room. Consider the following semantic rules:

- A professor may teach more than one course in a term;
- A professor may teach in more than one department;
- A course may be offered in more than one section;
- A course may be taught based on more than one book.

We assume that the reader of this paper is familiar with the database conceptual modeling (for example, with the EER method) and skip this phase. The relational database for such an environment may include the following relations:

- PROFESSOR ($\underline{P\#}$, PNAME, $\cdots$),
- COURSE ($\underline{C\#}$, CTITLE, $\cdots$),
- DEPARTMENT ($\underline{D\#}$, DTITLE, $\cdots$),
- BOOK ($\underline{B\#}$, BTITLE, $\cdots$),
- ROOM ($\underline{R\#}$, RCAP, $\cdots$),
- $\cdots$

The following relation shows the relationship among three entities, namely, PROFESSOR, COURSE and DEPARTMENT.

PCD ($\underline{P\#, C\#, D\#}$).

The semantic of the relation PCD is: Professor p teaches course c in department d. The above relationship can have some more attributes, e.g. YT: year-term and $S\#$: section number.

PCD_A ($P\#$, $C\#$, $D\#$, YT, $S\#$).

The semantic of the relation PCD_A is: Professor p teaches section s of course c in department d in year-term yt.

Figure 1 shows a sample body for PCD_A which has a $\mathcal{R}$. We categorized this relation as a Single and Strong (see the next section for details) $\mathcal{R}$ (denoted by σ^s) with the degree of five (i.e. $n = 5$) and its constraint degree is equal to 4 (i.e. $\#\mathcal{A}^{\mathcal{C}} = 4$ and $\#\mathcal{A}^{\mathcal{F}} = 1$). For instance, in the first tuple of the body of PCD_A given in Figure 1, we have: Professor $p = 1001$ teaches section $s = 2$ of course $c = 40100$ in department $d = 40$ in year-term $yt = 851$.

This relation and, in general, all Combinationally Constrained Relations of degree n and with Constraint Degree $n-1$ is in 6NF. In fact, it can be proved that every combination of $n-1$ attributes of such a relation is a candidate key. For example, the candidate keys of the relation shown in Figure 1 are $\{P\#, C\#, D\#, YT\}$, $\{P\#, C\#, D\#, S\#\}$, $\{P\#, C\#, YT, S\#\}$, $\{P\#, D\#, YT, S\#\}$, and $\{C\#, D\#, YT, S\#\}$. It is obvious that there is just one extra attribute in this relation in addition to its candidate key. On the other hand, it is in 5NF because it does not contain any Cyclic Constraint [4]. Therefore, according to the definitions given in [1,2], it is in 6NF. But, still, this relation has anomalies such as:

- Update operation: If the user wants to change the value of one of the attributes, s/he has to update four out of five tuples, otherwise the existing combinational constraint will be violated. This means that a tuple-level operation is transformed to a set-level operation.
- Delete operation: If the user deletes one or more tuples from such a relation, the combinational constraint will disappear. This means that the deletion operation is not possible at all unless all the tuples of the relation are deleted.
- Insert operation: If the user inserts an arbitrary tuple into this relation, the Simple Combinational Constraint will disappear and the relation will have a combinational constraint from a different category. It is possible to insert another set of tuples consisting

PCD_A	($P\#$,	$C\#$,	$D\#$,	YT,	$S\#$)
	1001	40100	40	851	2
	1001	40100	40	852	1
	1001	40100	50	851	1
	1001	40200	40	851	1
	1002	40100	40	851	1

Figure 1. σ^s - Single Strong $\mathcal{R}$.

of another $\mathcal{C}$ into this relation. In fact, in these cases, the relation will have a Multiple $\mathcal{R}$ (see next subsection for different categories of $\mathcal{R}$ relations).

It should be noted that the combinational constraint imposes a type of "domain constraint" [8] (vice versa is not true). From this point of view, it can be said that the above mentioned anomalies are due to the domain constraints too.

A General View

Now, we give a general view of the categorization that is introduced for the relations having combinational constraints. A $\mathcal{R}$, based on its constraint degree, can be categorized as Strong or Weak. In the case of Weak $\mathcal{R}$, it is possible to have more than one $\mathcal{C}$ in the relation R. We introduce two types of relations, namely Unary and Doubly (or Binary) Weak $\mathcal{R}$. A Doubly Weak $\mathcal{R}$ can be Symmetric or Asymmetric. A Symmetric or Asymmetric Doubly Weak $\mathcal{R}$ can be Pure or Impure. From another point of view, a $\mathcal{R}$ can be Single or Multiple. To find other kinds of $\mathcal{R}$, we combine the main categories of $\mathcal{R}$, namely Strong and Weak and Single and Multiple together. In the specification-generalization tree shown in Figure 2, all kinds of $\mathcal{R}$ are introduced. The formal definitions of all kinds of $\mathcal{R}$ are given next.

STRONG AND WEAK $\mathcal{R}$

A $\mathcal{R}$, based on its constraint degree, can be categorized as Strong or Weak. In the following, we give the definitions of Strong and Weak $\mathcal{R}$.

Definition 4

Strong $\mathcal{R}$

A $\mathcal{R}$ is strong when its constraint degree is equal to $n-1$, i.e.:

$$n \geq 3 \wedge (c = 1 \iff \#\mathcal{A}^{\mathcal{C}} = n-1 \wedge \#\mathcal{A}^{\mathcal{F}} = 1).$$

We represent such a constraint by $\mathcal{C}^s$ and the relation with such a constraint by $\mathcal{R}^s$.□

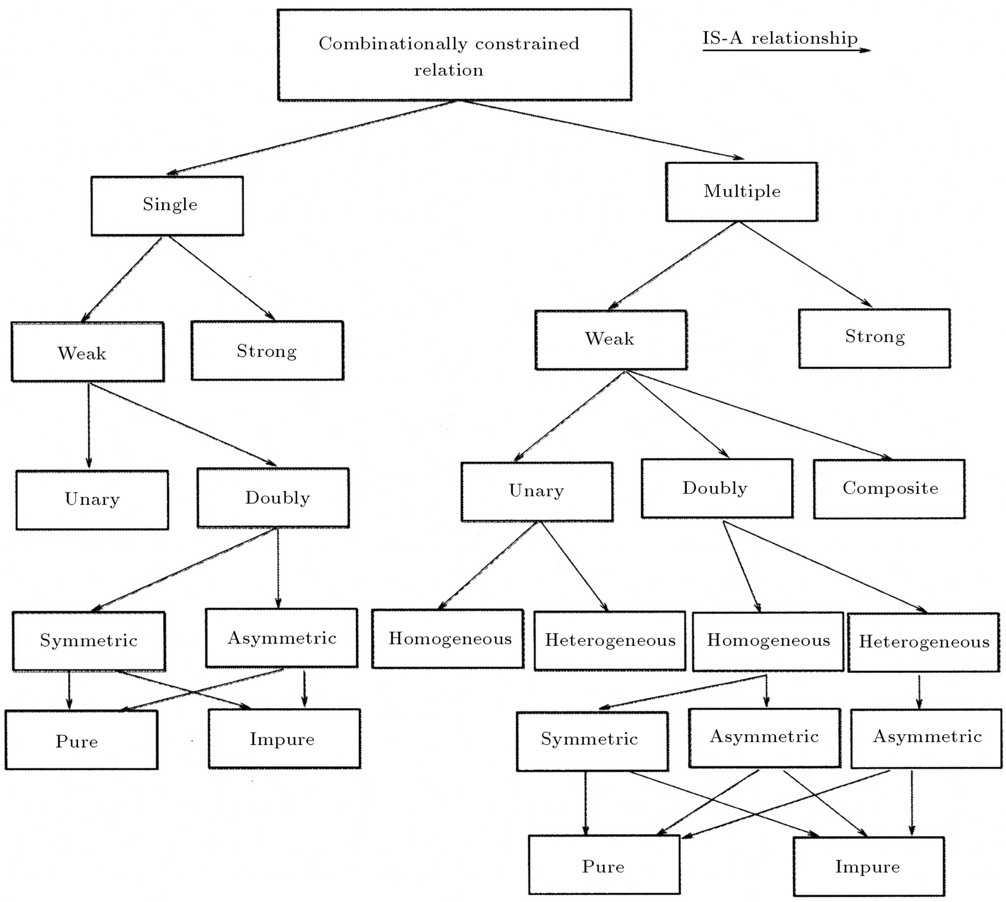

Figure 2. The specification-generalization tree of all kinds of $\mathcal{R}$

Example 1

In the relation $R_1(A, B, C, D)$ with the body shown in Figure 3, there is a $\mathcal{C}^s$ where $\#\mathcal{A}^\mathcal{C} = 3$ and $\#\mathcal{A}^\mathcal{F} = 1$. The cardinality of this relation is $C_3^4 = 4$ and, in $C_2^3 = 3$ tuples, the values of each combinationally constrained attribute are equal. Figure 3 shows the $\mathcal{A}^\mathcal{C}$ and $\mathcal{A}^\mathcal{F}$ of each tuple in the appropriate columns.$\lozenge$

Definition 5

Weak $\mathcal{R}$

A $\mathcal{R}$ is Weak when there are less than $n-1$ attributes under the $\mathcal{C}$, i.e.:

$$n \geq 4 \wedge 2 \leq \#\mathcal{A}^\mathcal{C} < n-1 \wedge 1 < \#\mathcal{A}^\mathcal{F} \leq n-2.$$

Such a constraint is denoted by $\mathcal{C}^w$ and the relation with such a constraint is represented by $\mathcal{R}^w$.$\square$

Example 2

In relation $R_2(A, B, C, D, E)$ with the body shown in Figure 4, there is a $\mathcal{C}^w$ where $\#\mathcal{A}^\mathcal{C} = 3$ and $\#\mathcal{A}^\mathcal{F} = 2$. The number of tuples of this relation is $C_3^5 = 10$ and in $C_2^4 = 6$ tuples, the values of each combinationally constrained attribute are equal.$\lozenge$

$\mathcal{R}^s$ Sample					$[\mathcal{A}^\mathcal{C}]_t$	$[\mathcal{A}^\mathcal{F}]_t$
R_1	$(A,$	$B,$	$C,$	$D)$		
	a_1	b_1	c_1	d_2	$\{A, B, C\}$	$\{D\}$
	a_1	b_1	c_2	d_1	$\{A, B, D\}$	$\{C\}$
	a_1	b_2	c_1	d_1	$\{A, C, D\}$	$\{B\}$
	a_2	b_1	c_1	d_1	$\{B, C, D\}$	$\{A\}$

Figure 3. A sample of Strong $\mathcal{R}(\mathcal{R}^s)$ ($n = 4$, $\#r = 4$, $\#\mathcal{A}^\mathcal{C} = 3$, $\#\mathcal{A}^\mathcal{F} = 1$).

