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# Solution of Convection-Dominated Problems on Irregular Meshes by Collocated Discrete Least Squares Mesh-Less (CDLSM) Method

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**Abstract.** *In this paper, a study is performed on the effect of irregularity of domain discretization on the performance of the CDLSM method for the solution of convection-dominated problems. The method is based on minimizing a least squares functional of the residuals of the governing differential equations and its boundary conditions over a set of collocation points. Four convection-dominated benchmark examples are solved using CDLSM method on three different sets of nodal distribution with different levels of irregularity and the results are presented. These experiments show that CDLSM method is capable of producing stable and accurate results for hyperbolic problems with shocked or high gradient solutions even on highly irregular mesh of nodes. Mesh-less methods as alternative numerical approaches to eliminate the well-known drawbacks of mesh-based methods have attracted much attention in the past decade due to their flexibility and their potentiality in negating the need for the human-labor intensive process of constructing geometric meshes in a domain. Exploiting this ability, however, requires that the method could solve the problem under consideration on unstructured distribution of nodes. This is particularly important when a refinement strategy is to be used to improve the performances of these methods.*

**Keywords:** *CDLSM; Meshless; Irregular mesh; Convection-dominated problems; Refinement strategy.*

## INTRODUCTION

In the past decades, a group of so-called mesh-free or meshless methods have become one of the hottest areas of research in computational mechanics. As their name implies, one common characteristic of all these methods is that they do not require the traditional mesh to construct the numerical formulation. Mesh-free methods possess a number of interesting properties. For example, they require node generation instead of mesh generation. In other words, there is no pre-specified connectivity or relationship among the nodes, thus the computational costs associated with mesh generation are highly reduced. Another attractive property of mesh-free methods is the computational ease of adding and subtracting nodes from the pre-existing nodes. The computational advantages of a mesh-free method suggest that they have potentials in solving a broad class of scientific and engineering problems.

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Various meshless methods have been developed and used to solve different problems including those encountered in the fluid mechanics discipline. Smooth Particle Hydrodynamics (SPH) introduced by Gingold and Monaghan [1], and used by Ataie and Shobeyri [2] and Ataie et al. [3], Reproducing Kernel Particle Method (RKPM) by Liu et al. [4], Element-Free Galerkin method (EFG) by Belytschko et al. [5], Mesh-less Local Petrov-Galerkin (MLPG) method by Atluri and Zhu [6], partition of unity by Melenk and Babuska [7], Hp-clouds by Duarte and Oden [8] and Finite Point (FP) method by Onate et al. [9] are all meshless methods from the point of view of the node interpolation, and have already been widely applied to various areas. The advantages of these meshless methods are apparent, however, serious limitations exist. For instance, the difficulties of imposition of essential boundary and treatment of material discontinuities, uncertain choice of the weight functions, difficulties in the integration of stiffness matrix, and complexity in algorithms for computing the interpolation functions are all major technical problems in these methods.

The meshless methods have been proposed to avoid the numerical difficulties of mesh entanglement in the Finite Element Method (FEM) which have been widely used in different engineering fields [10-12]. Meshless methods, however, have to pay for the high cost in the computational time, the enforcement of essential boundary condition and the treatment of material discontinuities. Special technologies, such as the Penalty method by Zhu and Atluri [13], transformation of approximate nodal values to actual nodal values by Cai and Zhu [14], nodal integration method by Beissel and Belytschko [15], and efficient computation of shape functions by Beitkopf et al. [16] have been proposed to overcome these problems.

Recently a family of collocation-based meshless methods are emerging in the literature. Collocation methods enjoy simplicity and efficiency when used as meshless methods but they suffer from stability problems. Some researchers have, therefore, attempted a hybridization of the collocation method with other discretization schemes as a remedy to the shortcomings of the collocation methods. Afshar and Arzani [17] developed Discrete Least Squares Mesh-less (DLSM) method for the solution of Poisson equation. In this method a fully least squares approach was used in both function approximation and the discretization of the governing differential equations. The meshless shape functions were derived using the Moving Least Squares (MLS) method of function approximation. The discretized equations were obtained via a discrete least squares method in which the sum of the squared residuals were minimized with respect to the unknown nodal parameters. While most of the existing meshless methods need background cells for numerical integration, DLSM did not require numerical integration procedure. This method had the additional advantages of producing symmetric, positive and definite matrices even for non-self adjoint operators as encountered in fluid flow problems.

Zhang et al. used the Least squares Collocation Meshless (LSCM) Method [18] to solve elliptic problems. In this method, a set of over determined system of equations, in which the number of the equations was greater than the number of unknowns, was constructed and solved by the least squares method. The solution of some steady and unsteady heat conduction problems were investigated by Liu et al. [19] using a Meshless Weighted Least Squares (MWLS) method. A sensitivity analysis on the MWLS parameters to solve the problems of a cantilever beam and an infinite plate with a central circular hole was performed by Pan et. al [20]. Armentano and Durán [21] carried out an error estimates for moving least square approximations used for the solution of 1-D convection-diffusion problems. Wang et al. [22] tested a point weighted least squares meshless method for the solution of 1-D and 2-D Poisson equations.

Firoozjaee and Afshar [23] proposed Collocated Discrete Least Squares Mesh-less (CDLSM) method to solve elliptic partial differential equations, and studied the effect of the collocation points on the convergence and accuracy of the method. CDLSM was later extended by Naisipour et al. [24] to solve elasticity problems on irregular distribution of nodal points. Afshar and Lashckarbolok [25] were first to use the CDLSM method for the solution of hyperbolic problems. They also suggested a posteriori error estimate and adaptive refinement strategy in conjunction with the CDLSM method for 1-D hyperbolic problems. More recently, Afshar et al. [26] examined the effect of the number of collocation points on the accuracy of CDLSM method for both transient and steady state one dimensional hyperbolic problems with uniform nodal spacing. CDLSM has also been used successfully to simulate free surface flows by Shobeyri and Afshar [27].

CDLSM has shown to have some similarity with MWLS as suggested by Liu et al. [19] and Pan et al. [20] who use least squares method for the discretization of the governing differential equations. A simple but very decisive difference, however, exists between these methods which is the use of the collocation points in the CDLSM method. In CDLSM, the least squares functional is formed at the collocation points while it is calculated at nodal points in MWLS. At least three advantages of the collocation points were shown in [26] which are as follows. First, the collocation points can stabilize the method, especially in problems with shocked solution. Second, the collocation points can improve the accuracy of the method even in problems with smooth solutions. Third, faster convergence can be achieved in steady-state problems using collocation points. It has recently, however, come to the attention of the authors, by one of the respected reviewers of the paper, that in an alternative formulation of the MWLS method proposed by Zhang et al. [28], a set of auxiliary points in addition to the nodal points were also used to eliminate the residuals of the governing equations.

In this paper, the CDLSM method is extended for the solution of one and two dimensional convection dominated problems and its performance for the solution of steady and transient problems on irregular distribution of nodes are investigated. Four test problems from the literature, namely nonlinear 1-D Burgers equation in transient form, 1-D dam break problem, 2-D pure convection problem and finally 2-D transient Burgers equation are solved using proposed CDLSM method on three set of node configurations with different level of irregularity and the results are presented and compared to the analytical results wherever available. These experiments show that the proposed CDLSM method is capable of producing stable and accurate results for the difficult problems considered.

## MOVING LEAST SQUARES (MLS) METHOD

Several techniques have been developed to construct meshless shape functions. The MLS approximation by Lancaster and Salkauskas [29], the Radial Point Interpolation Method (RPIM) by Liu and Gu [30] and the Kriging interpolation by Gu [31] are only some examples of existing methods. Amongst these methods, the MLS method has gained more popularity in the meshless community. In MLS, the function to be approximated is represented by:

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x}) a_i(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}). \qquad (1)$$

Here $\mathbf{p}^T(\mathbf{x})$ is a set of linearly independent polynomial basis and $\mathbf{a}(\mathbf{x})$ represents the unknown coefficients to be determined by the fitting algorithm. In Moving Least Square (MLS) approximation, at each point $\mathbf{x}$, $\mathbf{a}(\mathbf{x})$ is chosen to minimize the sum of weighted squared residuals defined by:

$$J = \frac{1}{2}\sum_{I=1}^{n} w(|\mathbf{x}-\mathbf{x}_I|)[\mathbf{p}^T(\mathbf{x}_I)\mathbf{a}(\mathbf{x}) - u_I]^2, \qquad (2)$$

where $u_I$ is nodal value of the function to be approximated, $n$ is the number of nodes and $w(|\mathbf{x}-\mathbf{x}_I|)$ is the weight function defined to have compact support. Many weight functions are established and used by different researchers. In this paper an exponential weight function is used as follows:

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & \text{for } r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for } \frac{1}{2} < r \le 1 \\ 0 & \text{for } r > 1 \end{cases} \qquad (3)$$

in which $r = s/s_{\max}$, $s = \|\mathbf{x}-\mathbf{x}_I\|$ and $s_{\max}$ is the radius of the support.