$\mathcal{R}^w$ Sample						$[\mathcal{A}^\mathcal{C}]_t$	$[\mathcal{A}^\mathcal{F}]_t$
R_2	$(A,$	$B,$	$C,$	$D,$	$E)$		
	a_1	b_1	c_1	d_2	e_2	$\{A, B, C\}$	$\{D, E\}$
	a_1	b_1	c_2	d_1	e_3	$\{A, B, D\}$	$\{C, E\}$
	a_1	b_1	c_3	d_3	e_1	$\{A, B, E\}$	$\{C, D\}$
	a_1	b_2	c_1	d_1	e_4	$\{A, C, D\}$	$\{B, E\}$
	a_1	b_3	c_1	d_4	e_1	$\{A, C, E\}$	$\{B, D\}$
	a_1	b_4	c_4	d_1	e_1	$\{A, D, E\}$	$\{B, C\}$
	a_2	b_1	c_1	d_1	e_5	$\{B, C, D\}$	$\{A, E\}$
	a_3	b_1	c_1	d_5	e_1	$\{B, C, E\}$	$\{A, D\}$
	a_4	b_1	c_5	d_1	e_1	$\{B, D, E\}$	$\{A, C\}$
	a_5	b_5	c_1	d_1	e_1	$\{C, D, E\}$	$\{A, B\}$

Figure 4. A sample of Weak $\mathcal{R}$ ($\mathcal{R}^w$) ($n = 5$, $\#r = 10$, $\#\mathcal{A}^\mathcal{C} = 3$, $\#\mathcal{A}^\mathcal{F} = 2$).

For understanding the $\mathcal{R}^w$, the frequency of indexes of attribute values in the tuples of the relation R_2 (shown in Figure 4) should be noticed. For example, the values of attribute A in six tuples have the index (1), since the values of each combinationally constrained attribute in six tuples must be equal.

Unary and Doubly $\mathcal{R}^w$

In the case of $\mathcal{R}^w$, it is possible to have more than one $\mathcal{C}$ in relation R. We now introduce two types of relation, namely Unary and Doubly (or Binary) $\mathcal{R}^w$. Theoretically, we can have a relation in which the number of $\mathcal{C}$s are more than two. We call these types of relation c-ary, i.e.:

$$c > 2 \wedge$$

$$(\forall i : 1 \cdots c \bullet (\#\mathcal{A}^\mathcal{C})_i < n-1 \wedge (\#\mathcal{A}^\mathcal{F})_i > 1),$$

but we will not consider them in this paper for the sake of brevity.

Definition 6

Unary $\mathcal{R}^w$

A $\mathcal{R}^w$ is unary if there is no other set of attributes distinct from the set of previously constrained attributes in which another $\mathcal{C}$ can be found. In other words, the values of each free attribute in at least two tuples are not equal. This proves that there is only one $\mathcal{C}^w$ in $\mathcal{R}^w$, i.e.:

$$c = 1 \iff \exists\, t_1, t_2 : r \mid t_1 \neq t_2 \bullet [\![\mathcal{A}^\mathcal{F}]\!]_{t_1} \neq [\![\mathcal{A}^\mathcal{F}]\!]_{t_2}.$$

The relation with such a constraint is denoted by $\mathcal{R}_u^w$.$\square$

Definition 7

Doubly $\mathcal{R}^w$

A $\mathcal{R}^w$ is doubly if there is only one set of attributes distinct from the set of previously constrained attributes in which another $\mathcal{C}$ can be found. In other words, the values of free attributes in $C_{\#\mathcal{A}^\mathcal{F}-1}^{n-1}$ tuples are equal. This means there are two $\mathcal{C}^w$ included in relation R, i.e.:

$$c = 2 \iff \exists\, r_1 : \mathbb{P}r \mid \#r_1 = C_{\#\mathcal{A}^\mathcal{F}-1}^{n-1} \bullet$$

$$\forall t_1, t_2 : r_1 \mid t_1 \neq t_2 \bullet [\![\mathcal{A}^\mathcal{F}]\!]_{t_1} = [\![\mathcal{A}^\mathcal{F}]\!]_{t_2}.$$

The relation having such a constraint is denoted by $\mathcal{R}_d^w$. The number of constrained attributes under constraint $\mathcal{C}_i$ is denoted by $(\#\mathcal{A}^\mathcal{C})_i$, $i \in \{1, 2\}$.$\square$

Symmetric and Asymmetric $\mathcal{R}_d^w$

A $\mathcal{R}_d^w$ can be Symmetric or Asymmetric. The following definitions specify them formally.

Definition 8

Symmetric $\mathcal{R}_d^w$
A $\mathcal{R}_d^w$ is Symmetric if the two $\mathcal{C}^w$s have distinct sets of constrained attributes with the same constraint degrees, i.e.:

$$(\#\mathcal{A}^{\mathcal{C}})_1 = (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r \bullet [(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset.$$

The relation having such constraints is denoted by $\mathcal{R}_d^{wx}$.□

Definition 9

Asymmetric $\mathcal{R}_d^w$
A $\mathcal{R}_d^w$ is Asymmetric if the two $\mathcal{C}^w$s have distinct sets of constrained attributes with different constraint degrees, i.e.:

$$(\#\mathcal{A}^{\mathcal{C}})_1 \neq (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r \bullet [(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset.$$

The relation having such constraints is represented by $\mathcal{R}_d^{wy}$.□

Example 3

Relation R_3 in Figure 5 has two $\mathcal{C}_d^w$s (i.e. $(\#\mathcal{A}^{\mathcal{C}})_1 = 3$, $(\#\mathcal{A}^{\mathcal{F}})_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})_2 = 2$, $(\#\mathcal{A}^{\mathcal{F}})_2 = 3$). Since $(\#\mathcal{A}^{\mathcal{C}})_1 \neq (\#\mathcal{A}^{\mathcal{C}})_2$, then it is an example of $\mathcal{R}_d^{wy}$.◊

It is obvious that in the presence of two $\mathcal{C}^w$s, each time that one of two $\mathcal{C}^w$s is considered, $(\#\mathcal{A}^{\mathcal{C}})_i$, $i \in \{1,2\}$ attributes in each tuple are under that $\mathcal{C}^w$ and $(\#\mathcal{A}^{\mathcal{F}})_i \leq n - (\#\mathcal{A}^{\mathcal{C}})_i$ attributes do not have that constraint and are free with respect to that $\mathcal{C}^w$.

Pure and Impure $\mathcal{R}_d^{wx}$ or $\mathcal{R}_d^{wy}$

A Symmetric or Asymmetric $\mathcal{R}_d^w$ can be Pure or Impure. For the definition of ImPure $\mathcal{R}_d^w$, we need to define the concept of absolutely free attributes.

Definition 10

Absolutely Free Attributes
In a Doubly Weak $\mathcal{R}$, Symmetric ($\mathcal{R}_d^{wx}$) or Asymmetric ($\mathcal{R}_d^{wy}$), there can be at least one extra attribute which is not under either of the two $\mathcal{C}^w$s. We call such attributes 'Absolutely free attributes' and the set of such attributes in a relation is denoted by $\mathcal{A}_a^{\mathcal{F}}$. The number of $\mathcal{A}_a^{\mathcal{F}}$ in a relation is shown by $\#\mathcal{A}_a^{\mathcal{F}}$ (the cardinality of $\mathcal{A}_a^{\mathcal{F}}$)□

The following definitions specify Pure and Impure $\mathcal{R}_d^w$ formally.

Definition 11

Pure $\mathcal{R}_d^w$
$\mathcal{R}_d^{wx}$ or $\mathcal{R}_d^{wy}$ is Pure if there is not an attribute in the relation under either of the two $\mathcal{C}^w$s, i.e.:

$$n \geq 4 \wedge (\#\mathcal{A}_a^{\mathcal{F}} = 0 \iff (\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i = n)).$$

Such a relation is denoted by $\mathcal{R}_{dp}^{wx}$ or $\mathcal{R}_{dp}^{wy}$.□

Definition 12

Impure $\mathcal{R}_d^w$
A $\mathcal{R}_d^{wx}$ or $\mathcal{R}_d^{wy}$ is Impure if there is at least one attribute in the relation which is not under the two

	$\mathcal{R}_d^{wy}$ Sample					$[(\mathcal{A}^{\mathcal{C}})_1]_t$	$[(\mathcal{A}^{\mathcal{F}})_1]_t$	$[(\mathcal{A}^{\mathcal{C}})_2]_t$	$[(\mathcal{A}^{\mathcal{F}})_2]_t$
R_3	$(A,$	$B,$	$C,$	$D,$	$E)$				
	a_1	b_1	c_1	d_2	e_2	$\{A,B,C\}$	$\{D,E\}$	$\{D,E\}$	$\{A,B,C\}$
	a_1	b_1	c_2	d_1	e_2	$\{A,B,D\}$	$\{C,E\}$	$\{C,E\}$	$\{A,B,D\}$
	a_1	b_1	c_2	d_2	e_1	$\{A,B,E\}$	$\{C,D\}$	$\{C,D\}$	$\{A,B,E\}$
	a_1	b_2	c_1	d_1	e_2	$\{A,C,D\}$	$\{B,E\}$	$\{B,E\}$	$\{A,C,D\}$
	a_1	b_2	c_1	d_2	e_1	$\{A,C,E\}$	$\{B,D\}$	$\{B,D\}$	$\{A,C,E\}$
	a_1	b_2	c_2	d_1	e_1	$\{A,D,E\}$	$\{B,C\}$	$\{B,C\}$	$\{A,D,E\}$
	a_2	b_1	c_1	d_1	e_2	$\{B,C,D\}$	$\{A,E\}$	$\{A,E\}$	$\{B,C,D\}$
	a_2	b_1	c_1	d_2	e_1	$\{B,C,E\}$	$\{A,D\}$	$\{A,D\}$	$\{B,C,E\}$
	a_2	b_1	c_2	d_1	e_1	$\{B,D,E\}$	$\{A,C\}$	$\{A,C\}$	$\{B,D,E\}$
	a_2	b_2	c_1	d_1	e_1	$\{C,D,E\}$	$\{A,B\}$	$\{A,B\}$	$\{C,D,E\}$

Figure 5. A sample of Asymmetric Doubly Weak $\mathcal{R}(\mathcal{R}_d^{wy})$ ($n = 5$, $\#r = 10$, $(\#\mathcal{A}^{\mathcal{C}})_1 = 3$, $(\#\mathcal{A}^{\mathcal{C}})_2 = 2$).