The coefficients $\mathbf{a}(\mathbf{x})$ are found by minimizing $J$ with respect to these coefficients. Carrying out the differentiation and setting it to zero leads to the following relation for the unknown parameters $\mathbf{a}(\mathbf{x})$:

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u}, \qquad (4)$$

where:

$$\mathbf{A} = \mathbf{P}^T\mathbf{W}(\mathbf{x})\mathbf{P}, \qquad (5)$$

$$\mathbf{B} = \mathbf{P}^T\mathbf{W}(\mathbf{x}), \qquad (6)$$

$$\mathbf{u}^T = (u_1, u_2, \cdots, u_n), \qquad (7)$$

$$\mathbf{P} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix}, \qquad (8)$$

and:

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w(|\mathbf{x}-\mathbf{x}_1|) & 0 & \cdots & 0 \\ 0 & w(|\mathbf{x}-\mathbf{x}_2|) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w(|\mathbf{x}-\mathbf{x}_n|) \end{bmatrix}. \qquad (9)$$

The approximation of the unknown function can now be written in the familiar form of:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I=1}^{n} \mathbf{N}_I(\mathbf{x})\mathbf{u}_I, \qquad (10)$$

where $\mathbf{N}_I(\mathbf{x})$ denote the shape function of node $I$ defined as:

$$\mathbf{N} = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}). \qquad (11)$$

MLS shape functions generally do not satisfy the Kronecker delta condition. Hence the parameters $\mathbf{u}_I$ cannot be treated like nodal values of the unknown function ($\mathbf{u}^h(\mathbf{x_i}) \neq \mathbf{u}_I$).

Generally, it is necessary to obtain the shape function derivatives. The spatial derivatives of the shape functions are obtained as:

$$\frac{d\mathbf{N}(\mathbf{x})}{dx} = \frac{d\mathbf{P}}{dx}\mathbf{A}^{-1}\mathbf{B} + \mathbf{P}\frac{d(\mathbf{A}^{-1})}{dx}\mathbf{B} + \mathbf{P}\mathbf{A}^{-1}\frac{d\mathbf{B}}{dx}. \qquad (12)$$

## COLLOCATED DISCRETE LEAST SQUARES MESHLESS (CDLSM) METHOD

Consider the general form of differential equations governing the transient convection-diffusion problems written in matrix form as:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}_i(\mathbf{u})\frac{\partial \mathbf{u}}{\partial \mathbf{x_i}} - k\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x_i}^2} = \mathbf{Q}(\mathbf{u}),$$

$$i = 1, 2, \quad \text{on} \quad \Omega. \qquad (13)$$

Subject to appropriate Dirichlet and Neuman boundary conditions:

$$\begin{aligned} \mathbf{u} &= \overline{\mathbf{u}} && \text{on } \Gamma_{\mathbf{u}}, \\ B(\mathbf{u}) &= g && \text{on } \Gamma_t. \end{aligned} \qquad (14)$$

Here, $\mathbf{u}$ denotes the problem unknown vector, $\mathbf{A_i}$ is the Jacobian matrix in the $i$th dimension which is generally a function of the unknown vector $\mathbf{u}$, $\mathbf{Q}$ is the source term vector, $k$ is the diffusion coefficient assumed to be independent from the spatial dimensions and $B$ is a differential operator defined on Neuman boundaries.

A semi-discretization is first carried out using the $\theta$ method in time as follows:

$$\mathbf{u^{n+1}} - \mathbf{u^n} + \Delta t\theta\left[\mathbf{A}_i^{n+1}(\mathbf{u})\frac{\partial \mathbf{u^{n+1}}}{\partial \mathbf{x_i}} - k\frac{\partial^2 \mathbf{u^{n+1}}}{\partial \mathbf{x_i^2}} - \mathbf{Q^{n+1}}\right] + \Delta t(1-\theta)\left[\mathbf{A}_i^n(\mathbf{u})\frac{\partial \mathbf{u^n}}{\partial \mathbf{x_i}} - k\frac{\partial^2 \mathbf{u^n}}{\partial \mathbf{x_i^2}} - \mathbf{Q^n}\right] = 0, \tag{15}$$

with $\frac{1}{2} \leq \theta \leq 1$ for the stability of the temporal discretization scheme. Assuming $\mathbf{Q} = \mathbf{S}\mathbf{u}$, the linearized residuals in the problem domain and its boundaries can now be defined as:

$$R_\Omega^{\mathbf{n+1}} = \mathbf{u^{n+1}} - \mathbf{u^n} + \Delta t\theta\left[\mathbf{A_i^n}(\mathbf{u})\frac{\partial \mathbf{u^{n+1}}}{\partial \mathbf{x_i}} - k\frac{\partial^2 \mathbf{u^{n+1}}}{\partial \mathbf{x_i^2}} - \mathbf{S^n u^{n+1}}\right] + \Delta t(1-\theta)\left[\mathbf{A_i^n}(\mathbf{u})\frac{\partial \mathbf{u^n}}{\partial \mathbf{x_i}} - k\frac{\partial^2 \mathbf{u^n}}{\partial \mathbf{x_i^2}} - \mathbf{S^n u^n}\right], \tag{16}$$