$\mathcal{C}^w$s, i.e.:

$$n \geq 5 \wedge (\#\mathcal{A}_a^{\mathcal{F}} > 0 \iff (\forall i : 1 \cdots 2 \bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i + (\#\mathcal{A}_a^{\mathcal{F}})_i = n)).$$

Such a relation is represented by $\mathcal{R}_{di}^{wx}$ or $\mathcal{R}_{di}^{wy}$. □

It should be noted that in the Impure $\mathcal{R}$, the cardinality of $\mathcal{R}$ is greater than or equal to $C_k^{n-\#\mathcal{A}_a^{\mathcal{F}}}$ and the values of constrained attributes in $C_{k-1}^{n-\#\mathcal{A}_a^{\mathcal{F}}-1}$ tuples are equal.

Example 4

The relation R_3 shown in Figure 5 is an example of $\mathcal{R}_{dp}^{wy}$ too. ◊

SINGLE AND MULTIPLE $\mathcal{R}$

A $\mathcal{R}$ can be Single or Multiple. In the following, we give the formal definitions of these kinds of $\mathcal{R}$. But first, we define the concept of a Module Relation.

Definition 13

Module Relation

A Module Relation is a set of tuples with a $\mathcal{C}$ (Strong or Weak) and is denoted by $\mathcal{M}$. The Strong $\mathcal{M}$ is a $\mathcal{M}$ having a $\mathcal{C}^s$ and the Weak $\mathcal{M}$ is a $\mathcal{M}$ having a $\mathcal{C}^w$, which are denoted by $\mathcal{M}^s$ and $\mathcal{M}^w$, respectively.□

In general, a $\mathcal{R}$ can include one or more than one $\mathcal{M}$. In the rest of the paper, we use the module symbols defined in Table 2.

Definition 14

Single $\mathcal{R}$

$\mathcal{R}$ is called Single if it contains just one $\mathcal{M}$, i.e. $m = 1 \wedge \#r = \#r_{\mathcal{M}}$. Such a relation is represented by σ.□

Table 2. Module Symbols

Symbol	Definition
m	The number of $\mathcal{M}$
m^s	The number of $\mathcal{M}^s$
m^w	The number of $\mathcal{M}^w$
$r_{\mathcal{M}}$	The body of $\mathcal{M}$
$\#r_{\mathcal{M}}$	The cardinality of $\mathcal{M}$

Definition 15

Multiple $\mathcal{R}$

A $\mathcal{R}$ is called Multiple if it contains more than one $\mathcal{M}$, i.e.:

$$m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j.$$

We denote such a relation by $(\mu)^m$, where $m > 1$. In other words we have $\bigcup_{j=1}^{m} \mathcal{M}^j \subseteq \mathcal{R}$ since $\mathcal{R}$ can have extra arbitrary tuples which are not necessarily under the $\mathcal{C}$s.□

A $(\mu)^m$ with $m = 1$ is σ, as defined in Definition 14, and we have $\sigma = (\mu)^1$. The general form of a $(\mu)^m$ with $m = 2$ and without extra tuples where, in each module $\#\mathcal{A}^{\mathcal{C}} = n - 1$, is shown in Figure 6. In other words, there are two $\mathcal{M}^s$ in this relation. It is obvious that the order of tuples is not significant. Figure 7 shows the same relation as Figure 6 with some extra tuples.

OTHER KINDS OF $\mathcal{R}$

To find other kinds of $\mathcal{R}$, we combine the main categories of $\mathcal{R}$, namely Strong and Weak and Single and Multiple together.

Kinds of Single $\mathcal{R}$

In the case of single relations (σ), we can have:

- Strong (σ^s), in which there is only one Strong $\mathcal{C}$ (i.e. $m^s = 1 \wedge m^w = 0$);
- Weak (σ^w), in which we can have Weak $\mathcal{C}$s, has two subcategories, namely Unary (i.e. $m^s = 0 \wedge m^w = 1$) and Doubly (i.e. $\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n - 1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1$), as follows:

$$\begin{array}{ll}
R \quad (A_1, \; A_2, \; \cdots, \; A_{n-1}, \; A_n) & \\
\left\{\begin{array}{lllll}
a_{1_1} & a_{2_1} & \cdots & a_{(n-1)_1} & a_{n_2} \\
a_{1_1} & a_{2_1} & \cdots & a_{(n-1)_2} & a_{n_1} \\
\vdots & & & & \\
a_{1_1} & a_{2_2} & \cdots & a_{(n-1)_1} & a_{n_1} \\
a_{1_2} & a_{2_1} & \cdots & a_{(n-1)_1} & a_{n_1}
\end{array}\right\} & \text{Module 1} \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\left\{\begin{array}{lllll}
b_{1_1} & b_{2_1} & \cdots & b_{(n-1)_1} & b_{n_2} \\
b_{1_1} & b_{2_1} & \cdots & b_{(n-1)_2} & b_{n_1} \\
\vdots & & & & \\
b_{1_1} & b_{2_2} & \cdots & b_{(n-1)_1} & b_{n_1} \\
b_{1_2} & b_{2_1} & \cdots & b_{(n-1)_1} & b_{n_1}
\end{array}\right\} & \text{Module 2}
\end{array}$$

Figure 6. General form of Multiple $\mathcal{R}((\mu)^2)(\#r = 2 * n)$.

$$
\begin{array}{cccccc}
R & (A_1, & A_2, & \cdots, & A_{n-1}, & A_n) \\
\hline
\end{array}
$$

$$
\left.\left\{\begin{array}{ccccc}
a_{1_1} & a_{2_1} & \cdots & a_{(n-1)_1} & a_{n_2} \\
a_{1_1} & a_{2_1} & \cdots & a_{(n-1)_2} & a_{n_1} \\
\vdots & & & & \\
a_{1_1} & a_{2_2} & \cdots & a_{(n-1)_1} & a_{n_1} \\
a_{1_2} & a_{2_1} & \cdots & a_{(n-1)_1} & a_{n_1}
\end{array}\right.\right\} \text{Module 1}
$$

$$
\left.\left\{\begin{array}{ccccc}
b_{1_1} & b_{2_1} & \cdots & b_{(n-1)_1} & b_{n_2} \\
b_{1_1} & b_{2_1} & \cdots & b_{(n-1)_2} & b_{n_1} \\
\vdots & & & & \\
b_{1_1} & b_{2_2} & \cdots & b_{(n-1)_1} & b_{n_1} \\
b_{1_2} & b_{2_1} & \cdots & b_{(n-1)_1} & b_{n_1}
\end{array}\right.\right\} \text{Module 2}
$$

$$
\left.\left\{\begin{array}{ccccc}
c_{1_1} & c_{2_1} & \cdots & c_{(n-1)_1} & c_{n_1} \\
c_{1_2} & c_{2_2} & \cdots & c_{(n-1)_2} & c_{n_2} \\
\vdots & & & & \\
c_{1_x} & c_{2_x} & \cdots & c_{(n-1)_x} & c_{n_x}
\end{array}\right.\right\} \text{Extra Tuples}
$$

Figure 7. General form of Multiple $\mathcal{R}$ with extra tuples $((\mu)^2)(\#r = 2 * n + x)$.

- ⋆ Unary (σ_u^w), in which there is only one Weak $\mathcal{C}$;
- ⋆ Doubly (σ_d^w), in which there are two Weak $\mathcal{C}$s. This kind of $\mathcal{R}$ can be:
  - * Symmetric (σ_d^{wx}), which means the $\#\mathcal{A}^\mathcal{C}$s of $\mathcal{C}$s are equal;
  - * Asymmetric (σ_d^{wy}), when the $\#\mathcal{A}^\mathcal{C}$s of $\mathcal{C}$s are not equal.

  These two later kinds can be both:
  - * Pure (σ_{dp}^{wx} or σ_{dp}^{wy}), in which no extra attribute is available;
  - * Impure (σ_{di}^{wx} or σ_{di}^{wy}), in which at least one extra attribute can be found.

We will discuss the quantitative properties of the various kinds of Single $\mathcal{R}$ in the next section.

The general form of a σ^s is shown in Figure 8, where a_{i_1} and a_{i_2} are two distinct values of the attribute A_i from domain D_i. The constrained and free attributes of each tuple are shown in Figure 8.