$$R_{\Gamma_t}^{n+1} = B(\mathbf{u}_k) - g(\mathbf{x_k}), \tag{17}$$

$$R_{\Gamma_u}^{n+1} = \mathbf{u^{n+1}} - \bar{\mathbf{u}}. \tag{18}$$

The philosophy of least squares is to find an approximate solution that minimizes the least squares functional to be defined later. Assume that the problem domain and boundaries are discretized by some nodal points, and their number is $n$. Beside the nodal points, the collocation points are used in the problem domain and on its boundaries to construct the least squares functional. The total number of collocation points is $M$ comprised of $M_d$ internal collocation points, $M_u$ collocation points on the Dirichlet boundary and $M_t$ collocation points on Neuman boundary, i.e.:

$$M = M_d + M_u + M_t. \tag{19}$$

The approximate value of the function $\mathbf{u}$ at a collocation point $\mathbf{x_k}$ can be obtained through interpolation:

$$\mathbf{u}(\mathbf{x_k}) = \sum_{i=1}^{n_k} N_i(\mathbf{x_k}).\mathbf{u_i}, \tag{20}$$

where $n_k$ is the number of nodal points having $\mathbf{x_k}$ in their support domain. Substituting Equation 20 into Equations 16, 17 and 18 leads to the differential equation residual $R^d$, the Neuman boundary condition residual $R^t$ and the Dirichlet boundary condition residual $R^u$ defined as:

$$R_k^{(d)} = L(\mathbf{u_k}) - \mathbf{f}(\mathbf{x_k}) = \sum_{j=1}^{n_k} L(N_j(\mathbf{x_k}))\mathbf{u_j} - \mathbf{f}(\mathbf{x_k}), \quad (k = 1 \sim M), \tag{21}$$

with

$$L(\cdot) = (\cdot) + \Delta t\theta\left[\mathbf{A}_i^n(\mathbf{u})\frac{\partial(\cdot)}{\partial \mathbf{x}_i} - k\frac{\partial^2(\cdot)}{\partial \mathbf{x_i^2}} - \mathbf{S}^n(\cdot)\right], \tag{22}$$

$$\mathbf{f} = \mathbf{u}^n - \Delta t(1-\theta)\left[\mathbf{A}_i^n\frac{\partial \mathbf{u}^n}{\partial \mathbf{x}_i} - k\frac{\partial^2 \mathbf{u}^n}{\partial \mathbf{x}_i^2} - \mathbf{Q}^n\right], \tag{23}$$

$$R_k^{(t)} = B(\mathbf{u}_k) - g(\mathbf{x}_k) = \sum_{j=1}^{n_k} B(N_j(\mathbf{x}_k))\mathbf{u}_j - g(\mathbf{x_k}), \quad (k = 1 \sim M_t), \tag{24}$$

$$R_k^{(u)} = \mathbf{u_k} - \bar{\mathbf{u}} = \sum_{j=1}^{n_k}(N_j(x_k))\mathbf{u_j} - \bar{\mathbf{u}}, \quad (k = 1 \sim M_u). \tag{25}$$

Now the least squares functional of all residuals at all collocation points can be constructed as:

$$J = \frac{1}{2}\left(\sum_{k=1}^{Md}[R_k^{(d)}]^2 + \alpha\sum_{k=1}^{M_t}[R_k^{(t)}]^2 + \beta.\sum_{k=1}^{M_u}[R_k^{(u)}]^2\right). \tag{26}$$

The factors $\alpha$ and $\beta$ in the above equation are meant to represent the relative weight of the boundary residuals with respect to the domain residual.

Minimization of Equation 26 with respect to the nodal parameters $u_i$ leads to:

$$\frac{\partial J}{\partial \mathbf{u}_i} = \sum_{k=1}^{M}\frac{\partial R_k^{(d)}}{\partial \mathbf{u}_i}[R_k^{(d)}] + \alpha\sum_{k=1}^{M_t}\frac{\partial R_k^{(t)}}{\partial \mathbf{u}_i}[R_k^{(t)}] + \beta\sum_{k=1}^{M_u}\frac{\partial R_k^{(u)}}{\partial \mathbf{u}_i}[R_k^{(u)}] = 0. \tag{27}$$

Substituting Equations 21, 24 and 25 into Equation 27 yields the final system of equations:

$$\mathbf{KU} = \mathbf{F}. \tag{28}$$

The typical components of the matrix $\mathbf{K}$ and right hand side vector $\mathbf{F}$ are defined as:

$$K_{lm} = \sum_{i=1}^{M_d}[L(N_l)]_i^T[L(N_m)]_i + \alpha\sum_{i=1}^{M_t}[B(N_l)]_i^T[B(N_m)]_i + \beta\sum_{i=1}^{M_u}[(N_l)]_i^T[(N_m)]_i, \quad l, m = \mathbf{1}, \cdots, n, \tag{29}$$

$$F_l = -\sum_{i=1}^{M_d}[L(N_l)]_i^T \mathbf{f}_i + \alpha \sum_{i=1}^{M_t}[B(N_l)]_i^T g_i + \beta \sum_{i=1}^{M_u}[(N_l)]_i^T(\bar{\mathbf{u}}),$$

$$l = \mathbf{1}, \cdots, n. \quad (30)$$

The system of equations can now be formed and solved at each time step and the required solution produced in a time marching manner until a steady state solution is reached if a steady state solution is desired. It should be noted here that the proposed method is stable for any time and space step sizes due to the implicit nature of the method. The stiffness matrix $\mathbf{K}$ in Equation 29 can be seen to be symmetric and positive-definite. Therefore, the final system of equations can be solved using efficient iterative procedure such as conjugate gradient methods.

## NUMERICAL EXAMPLES

In this section, a set of transient and steady state hyperbolic problems are solved on a series of nodal distributions with different level of irregularity and the results are compared to assess the effect of mesh irregularity on the performance of the proposed CDLSM method. In 1-D problems, the standard deviation of the nodal spacing is considered as a measure of mesh irregularity while for 2-D problems, the average value of the absolute difference between the radius of support domain on irregular mesh with those on regular mesh is considered as the mesh irregularity.

The mesh irregularity index can be defined mathematically as:

For one-dimensional problems:

$$I_{\text{indx}} = \left(\frac{1}{M_n - 1}\sum_{i=1}^{M_n}(x_i^{\text{reg}} - x_i^{\text{irreg}})^2\right)^{\frac{1}{2}}.$$

For two-dimensional problems:

$$I_{\text{indx}} = \frac{1}{M_n}\sum_{i=1}^{M_n}(abs(s_{\max}^{\text{reg}}|_i - s_{\max}^{\text{irreg}}|_i)),$$

where $M_n$ is the number of nodal points in the computational domain, $x_i^{\text{reg}}$, $s_{\max}^{\text{reg}}|_i$, $x_i^{\text{irreg}}$ and $s_{\max}^{\text{irreg}}|_i$ are the positions of nodal points and the radius of support domain for regular and irregular nodal distributions, respectively.

The accuracy of MLS interpolation greatly deadens on how to define support domain for the point of interest. Therefore, an efficient method to choose support domain is required for accurate and efficient approximation. For one-dimensional problems, the size of the support domain ($s_{\max}$) is defined so that at least two nodes be in the support domain. For two-dimensional problems, the radius of the support domain is defined by:

$$s_{\max} = \alpha_s d_c,$$

where $\alpha_s$ is a user-defined coefficient and $d_c$ is a measure of the average nodal spacing. For irregular nodal point distributions $d_c$ is chosen as the average distance of the five nearest nodal points to the point under consideration. Generally, $\alpha_s = 2.0 \sim 3.0$ leads to accurate results for many problems [32]. Here, $\alpha_s = 2.0$ and 2.7 are used for pure convection and the Burgers problems, respectively.

It also should be noted that polynomial basis of order zero ($P = [1]$) and order two ($p = [1, x, y, xy, x^2, y^2]$) are used for 1-D and 2-D problems, respectively, to construct MLS shape functions.

### Transient 1-D Burgers Equation

This is a problem governed by the inviscid Burgers equation defined by following parameters of Equation 13:

$$\mathbf{A} = u, \qquad k = 0, \qquad \mathbf{Q} = 0.$$

The problem is solved on the domain $0 \le x \le 1$ with the following initial and boundary conditions:

$$u(0) = 2, \qquad 0 \le x \le 0.5,$$

$$u(0) = 0, \qquad 0.5 < x \le 1,$$

$$u(t) = 2, \qquad x = 0.0,$$

$$u(t) = 0, \qquad x = 1.$$

The exact solution to this problem is represented by a discontinuity moving with velocity of 1 m/s. Burgers equation is a simple non-linear model representing physical problems described by the convection-diffusion and convection-reaction process. Many physical problems such as sound and shock waves in viscous medium and magnetohydrodynamic waves can be described by Burgers equation.

This problem is solved on a mesh of 61 nodal points with 121 distributed collocation points, 61 of them coinciding with the nodal points and each of the remaining collocation points is located between two nodal points. The problem is solved on three meshes of 0.0, 0.0078, and 0.0098 irregularity using a time step size of 0.003. The mesh of nodes and the corresponding results are shown in Figures 1 to 3 and