Kinds of Multiple $\mathcal{R}$

In the case of Multiple relations $((\mu)^m)$ we can have:

- ∘ Strong $((\mu^s)^m)$ in which all module relations are Strong. We gave the general form of a Multiple $\mathcal{R}$ with two $\mathcal{M}^s$ in Figure 6;
- ∘ Weak $((\mu^w)^m)$ in which at least one of the module relations is Weak. This subcategory of Multiple relations can be:
  - ⋆ Unary $((\mu_u^w)^m)$ in which all module relations are unary Weak;
  - ⋆ Doubly $((\mu_d^w)^m)$ in which all module relations are doubly Weak;
  - ⋆ Composite $((\mu_c^w)^m)$ in which the module relations can have different types, namely Strong or Weak.

  The Unary or Doubly Weak subcategories can be categorized in two different subcategories as follows:
  - ⋆ Homogeneous $((\mu_{uh}^w)^m$ or $(\mu_{dh}^w)^m)$ in which all module relations have the same properties;
  - ⋆ Heterogeneous $((\mu_{ut}^w)^m$ or $(\mu_{dt}^w)^m)$ when module relations can have different properties.

  The Homogeneous Doubly Weak subcategory can be:
  - ⋆ Symmetric $((\mu_{dh}^{wx})^m)$ in which all module relations are Symmetric;
  - ⋆ Asymmetric $((\mu_{dh}^{wy})^m)$ in which all module relations are Asymmetric with the same properties.

  and each one of them can be:
  - ⋆ Pure $((\mu_{dhp}^{wx})^m$ or $(\mu_{dhp}^{wy})^m)$;
  - ⋆ Impure $((\mu_{dhi}^{wx})^m$ or $(\mu_{dhi}^{wy})^m)$.

  Notice that all Heterogeneous Doubly Weak relations are, by definition, Asymmetric $((\mu_{dt}^{wy})^m)$ and they can be categorized as:

General Form of σ^s	$[\mathcal{A}^\mathcal{C}]_t$	$[\mathcal{A}^\mathcal{F}]_t$
$R \quad (A_1, \; A_2, \; \cdots, \; A_{n-1}, \; A_n)$		
$a_{1_1} \; a_{2_1} \; \cdots \; a_{(n-1)_1} \; a_{n_2}$	$\{A_1, \cdots, A_{n-1}\}$	$\{A_n\}$
$a_{1_1} \; a_{2_1} \; \cdots \; a_{(n-1)_2} \; a_{n_1}$	$\{A_1, \cdots, A_{n-2}, A_n\}$	$\{A_{n-1}\}$
$\vdots$		
$a_{1_1} \; a_{2_2} \; \cdots \; a_{(n-1)_1} \; a_{n_1}$	$\{A_1, A_3, \cdots, A_n\}$	$\{A_2\}$
$a_{1_2} \; a_{2_1} \; \cdots \; a_{(n-1)_1} \; a_{n_1}$	$\{A_2, \cdots, A_n\}$	$\{A_1\}$

Figure 8. General form of Single and Strong $\mathcal{R}(\sigma^s)$ $(\#r = n, \#\mathcal{A}^\mathcal{C} = n-1, \#\mathcal{A}^\mathcal{F} = 1)$.

- ⋆ Pure $((\mu^{wy}_{dtp})^m)$;
- ⋆ Impure $((\mu^{wy}_{dti})^m)$.

We will give the quantitative properties of various kinds of Multiple $\mathcal{R}$ in the next section.

PROPERTIES OF $\mathcal{R}$

We distinguish two classes of properties for Combinationally Constrained Relations, namely quantitative and qualitative properties. We discuss the first class of these properties in this paper. Some of the quantitative properties are as follows:

- c: the number of combinational constraints;
- m: the number of Module relations;
- $\#r$: the number of tuples in the body of a relation;
- $\#r_{\mathcal{M}}$: the number of tuples in the body of a Module relation;
- m^s: the number of Strong Module relations;
- m^w: the number of Weak Module relations;
- $\#\mathcal{A}^{\mathcal{C}}$: the number of constrained attributes;
- $\#\mathcal{A}^{\mathcal{F}}$: the number of free attributes;
- $\#\mathcal{A}^{\mathcal{F}}_a$: the number of "absolutely free attributes".

There are some other quantitative properties such as the number of candidate keys and the cardinality of relation, which are not discussed here, since they are related to some qualitative properties including the candidate keys, the level of normality and decomposability of the relations. In the Appendices, we specify properties of various kinds of $\mathcal{R}$ using predicates based on the above mentioned quantitative properties, formally. These properties are the conjunction of the following properties for each kind appropriately:

- Strong: $n \geq 3 \wedge (c = 1 \iff \#\mathcal{A}^{\mathcal{C}} = n-1 \wedge \#\mathcal{A}^{\mathcal{F}} = 1)$;
- Single: $m = 1 \wedge \#r = \#r_{\mathcal{M}}$;
- Single Strong: $m^s = 1 \wedge m^w = 0$;
- Single Weak: $m^s = 0 \wedge m^w = 1$;
- Single Unary: $c = 1 \iff \exists t_1, t_2 : r | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} \neq [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2}$;
- Weak Unary: $n \geq 4 \wedge 2 \leq \#\mathcal{A}^{\mathcal{C}} < n-1 \wedge 1 < \#\mathcal{A}^{\mathcal{F}} \leq n-2$;
- Weak Doubly: $\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1$;
- Single Doubly: $c = 2 \iff \exists r_1 : \mathbb{P}r | \#r_1 = C^{n-1}_{\#\mathcal{A}^{\mathcal{F}}-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} = [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2}$;
- Weak N-ary: $\forall i : 1 \cdots c \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1$;
- Single N-ary: $c > 2$;
- Single Pure: $n \geq 4 \wedge (\#\mathcal{A}^{\mathcal{F}}_n = 0 \iff (\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i = n))$;
- Single Impure: $n \geq 5 \wedge (\#\mathcal{A}^{\mathcal{F}}_n > 0 \iff (\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i + (\#\mathcal{A}^{\mathcal{F}}_a)_i = n))$;
- Single Symmetric: $(\#\mathcal{A}^{\mathcal{C}})_1 = (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r \bullet [(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset$;
- Single Asymmetric: $(\#\mathcal{A}^{\mathcal{C}})_1 \neq (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r \bullet [(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset$;
- Multiple: $m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j$;
- Multiple Strong: $n \geq 3 \wedge m^s > 1 \wedge m^w = 0 \wedge \forall j : 1 \cdots m \bullet (c)^j = 1 \wedge (\#\mathcal{A}^{\mathcal{C}})^j = n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = 1$;
- Multiple Weak: $n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j > 1) \wedge \bigcup_{j=1}^{m} (r_{\mathcal{M}})^j \subseteq r$;
- Multiple Unary: $\forall j : 1 \cdots m \bullet ((c)^j = 1 \iff \exists t_1, t_2 : r | t_1 \neq t_2 \bullet [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} \neq [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge (\#\mathcal{A}^{\mathcal{F}}_a)^j = 0$;
- Multiple Homogeneous Unary: $\exists c_1, f_1 : 1 \cdots n | c_1 + f_1 = n \bullet \forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = f_1$;
- Multiple Heterogeneous Unary: $\exists c_1, f_1 : 1 \cdots n | c_1 + f_1 = n \bullet \forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = f_1$;
- Multiple Doubly: $\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j | \#r_1 = C^{n-1}_{(\#\mathcal{A}^{\mathcal{F}})^j - 1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}$;
- Multiple Homogeneous Doubly: $\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet \forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j_1 = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j_1 = f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})^j_2 = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})^j_2 = f_2$;
- Multiple Heterogeneous Doubly: $\neg\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet \forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j_1 = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j_1 = f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})^j_2 = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})^j_2 = f_2$;
- Multiple Symmetric: $\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j_1 = (\#\mathcal{A}^{\mathcal{C}})^j_2 \wedge (\#\mathcal{A}^{\mathcal{F}})^j_1 = (\#\mathcal{A}^{\mathcal{F}})^j_2 \wedge \forall t : (r_{\mathcal{M}})^j \bullet [(\mathcal{A}^{\mathcal{C}})^j_1]_t \cap [(\mathcal{A}^{\mathcal{C}})^j_2]_t = \emptyset$;
- Multiple Asymmetric: $\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j_1 \neq (\#\mathcal{A}^{\mathcal{C}})^j_2 \wedge \forall t : (r_{\mathcal{M}})^j \bullet [(\mathcal{A}^{\mathcal{C}})^j_1]_t \cap [(\mathcal{A}^{\mathcal{C}})^j_2]_t = \emptyset$;
- Multiple Pure: $\#\mathcal{A}^{\mathcal{F}}_a = 0 \iff \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})^j_i + (\#\mathcal{A}^{\mathcal{F}})^j_i = n$;
- Multiple Impure: $n \geq 5 \wedge (\#\mathcal{A}^{\mathcal{F}}_a > 0 \iff \exists a : 1 \cdots n \bullet (\#\mathcal{A}^{\mathcal{F}}_a = a \wedge \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})^j_i + (\#\mathcal{A}^{\mathcal{C}})^j_i + a = n))$;
- Multiple Composite: $(\exists j : 1 \cdots m \bullet (c)^j = 1) \wedge (\exists j : 1 \cdots m \bullet (c)^j = 2)$.

CASE STUDY

Here, we continue the university example given earlier. We can add another attribute, namely DY: day of the week to PCD_A and form new relationship PCD_B .

$$PCD_B(P\#, C\#, D\#, YT, S\#, DY).$$

The semantic of the relation PCD_B is: Professor p teaches section s of course c in department d in year-term yt on day dy of the week. And finally, the attribute $R\#$: The room number is added to the above relationship.

$$PCD_C(P\#, C\#, D\#, YT, S\#, DY, R\#).$$

The semantic of the relation PCD_C is: Professor p teaches section s of course c in department d in year-term yt on day dy of the week in room r.

As stated before, the number of attributes in the keys of PCD_A, PCD_B and PCD_C depends on the number of $\mathcal{C}$s, the Constraint Degrees of $\mathcal{C}$s and the tuples in the bodies of each relation.

In the following, we will show that the different bodies of these relations satisfy the constraints of the Combinationally Constraint Relation.