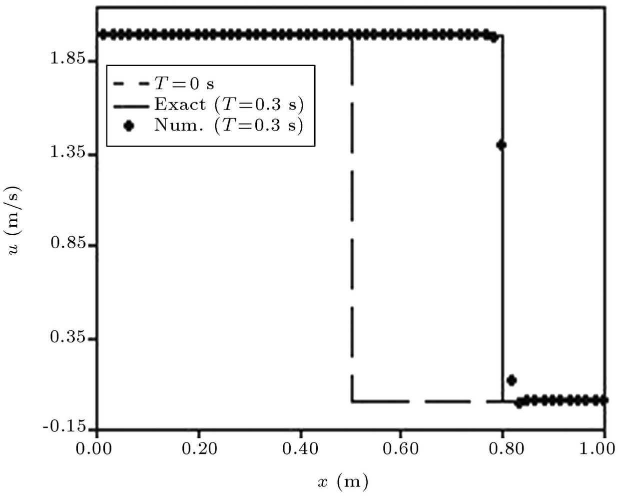


**Figure 1.** Solution of steady Burgers problem on a mesh of 0.0 irregularity (uniform mesh).

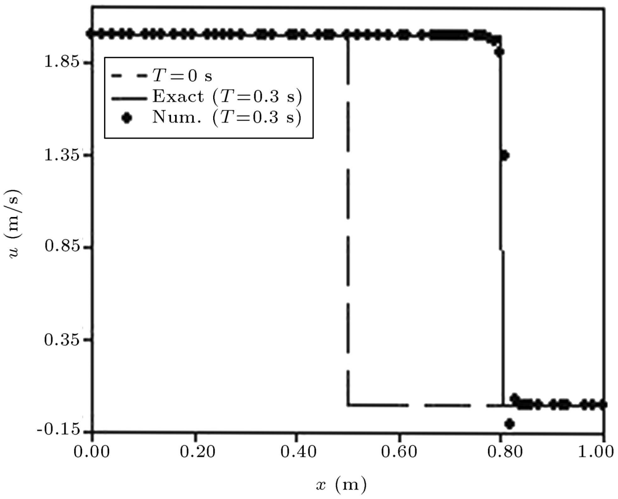


**Figure 2.** Solution of steady Burgers problem on a mesh of 0.0078 irregularity.

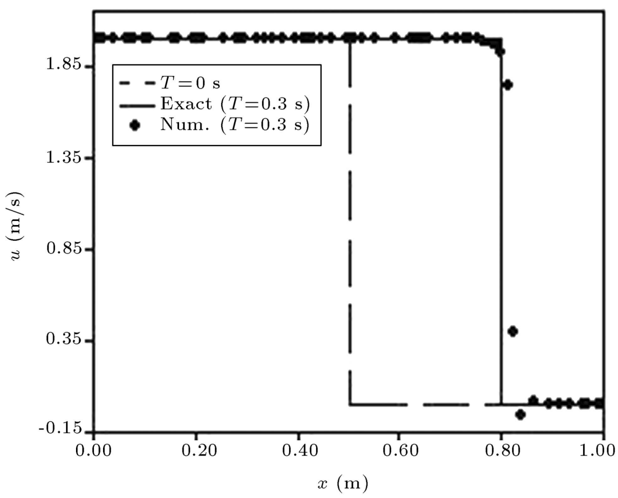


**Figure 3.** Solution of steady Burgers problem on a mesh of 0.0098 irregularity.

compared with the analytical solution at time of 0.3 s. The results clearly show the ability of the proposed CDLSM method to correctly capture the shock even for highly irregular mesh of Figure 3.

## Breaking of a Dam

The non-linear shallow-water equations in one dimension governing the breaking of a dam problem can be defined by the following parameters of Equation 13:

$$\mathbf{u} = \begin{bmatrix} H+\eta \\ (H+\eta)u \end{bmatrix}, \qquad \mathbf{k} = 0,$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -u^2 + g(H+\eta) & \mathbf{2}u \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 \\ g(H+\eta)^{dH/dx} \end{bmatrix},$$

where $H$ is the depth, $\eta$ is the surface elevation, $u$ is the velocity, $g$ is the acceleration due to gravity and $\frac{dH}{dx} = 0$ is the bed slope. The breaking of a dam is a significant practical problem in civil engineering. It is necessarily required to predict the fluid flow induced by breaking of dam for designing a dam and its surrounding environment. Dam break flow is an ideal test problem to examine the accuracy of numerical approaches, and it can be simulated by removal of a barrier holding a body of water at rest in numerical simulations.

The problem of a propagating jump discontinuity due to the breaking of a dam was computed by Lonher et al. [33] using a Taylor-Galerkin finite element method, Carrey and Jiang [34], Zienkiewicz and Taylor [35] and Afshar and Morgan [36] using least square finite element schemes. In the present study, the initial condition of $u = 0$, $\eta = 2$ for $0 \leq x \leq 20$ and $u = 0$, $\eta = 0$ for $20 < x \leq 40$ was used. The depth $H$ and $g$ are assumed constant and equal to unity. The problem is solved here on a mesh of 81 nodal points with 321 distributed collocation points, 81 of them coinciding with the nodal points. The problem is solved on three meshes of 0.0, 0.22, and 0.41 irregularity using a time step size of 0.1. The mesh of nodes and the corresponding results are shown in Figures 4 to 6 and compared with the results of Zienkiewicz and Taylor [35]. The results again show that proposed CDLSM method can handle propagating shocked solution on highly irregular meshes.

## Two-Dimensional Pure Convection Problem

Pure convection problems in 2-D can be described by the following parameters of Equation 13:

$$A_1(\mathbf{u}) = A_2(\mathbf{u}) = 1,$$

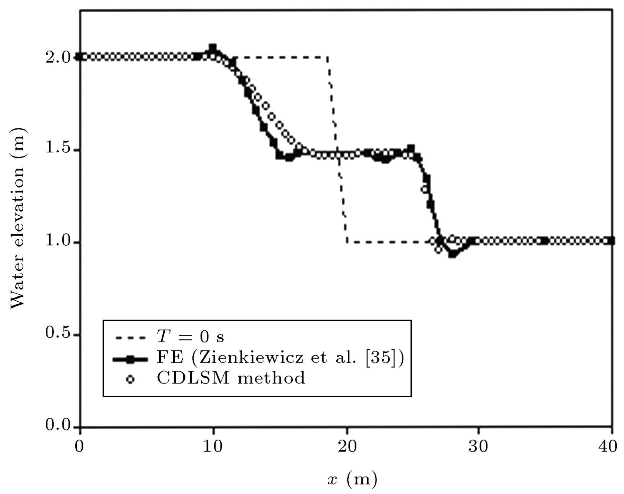


**Figure 4.** Solution of dam break equation problem at $T = 5$ s on a mesh 0.0 irregularity (uniform mesh).

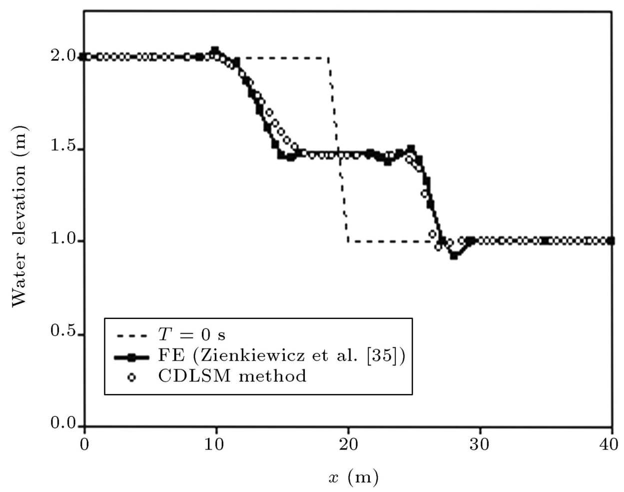


**Figure 5.** Solution of dam break equation problem at $T = 5$ s on a mesh 0.22 irregularity.

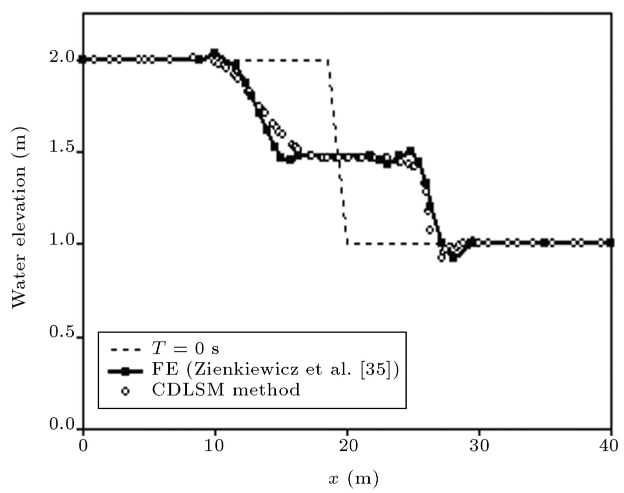


**Figure 6.** Solution of dam break equation problem at $T = 5$ s on a mesh 0.41 irregularity.

where $A_1$ and $A_2$ are constant coefficient representing the components of the velocity field along $x$ and $y$ axes, respectively. The boundary conditions of the problem are defined as:

$$\begin{cases} u = 0 & x = 0, \quad 0 \le y \le 1 \\ u = 1 & y = 0, \quad 0 \le x \le 1 \end{cases}$$

Viscosity effects are neglected in pure convection problems and, therefore, the mathematical solution of these types of problems can be sharp fronts and discontinuities. Viscosity plays a significant role to smooth the sharp discontinuities in the real physical phenomena.

The problem is first solved on a regular mesh of 441 nodal points ($\Delta x = \Delta y = 0.05$) using 841 uniformly distributed collocation points 441 of which coinciding with the nodal points. Figure 7 shows the distribution of nodal points along with the contour lines of solution obtained. This solution was obtained using a time step size of 0.015 and $\theta = 0.5$.