Figure 1 shows a sample body for PCD_A, which is a Single and Strong $\mathcal{R}(\sigma^s)$ with the degree of five (i.e. $n = 5$). It is single, since it contains only one $\mathcal{M}$ (i.e. $m = 1$). It is strong, since it has just one $\mathcal{C}$ (i.e. $c = 1$) and its constraint degree is equal to 4 (i.e. $\#\mathcal{A}^{\mathcal{C}} = 4$ and $\#\mathcal{A}^{\mathcal{F}} = 1$).

A sample body for PCD_A, which is a Multiple and Strong $\mathcal{R}((\mu^s)^2)$, is shown in Figure 9. It is Multiple, since it contains two $\mathcal{M}$s (i.e. $m = 2$). It is strong, since it has just one $\mathcal{C}$ in all module relations (i.e. $(c)^1 = 1 \wedge (c)^2 = 1$) and the constraint degrees of all module relations are equal to 4 (i.e. $(\#\mathcal{A}^{\mathcal{C}})^1 = (\#\mathcal{A}^{\mathcal{C}})^2 = 4$, and $(\#\mathcal{A}^{\mathcal{F}})^1 = (\#\mathcal{A}^{\mathcal{F}})^2 = 1$). The first module is exactly the same as the module given in Figure 1. The second module contains exactly the same constraint as the first module. However, the values of attributes in the second module are different from those of the first module.

In Figure 10, a sample body for PCD_A, which is a Single Unary Weak $\mathcal{R}(\sigma_u^w)$, is given. It is single, since it contains only one $\mathcal{M}$ (i.e. $m = 1$). It is unary, since it has just one $\mathcal{C}$ in the module relation (i.e. $c = 1$) and the constraint degree of the module relation is equal to 3 (i.e. $\#\mathcal{A}^{\mathcal{C}} = 3$, and $\#\mathcal{A}^{\mathcal{F}} = 2$). It can be seen that the values of each constrained attribute in six (C_2^4) tuples are equal.

A Multiple Homogeneous Unary Weak $\mathcal{R}((\mu_{uh}^w)^2)$ is given in Figure 11. It is Multiple since it contains two $\mathcal{M}$s (i.e. $m = 2$). It is unary, since it has just one $\mathcal{C}$ in all module relations (i.e. $(c)^1 = 1 \wedge (c)^2 = 1$) and the constraint degrees of all module relations are equal to 3 (i.e. $(\#\mathcal{A}^{\mathcal{C}})^1 = (\#\mathcal{A}^{\mathcal{C}})^2 = 3$, and $(\#\mathcal{A}^{\mathcal{F}})^1 = (\#\mathcal{A}^{\mathcal{F}})^2 = 2$), so the relation is homogeneous. We repeat the body of relations in Figure 10 as the first module and the second module has the same constraint as that of the first module.

Figure 12 shows a sample body for PCD_A, which is a Multiple Heterogeneous Unary Weak $\mathcal{R}((\mu_{uh}^w)^2)$. It is Multiple, since it contains two $\mathcal{M}$s (i.e. $m = 2$). It is unary, since it has just one $\mathcal{C}$ in all module relations (i.e. $(c)^1 = 1 \wedge (c)^2 = 1$), but the constraint degree of the first module relation is equal to 3 (i.e. $(\#\mathcal{A}^{\mathcal{C}})^1 = 3$, and $(\#\mathcal{A}^{\mathcal{F}})^1 = 2$), and the constraint degree of the second module relation is equal to 2 (i.e. $(\#\mathcal{A}^{\mathcal{C}})^2 = 2$, and $(\#\mathcal{A}^{\mathcal{F}})^2 = 3$), so the relation is heterogeneous. The first module is again the same as the body of relation given in Figure 10. In the second module, the properties of the constraint are changed and as can be seen, the values of constrained attributes in four (C_1^4) tuples are equal.

PCD_A

(P#,	C#,	D#,	YT,	S#)
1001	40100	40	851	2
1001	40100	40	852	1
1001	40100	50	851	1
1001	40200	40	851	1
1002	40100	40	851	1
...	...	...	...	...
1003	60100	60	861	4
1003	60100	60	862	3
1003	60100	70	861	3
1003	60200	60	861	3
1004	60100	60	861	3

Figure 9. $(\mu^s)^2$-Multiple Strong $\mathcal{R}$.

PCD_A

(P#,	C#,	D#,	YT,	S#)
1001	40100	50	842	2
1001	40100	60	851	3
1001	40100	70	861	1
1001	40200	50	851	4
1001	40300	50	862	1
1001	40400	80	851	1
1002	40100	50	851	5
1003	40100	50	841	1
1004	40100	90	851	1
1005	40500	50	851	1

Figure 10. σ_u^w-Single Unary Weak $\mathcal{R}$.

PCD_A	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#)$
	1001	40100	50	842	2
	1001	40100	60	851	3
	1001	40100	70	861	1
	1001	40200	50	851	4
	1001	40300	50	862	1
	1001	40400	80	851	1
	1002	40100	50	851	5
	1003	40100	50	841	1
	1004	40100	90	851	1
	1005	40500	50	851	1
	......	......	......	......	......
	1006	50100	10	842	2
	1006	50100	20	871	3
	1006	50100	30	861	6
	1006	50200	10	871	4
	1006	50300	10	862	6
	1006	50400	40	871	6
	1007	50100	10	871	5
	1008	50100	10	841	6
	1009	50100	50	871	6
	1010	50500	10	851	6

Figure 11. $(\mu_{uh}^{w})^2$-Multiple Homogeneous Unary Weak $\mathcal{R}$.

PCD_A	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#)$
	1001	40100	50	842	2
	1001	40100	60	851	3
	1001	40100	70	861	1
	1001	40200	50	851	4
	1001	40300	50	862	1
	1001	40400	80	851	1
	1002	40100	50	851	5
	1003	40100	50	841	1
	1004	40100	90	851	1
	1005	40500	50	851	1
	......	......	......	......	......
	1006	60100	20	831	2
	1006	60200	10	832	3
	1006	60300	30	871	4
	1006	60400	40	841	6
	1007	60100	10	852	2
	1008	60100	50	871	3
	1009	60100	60	861	6
	1010	60500	10	871	4
	1011	60600	10	852	6
	1012	60700	70	871	6

Figure 12. $(\mu_{ut}^{w})^2$-Multiple Heterogeneous Unary Weak $\mathcal{R}$.

A Single Pure Asymmetric Doubly Weak $\mathcal{R}(\sigma_{dp}^{wy})$ is shown in Figure 13. It is single, since it contains only one $\mathcal{M}$ (i.e. $m = 1$). It is doubly, since it has two $\mathcal{C}$s in the module relation (i.e. $c = 2$), but, the constraint degrees of $(\mathcal{C})_1$ and $(\mathcal{C})_2$ are not equal (i.e. $(\#\mathcal{A}^{\mathcal{C}})_1 = 3$, $(\#\mathcal{A}^{\mathcal{F}})_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})_2 = 2$ and $(\#\mathcal{A}^{\mathcal{F}})_2 = 3$), so it is Asymmetric. It is Pure, since there are no extra attributes in the relation. Because of the first constraint, $(\mathcal{C})_1$, the values of the constraint attributes in six (C_2^4) tuples are equal and because of $(\mathcal{C})_2$, the values of constraint attributes in four (C_1^4) tuples are equal.

PCD_A	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#)$
	1001	40100	40	841	2
	1001	40100	50	851	2
	1001	40100	50	841	1
	1001	40200	40	851	2
	1001	40200	40	841	1
	1001	40200	50	851	1
	1002	40100	40	851	2
	1002	40100	40	841	1
	1002	40100	50	851	1
	1002	40200	40	851	1

Figure 13. σ_{dp}^{wy}-Single Pure Asymmetric Doubly Weak $\mathcal{R}$.

To show the impurity property of relations, we use the relation PCD_B. Figure 14 shows a sample body for PCD_B, which contains a Single Impure Asymmetric Doubly Weak $\mathcal{R}(\sigma_{di}^{wy})$ with the degree of six (i.e. $n = 6$). It is exactly the same as the relation given in Figure 13, except that it has an extra attribute, namely DY. The values given for this attribute do not satisfy any kind of combinational constraint, so the relation is an Impure relation.

A Single Pure Symmetric Doubly Weak $\mathcal{R}(\sigma_{dp}^{wx})$ is given in Figure 15. It is single, since it contains only one $\mathcal{M}$ (i.e. $m = 1$). It is doubly, since it has two $\mathcal{C}$s in the module relation (i.e. $c = 2$) and the constraint degrees of both $\mathcal{C}$s are equal to 3 (i.e. $(\#\mathcal{A}^{\mathcal{C}})_1 = 3$, $(\#\mathcal{A}^{\mathcal{F}})_1 = 3$, $(\#\mathcal{A}^{\mathcal{C}})_2 = 3$, and $(\#\mathcal{A}^{\mathcal{F}})_2 = 3$), so it is Symmetric. It is Pure, since there is no extra attribute in the relation. It can be seen that because of $(\mathcal{C})_1$ and (C_2), the values of constraint attributes in two distinct sets of tuples with the cardinality of 10 (C_2^5) are equal.

Again, we use the relation PCD_A (Figure 16) to show a Multiple Pure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$ ($(\mu_{dhp}^{wy})^2$). It is Multiple, since it contains two $\mathcal{M}$s (i.e. $m = 2$). It is doubly, since it has two $\mathcal{C}$s in all module relations (i.e. $(c)^1 = 2 \wedge (c)^2 = 2$).

PCD_B	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#,$	$DY)$
	1001	40100	40	841	2	1
	1001	40100	50	851	2	2
	1001	40100	50	841	1	3
	1001	40200	40	851	2	4
	1001	40200	40	841	1	5
	1001	40200	50	851	1	6
	1002	40100	40	851	2	7
	1002	40100	40	841	1	2
	1002	40100	50	851	1	3
	1002	40200	40	851	1	1

Figure 14. σ^{wy}_{di}-Single Impure Asymmetric Doubly Weak $\mathcal{R}$.

PCD_B	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#,$	$DY)$
	1001	40101	41	852	2	2
	1001	40101	42	851	2	2
	1001	40101	42	852	1	2
	1001	40101	42	852	2	1
	1001	40102	41	851	2	2
	1001	40102	41	852	1	2
	1001	40102	41	852	2	1
	1001	40102	42	851	1	2
	1001	40102	42	851	2	1
	1001	40102	42	852	1	1
	1002	40101	41	851	2	2
	1002	40101	41	852	1	2
	1002	40101	41	852	2	1
	1002	40101	42	851	1	2
	1002	40101	42	851	2	1
	1002	40101	42	852	1	1
	1002	40102	41	851	1	2
	1002	40102	41	851	2	1
	1002	40102	41	852	1	1
	1002	40102	42	851	1	1

Figure 15. σ^{wx}_{dp}-Single Pure Symmetric Doubly Weak $\mathcal{R}$.

PCD_A	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#)$
	1001	40100	40	841	2
	1001	40100	50	851	2
	1001	40100	50	841	1
	1001	40200	40	851	2
	1001	40200	40	841	1
	1001	40200	50	851	1
	1002	40100	40	851	2
	1002	40100	40	841	1
	1002	40100	50	851	1
	1002	40200	40	851	1
	....	....	....	....	....
	1003	40300	60	841	2
	1003	40300	70	851	2
	1003	40300	70	841	1
	1003	40400	60	851	2
	1003	40400	60	841	1
	1003	40400	70	851	1
	1004	40300	60	851	2
	1004	40300	60	841	1
	1004	40300	70	851	1
	1004	40400	60	851	1

Figure 16. $(\mu^{wy}_{dhp})^2$-Multiple Pure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$.

PCD_B	$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#,$	$DY)$
	1001	40100	40	841	2	1
	1001	40100	50	851	2	2
	1001	40100	50	841	1	3
	1001	40200	40	851	2	4
	1001	40200	40	841	1	5
	1001	40200	50	851	1	6
	1002	40100	40	851	2	7
	1002	40100	40	841	1	1
	1002	40100	50	851	1	2
	1002	40200	40	851	1	3
	....	....	....	....	....	....
	1003	40300	60	841	2	4
	1003	40300	70	851	2	5
	1003	40300	70	841	1	6
	1003	40400	60	851	2	7
	1003	40400	60	841	1	1
	1003	40400	70	851	1	2
	1004	40300	60	851	2	3
	1004	40300	60	841	1	4
	1004	40300	70	851	1	5
	1004	40400	60	851	1	6

Figure 17. $(\mu^{wy}_{dhi})^2$-Multiple Impure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$.

The constraint degrees of $(\mathcal{C})^1_1$ and $(\mathcal{C})^1_2$ are not qual (i.e. $(\#\mathcal{A}^{\mathcal{C}})^1_1 = 3$, $(\#\mathcal{A}^{\mathcal{F}})^1_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})^1_2 = 2$ and $(\#\mathcal{A}^{\mathcal{F}})^1_2 = 3$). The constraint degrees of $(\mathcal{C})^2_1$ and $(\mathcal{C})^2_2$ are not qual either (i.e. $(\#\mathcal{A}^{\mathcal{C}})^2_1 = 3$, $(\#\mathcal{A}^{\mathcal{F}})^2_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})^2_2 = 2$ and $(\#\mathcal{A}^{\mathcal{F}})^2_2 = 3$), so it is Asymmetric. It is Pure, since there is no extra attributes in the relation.

Figure 17 shows a Multiple Impure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$ ($(\mu^{wy}_{dhi})^2$). It is exactly the same as the relation given in Figure 16, except that it has an extra attribute, namely DY, and it does not participate in any constraint, so it is an Impure relation.

Figure 18 shows a sample body for PCD_B which is a Multiple Pure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}((\mu^{wy}_{dtp})^2)$. It is Multiple, since it contains two $\mathcal{M}$s (i.e. $m = 2$). It is doubly, since it has two $\mathcal{C}$s in all module relations (i.e. $(c)^1 = 2 \wedge (c)^2 = 2$). The constraint degrees of $(\mathcal{C})^1_1$ and $(\mathcal{C})^1_2$ are equal to 4 (i.e. $(\#\mathcal{A}^{\mathcal{C}})^1_1 = 4$, $(\#\mathcal{A}^{\mathcal{F}})^1_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})^1_2 = 4$ and $(\#\mathcal{A}^{\mathcal{F}})^1_2 = 2$). But, the constraint degrees of $(\mathcal{C})^2_1$ and $(\mathcal{C})^2_2$ are not equal (i.e. $(\#\mathcal{A}^{\mathcal{C}})^2_1 = 4$, $(\#\mathcal{A}^{\mathcal{F}})^2_1 = 2$, $(\#\mathcal{A}^{\mathcal{C}})^2_2 = 5$ and $(\#\mathcal{A}^{\mathcal{F}})^2_2 = 1$)), so it is heterogeneous and Asymmetric.

PCD_B

$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#,$	$DY)$
1001	40101	41	852	2	2
1001	40101	42	851	2	2
1001	40101	42	852	1	2
1001	40101	42	852	2	1
1001	40102	41	851	2	2
1001	40102	41	852	1	2
1001	40102	41	852	2	1
1001	40102	42	851	1	2
1001	40102	42	851	2	1
1001	40102	42	852	1	1
1002	40101	41	851	2	2
1002	40101	41	852	1	2
1002	40101	41	852	2	1
1002	40101	42	851	1	2
1002	40101	42	851	2	1
1002	40101	42	852	1	1
1002	40102	41	851	1	2
1002	40102	41	851	2	1
1002	40102	41	852	1	1
1002	40102	42	851	1	1
...	...	...	...	...	...
1003	60103	30	831	2	2
1003	60103	30	841	3	2
1003	60103	30	841	2	3
1003	60103	40	831	3	2
1003	60103	40	831	2	3
1003	60103	40	841	3	3
1003	60104	30	831	2	2
1003	60104	40	831	3	3
1003	60104	30	831	2	3
1003	60103	40	841	3	2
1004	60103	30	831	2	3
1004	60103	30	841	3	3
1004	60104	30	831	3	3
1004	60104	30	831	3	3
1004	60103	30	831	2	3

Figure 18. $(\mu^{wy}_{dtp})^2$-Multiple Impure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}$.

It is Pure since there are no extra attributes in the relation.

We use the relation PCD_C to show a Multiple Impure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}((\mu^{wy}_{dti})^2)$. Figure 19 shows such a relation with the degree of seven (i.e. $n = 7$). It is exactly the same as the relation given in Figure 18, except that it has an extra attribute, namely $R\#$, so it is an Impure relation.

CONCLUSION AND FUTURE WORKS

In this paper, we introduced the concept of Combinationally Constrained Relations and distinguished two basic types of combinational constraints, namely

PCD_C

$(P\#,$	$C\#,$	$D\#,$	$YT,$	$S\#,$	$DY,$	$R\#)$
1001	40101	41	852	2	2	1
1001	40101	42	851	2	2	2
1001	40101	42	852	1	2	3
1001	40101	42	852	2	1	4
1001	40102	41	851	2	2	5
1001	40102	41	852	1	2	6
1001	40102	41	852	2	1	7
1001	40102	42	851	1	2	8
1001	40102	42	851	2	1	9
1001	40102	42	852	1	1	1
1002	40101	41	851	2	2	2
1002	40101	41	852	1	2	3
1002	40101	41	852	2	1	4
1002	40101	42	851	1	2	5
1002	40101	42	851	2	1	6
1002	40101	42	852	1	1	7
1002	40102	41	851	1	2	8
1002	40102	41	851	2	1	9
1002	40102	41	852	1	1	1
1002	40102	42	851	1	1	2
...	...	...	...	...	...	...
1003	60103	30	831	2	2	3
1003	60103	30	841	3	2	4
1003	60103	30	841	2	3	5
1003	60103	40	831	3	2	6
1003	60103	40	831	2	3	7
1003	60103	40	841	3	3	8
1003	60104	30	831	2	2	9
1003	60104	40	831	3	3	1
1003	60104	30	831	2	3	2
1003	60103	40	841	3	2	3
1004	60103	30	831	2	3	4
1004	60103	30	841	3	3	5
1004	60104	30	831	3	3	6
1004	60104	30	831	3	3	7
1004	60103	30	831	2	3	8

Figure 19. $(\mu^{wy}_{dti})^2$-Multiple Impure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}$.

Weak and Strong. Then, we classified the relations having such a constraint. In general, we have found 16 types of Combinationally Constrained Relation and investigated their quantitative properties.

In our future work, we will study the qualitative properties of these relations and show that some of these relations are in 5NF and even in 6NF, but they contain lots of redundancies and some of them are in 4NF and can be decomposed into 5NF or 6NF relations.

We will research the following issues in our future works:

- Discussing the qualitative properties of various types of $\mathcal{R}$ including the normality form of the above mentioned relations;
- Determining the decomposition algorithms for each one of these relations when possible;
- Defining the set of semantical dependencies among module relations of Multiple $\mathcal{R}$ and determining the effect of these dependencies on their qualitative properties, especially the level of normality.

NOMENCLATURE

Basic

$\mathbb{P}$	power set
$\emptyset$	empty set
$[A]_t$	attribute(s) A of tuple t
$[\![A]\!]_t$	value of attribute(s) A in tuple t
n	degree of relation
$\mathcal{A}^{\mathcal{C}}$	the set of "constrained attributes" of a tuple
$\#\mathcal{A}^{\mathcal{C}}$	constraint degree (the cardinality of $\mathcal{A}^{\mathcal{C}}$)
$\mathcal{A}^{\mathcal{F}}$	the set of free attributes of a tuple
$\#\mathcal{A}^{\mathcal{F}}$	the cardinality of $\mathcal{A}^{\mathcal{F}}$
$\mathcal{A}^{\mathcal{F}}_a$	the set of absolutely free attributes
$\#\mathcal{A}^{\mathcal{F}}_a$	the cardinality of $\mathcal{A}^{\mathcal{F}}_a$

Constraint

$\mathcal{C}$	combinational constraint
$\mathcal{C}^s$	Strong $\mathcal{C}$
$\mathcal{C}^w$	Weak $\mathcal{C}$

Relation

$\mathcal{R}$	Combinationally Constrained Relation
r	body of $\mathcal{R}$
c	the number of $\mathcal{C}$ in a r
$\#r$	cardinality of $\mathcal{R}$ (the number of tuples in r)
$\mathcal{R}^s$	Strong $\mathcal{R}$
$\mathcal{R}^w$	Weak $\mathcal{R}$
$\mathcal{R}^w_u$	Unary Weak $\mathcal{R}$
$\mathcal{R}^w_d$	Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wx}_d$	Symmetric Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wy}_d$	Asymmetric Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wx}_{dp}$	Pure Symmetric Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wy}_{dp}$	Pure Asymmetric Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wx}_{di}$	Impure Symmetric Doubly Weak $\mathcal{R}$
$\mathcal{R}^{wy}_{di}$	Impure Asymmetric Doubly Weak $\mathcal{R}$

Module Relation

$\mathcal{M}$	Module Relation
$\mathcal{M}^s$	Strong Module Relation
$\mathcal{M}^w$	Weak Module Relation
m	the number of $\mathcal{M}$
m^s	the number of $\mathcal{M}^s$
m^w	the number of $\mathcal{M}^w$
$r_{\mathcal{M}}$	the body of $\mathcal{M}$
$\#r_{\mathcal{M}}$	the cardinality of $\mathcal{M}$

Single Combinationally Constrained Relation

σ	Single $\mathcal{R}$
σ^s	Strong σ
σ^w	Weak σ
σ^w_u	Unary σ^w
σ^w_d	Doubly σ^w
σ^{wx}_d	Symmetric σ^w_d
σ^{wy}_d	Asymmetric σ^w_d
σ^{wx}_{dp}	Pure σ^{wx}_d
σ^{wy}_{dp}	Pure σ^{wy}_d
σ^{wx}_{di}	Impure σ^{wx}_d
σ^{wy}_{di}	Impure σ^{wy}_d

Multiple Combinationally Constrained Relation

$(\mu)^m$	Multiple $\mathcal{R}$ with m Module Relations
$(\mu^s)^m$	Strong $(\mu)^m$
$(\mu^w)^m$	Weak $(\mu)^m$
$(\mu^w_u)^m$	Unary $(\mu^w)^m$
$(\mu^w_d)^m$	Doubly $(\mu^w)^m$
$(\mu^w_c)^m$	Composite $(\mu^w)^m$
$(\mu^w_{uh})^m$	Homogeneous $(\mu^w_u)^m$
$(\mu^w_{dh})^m$	Homogeneous $(\mu^w_d)^m$
$(\mu^w_{ut})^m$	Heterogeneous $(\mu^w_u)^m$
$(\mu^w_{dt})^m$	Heterogeneous $(\mu^w_d)^m$

$(\mu_{dh}^{wx})^m$	Symmetric $(\mu_{dh}^{w})^m$
$(\mu_{dh}^{wy})^m$	Asymmetric $(\mu_{dh}^{w})^m$
$(\mu_{dhp}^{wx})^m$	Pure $(\mu_{dh}^{wx})^m$
$(\mu_{dhp}^{wy})^m$	Pure $(\mu_{dh}^{wy})^m$
$(\mu_{dhi}^{wx})^m$	Impure $(\mu_{dh}^{wx})^m$
$(\mu_{dhi}^{wy})^m$	Impure $(\mu_{dh}^{wy})^m$
$(\mu_{dtp}^{wy})^m$	Pure Asymmetric $(\mu_{dt}^{w})^m$
$(\mu_{dti}^{wy})^m$	Impure Asymmetric $(\mu_{dt}^{w})^m$

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APPENDIX A

Properties of Different Kinds of Single $\mathcal{R}$

In the following, we give the specifications of various kinds of Single $\mathcal{R}$ using the quantitative properties.

σ^s (Single Strong $\mathcal{R}$):

$$m = 1 \wedge \#r = \#r_{\mathcal{M}} \wedge$$

$$m^s = 1 \wedge m^w = 0 \wedge$$

$$n \geq 3 \wedge (c = 1 \iff \#\mathcal{A}^{\mathcal{C}} = n - 1 \wedge \#\mathcal{A}^{\mathcal{F}} = 1).$$

σ_u^w (Single Unary Weak $\mathcal{R}$):

$$(m = 1 \wedge \#r = \#r_{\mathcal{M}}) \wedge$$

$$(c{=}1 \iff \exists t_1, t_2 : r | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} \neq [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2}) \wedge$$

$$m^s = 0 \wedge m^w = 1 \wedge$$

$$n \geq 4 \wedge 2 \leq \#\mathcal{A}^{\mathcal{C}} < n - 1 \wedge 1 < \#\mathcal{A}^{\mathcal{F}} \leq n - 2.$$

σ_{dp}^{wx} (Single Pure Symmetric Doubly Weak $\mathcal{R}$):

$((m = 1 \wedge \#r = \#r_{\mathcal{M}})\wedge$

$m^s = 0 \wedge m^w = 1\wedge$

$(c{=}2 \iff \exists r_1 : \mathbb{P}r | \#r_1 = C^{n-1}_{\#\mathcal{A}^{\mathcal{F}}-1}\bullet$

$\forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} = [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2})\wedge$

$(\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1))\wedge$

$((\#\mathcal{A}^{\mathcal{C}}_1 = (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r\bullet$

$[(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset)\wedge$

$(n \geq 4 \wedge (\#\mathcal{A}^{\mathcal{F}}_a = 0 \iff (\forall i : 1 \cdots 2\bullet$

$(\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i = n))).$

σ^{wx}_{di} (Single Impure Symmetric Doubly Weak $\mathcal{R}$):

$((m = 1 \wedge \#r = \#r_{\mathcal{M}})\wedge$

$m^s = 0 \wedge m^w = 1\wedge$

$(c{=}2 \iff \exists r_1 : \mathbb{P}r | \#r_1 = C^{n-1}_{\#\mathcal{A}^{\mathcal{F}}-1}\bullet$

$\forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} = [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2})\wedge$

$(\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1))\wedge$

$((\#\mathcal{A}^{\mathcal{C}})_1 = (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r\bullet$

$[(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset)\wedge$

$(n \geq 5 \wedge (\#\mathcal{A}^{\mathcal{F}}_a > 0 \iff (\forall i : 1 \cdots 2\bullet$

$(\#\mathcal{A}^{c})_i + (\#\mathcal{A}^{\mathcal{F}})_i + (\#\mathcal{A}^{\mathcal{F}}_a)_i = n))).$

σ^{wy}_{dp} (Single Pure Asymmetric Doubly Weak $\mathcal{R}$):

$((m = 1 \wedge \#r = \#r_{\mathcal{M}})\wedge$

$m^s = 0 \wedge m^w = 1\wedge$

$(c = 2 \iff \exists r_1 : \mathbb{P}r | \#r_1 = C^{n-1}_{\#\mathcal{A}^{\mathcal{F}}-1}\bullet$

$\forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} = [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2})\wedge$

$(\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1))\wedge$

$((\#\mathcal{A}^{\mathcal{C}})_1 \neq (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r\bullet$

$[(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset)\wedge$

$(n \geq 4 \wedge (\#\mathcal{A}^{\mathcal{F}}_a = 0 \iff (\forall i : 1 \cdots 2\bullet$

$(\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i = n))).$

σ^{wy}_{di} (Single Impure Asymmetric Doubly Weak $\mathcal{R}$):

$((m = 1 \wedge \#r = \#r_{\mathcal{M}})\wedge$

$m^s = 0 \wedge m^w = 1\wedge$

$(c = 2 \iff \exists r_1 : \mathbb{P}r | \#r_1 = C^{n-1}_{\#\mathcal{A}^{\mathcal{F}}-1}\bullet$

$\forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_1} = [\![\mathcal{A}^{\mathcal{F}}]\!]_{t_2})\wedge$

$(\forall i : 1 \cdots 2 \bullet (\#\mathcal{A}^{\mathcal{C}})_i < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})_i > 1))\wedge$

$((\#\mathcal{A}^{\mathcal{C}})_1 \neq (\#\mathcal{A}^{\mathcal{C}})_2 \wedge \forall t : r\bullet$

$[(\mathcal{A}^{\mathcal{C}})_1]_t \cap [(\mathcal{A}^{\mathcal{C}})_2]_t = \emptyset)\wedge$

$(n \geq 5 \wedge (\#\mathcal{A}^{\mathcal{F}}_a > 0 \iff (\forall i : 1 \cdots 2\bullet$

$(\#\mathcal{A}^{\mathcal{C}})_i + (\#\mathcal{A}^{\mathcal{F}})_i + (\#\mathcal{A}^{\mathcal{F}}_a)_i = n))).$

APPENDIX B

Properties of Different Kinds of Multiple $\mathcal{R}$

In the following, we give the specifications of various kinds of Multiple $\mathcal{R}$ using predicates based on the quantitative properties.

$(\mu^s)^m$ (Multiple Strong $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j)\wedge$$

$(n \geq 3 \wedge m^s > 1 \wedge m^w = 0 \wedge \forall j : 1 \cdots m \bullet$

$(c)^j = 1 \wedge (\#\mathcal{A}^{\mathcal{C}})^j = n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = 1).$

$(\mu^w_{uh})^m$ (Multiple Homogeneous Unary Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j)\wedge$$

$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m\bullet$

$(\#\mathcal{A}^{\mathcal{C}})^j < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$

$$> 1) \wedge \bigcup_{j=1}^{m} (r_{\mathcal{M}})^j \subseteq r)\wedge$$

$(\forall j : 1 \cdots m \bullet ((c)^j = 1 \iff \exists t_1, t_2 : r | t_1 \neq t_2\bullet$

$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} \neq [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge (\#\mathcal{A}^{\mathcal{F}}_a)^j = 0)\wedge$

$(\exists c_1, f_1 : 1 \cdots n | c_1 + f_1 = n \bullet \forall j : 1 \cdots m\bullet$

$(\#\mathcal{A}^{\mathcal{C}})^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = f_1).$

$(\mu_{ut}^{w})^m$ (Multiple Heterogeneous Unary Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m}(\#r_{\mathcal{M}})^j)\wedge$$
$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m\bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})^j < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$$
$$> 1) \wedge \bigcup_{j=1}^{m}(r_{\mathcal{M}})^j \subseteq r)\wedge$$
$$(\forall j : 1 \cdots m \bullet ((c)^j = 1 \iff \exists t_1, t_2 : r | t_1 \neq t_2\bullet$$
$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} \neq [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge (\#\mathcal{A}_n^{\mathcal{F}})^j = 0)\wedge$$
$$(\neg\exists c_1, f_1 : 1 \cdots n | c_1 + f_1 = n \bullet \forall j : 1 \cdots m\bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j = f_1).$$

$(\mu_{dhp}^{wx})^m$ (Multiple Pure Symmetric Homogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m}(\#r_{\mathcal{M}})^j)\wedge$$
$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m\bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})^j < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$$
$$> 1) \wedge \bigcup_{j=1}^{m}(r_{\mathcal{M}})^j \subseteq r)\wedge$$
$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$
$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j-1}^{n-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2\bullet$$
$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2})\wedge$$
$$(\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n\bullet$$
$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})_1^j$$
$$= f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j = f_2)\wedge$$
$$(\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = (\#\mathcal{A}^{\mathcal{C}})_2^j\wedge$$
$$(\#\mathcal{A}^{\mathcal{F}})_1^j = (\#\mathcal{A}^{\mathcal{F}})_2^j \wedge \forall t : (r_{\mathcal{M}})^j\bullet$$
$$[(\mathcal{A}^{\mathcal{C}})_1^j]_t \cap [(\mathcal{A}^{\mathcal{C}})_2^j]_t = \emptyset)\wedge$$
$$(\#\mathcal{A}_a^{\mathcal{F}} = 0 \iff \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})_i^j + (\#\mathcal{A}^{\mathcal{F}})_i^j = n).$$

$(\mu_{dhi}^{wx})^m$ (Multiple Impure Symmetric Homogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m}(\#r_{\mathcal{M}})^j)\wedge$$
$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m\bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})^j < n-1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$$
$$> 1) \wedge \bigcup_{j=1}^{m}(r_{\mathcal{M}})^j \subseteq r)\wedge$$
$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$
$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j-1}^{n-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2\bullet$$
$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2})\wedge$$
$$(\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n\bullet$$
$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})_1^j$$
$$= f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j = f_2)\wedge$$
$$(\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = (\#\mathcal{A}^{\mathcal{C}})_2^j\wedge$$
$$(\#\mathcal{A}^{\mathcal{F}})_1^j = (\#\mathcal{A}^{\mathcal{F}})_2^j \wedge \forall t : (r_{\mathcal{M}})^j\bullet$$
$$[(\mathcal{A}^{\mathcal{C}})_1^j]_t \cap [(\mathcal{A}^{\mathcal{C}})_2^j]_t = \emptyset)\wedge$$
$$(n \geq 5 \wedge (\#\mathcal{A}_a^{\mathcal{F}} > 0 \iff \exists a : 1 \cdots n\bullet$$
$$(\#\mathcal{A}_a^{\mathcal{F}} = a \wedge \forall j : 1 \cdots m; i : 1 \cdots 2\bullet$$
$$(\#\mathcal{A}^{\mathcal{C}})_i^j + (\#\mathcal{A}^{\mathcal{C}})_i^j + a = n))).$$

$(\mu_{dhp}^{wy})^m$ (Multiple Pure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m}(\#r_{\mathcal{M}})^j)\wedge$$
$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$
$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j-1}^{n-1}\bullet$$
$$\forall t_1, t_2 : r_1 | t_1 \neq t_2\bullet$$
$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2})\wedge$$

$$(\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet$$

$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})_1^j$$

$$= f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j = f_2) \wedge$$

$$(\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j \neq (\#\mathcal{A}^{\mathcal{C}})_2^j \wedge \forall t : (r_{\mathcal{M}})^j \bullet$$

$$[(\mathcal{A}^{\mathcal{C}})_1^j]_t \cap [(\mathcal{A}^{\mathcal{C}})_2^j]_t = \emptyset) \wedge$$

$$(\#\mathcal{A}_n^{\mathcal{F}} = 0 \iff \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})_i^j + (\#\mathcal{A}^{\mathcal{F}})_i^j = n)$$

$(\mu_{dhi}^{wy})^m$ (Multiple Impure Asymmetric Homogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j \wedge$$

$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})^j < n - 1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j > 1) \wedge \bigcup_{j=1}^{m}$$

$$(r_{\mathcal{M}})^j \subseteq r) \wedge$$

$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$

$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j - 1}^{n-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet$$

$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge$$

$$(\exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet$$

$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = c_1 \wedge$$

$$(\#\mathcal{A}^{\mathcal{F}})_1^j = f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j$$

$$= f_2) \wedge$$

$$(\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j \neq (\#\mathcal{A}^{\mathcal{C}})_2^j \wedge \forall t : (r_{\mathcal{M}})^j \bullet$$

$$[(\mathcal{A}^{\mathcal{C}})_1^j]_t \cap [(\mathcal{A}^{\mathcal{C}})_2^j]_t = \emptyset) \wedge$$

$$(n \geq 5 \wedge (\#\mathcal{A}_n^{\mathcal{F}} > 0 \iff \exists a : 1 \cdots n \bullet$$

$$(\#\mathcal{A}_a^{\mathcal{F}} = a \wedge \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})_i^j + (\#\mathcal{A}^{\mathcal{C}})_i^j + a = n))).$$

$(\mu_{dtp}^{wy})^m$ (Multiple Pure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j) \wedge$$

$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})^j < n - 1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$$

$$> 1) \wedge \bigcup_{j=1}^{m} (r_{\mathcal{M}})^j \subseteq r) \wedge$$

$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$

$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j - 1}^{n-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet$$

$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge$$

$$(\neg \exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet$$

$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j = c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})_1^j$$

$$= f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j = c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j = f_2) \wedge$$

$$(\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j \neq (\#\mathcal{A}^{\mathcal{C}})_2^j \wedge \forall t : (r_{\mathcal{M}})^j \bullet$$

$$[(\mathcal{A}^{\mathcal{C}})_1^j]_t \cap [(\mathcal{A}^{\mathcal{C}})_2^j]_t = \emptyset) \wedge$$

$$(\#\mathcal{A}_a^{\mathcal{F}} = 0 \iff \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})_i^j + (\#\mathcal{A}^{\mathcal{F}})_i^j = n).$$

$(\mu_{dti}^{wy})^m$ (Multiple Impure Asymmetric Heterogeneous Doubly Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j) \wedge$$

$$(n \geq 4 \wedge m^s \geq 0 \wedge m^w \geq 1 \wedge (\exists j : 1 \cdots m \bullet$$

$$(\#\mathcal{A}^{\mathcal{C}})^j < n - 1 \wedge (\#\mathcal{A}^{\mathcal{F}})^j$$

$$> 1) \wedge \bigcup_{j=1}^{m} (r_{\mathcal{M}})^j \subseteq r) \wedge$$

$$(\forall j : 1 \cdots m \bullet (c)^j = 2 \iff \exists r_1 : \mathbb{P}(r_{\mathcal{M}})^j |$$

$$\#r_1 = C_{(\#\mathcal{A}^{\mathcal{F}})^j - 1}^{n-1} \bullet \forall t_1, t_2 : r_1 | t_1 \neq t_2 \bullet$$

$$[\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_1} = [\![(\mathcal{A}^{\mathcal{F}})^j]\!]_{t_2}) \wedge$$

$$(\neg \exists c_1, f_1, c_2, f_2 : 1 \cdots n | c_1 + f_1 = n \wedge c_2 + f_2 = n \bullet$$

$$\forall j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})_1^j$$

$$= c_1 \wedge (\#\mathcal{A}^{\mathcal{F}})_1^j = f_1 \wedge (\#\mathcal{A}^{\mathcal{C}})_2^j$$

$$= c_2 \wedge (\#\mathcal{A}^{\mathcal{F}})_2^j = f_2) \wedge$$

$(\exists j : 1 \cdots m \bullet (\#\mathcal{A}^{\mathcal{C}})^j_1 \neq (\#\mathcal{A}^{\mathcal{C}})^j_2 \wedge \forall t : (r_{\mathcal{M}})^j \bullet$

$[(\mathcal{A}^{\mathcal{C}})^j_1]_t \cap [(\mathcal{A}^{\mathcal{C}})^j_2]_t = \emptyset) \wedge$

$(n \geq 5 \wedge (\#\mathcal{A}^{\mathcal{F}}_a > 0 \iff \exists a : 1 \cdots n \bullet$

$(\#\mathcal{A}^{\mathcal{F}}_a = a \wedge \forall j : 1 \cdots m; i : 1 \cdots 2 \bullet$

$(\#\mathcal{A}^{\mathcal{C}})^j_i + (\#\mathcal{A}^{\mathcal{C}})^j_i + a = n))).$

$(\mu^w_c)^m$ (Multiple Composite Weak $\mathcal{R}$):

$$(m > 1 \wedge \#r \geq \sum_{j=1}^{m} (\#r_{\mathcal{M}})^j) \wedge$$

$((\exists j : 1 \cdots m \bullet (c)^j = 1) \wedge (\exists j : 1 \cdots m \bullet$

$(c)^j = 2)).$