The problem is also solved on two meshes with irregularity indexes of 0.011 and 0.0195 using the same computational parameters as used on the uniform mesh. Again 841 collocation points are used as in the case of uniform mesh so that the computational effort is the same as that of uniform mesh. 441 of the collocation points coincided with the nodal points and the remaining 400 collocation points were distributed uniformly on the computational domain. Figures 8 and 9 show the nodal distribution and the solution contours on two irregular meshes used. The solutions along $y = 0.5$ obtained using different meshes are compared in Figure 10 showing that the accuracy of the results produced by proposed CDLSM method is not affected much by the irregularity of the meshes used.

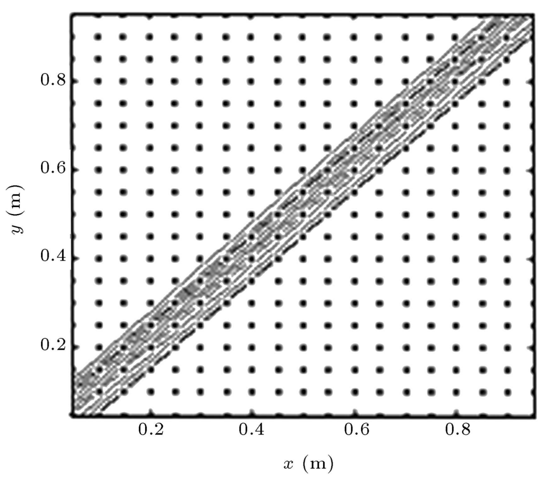


**Figure 7.** Solution of pure convection problem on uniform mesh.

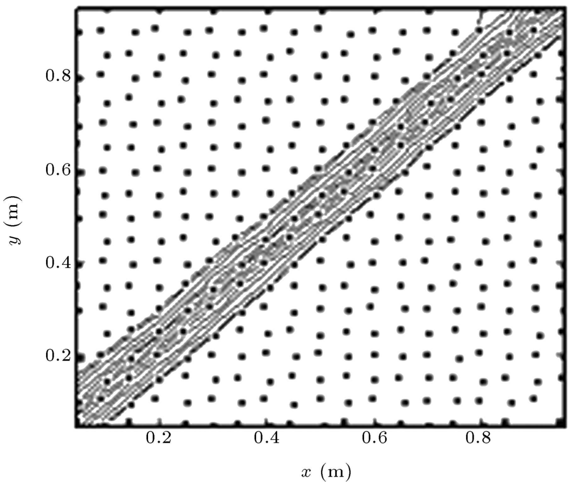


**Figure 8.** Solution of pure convection problem on a mesh 0.011 irregularity.

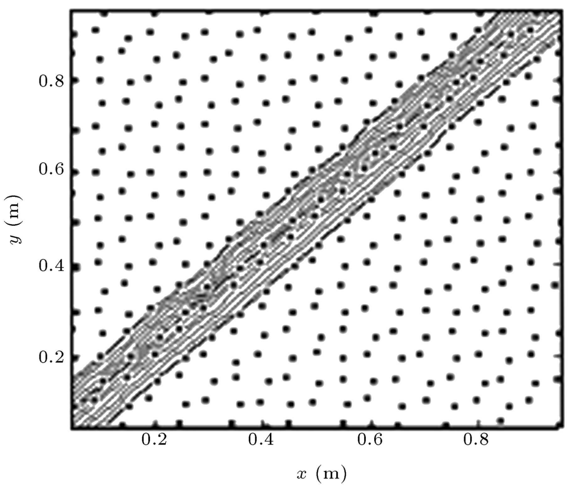


**Figure 9.** Solution of pure convection problem on a mesh 0.0195 irregularity.

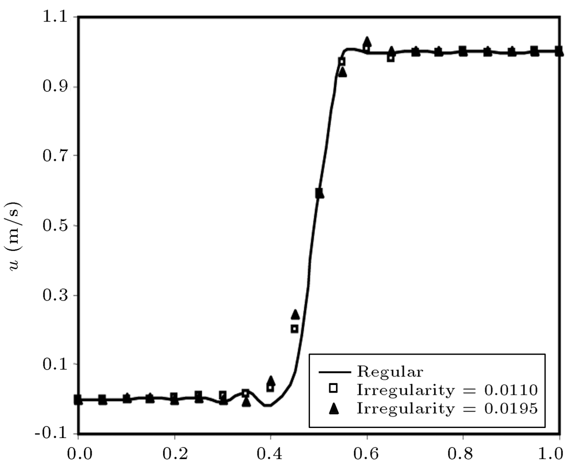


**Figure 10.** Comparison of solution of pure convection problem on different meshes.

## 2-D Viscous Burgers Equation

2-D viscous Burgers equation can be represented by the following parameters of Equation 13.

$$\mathbf{A_1}(\mathbf{u}) = \mathbf{A_2}(\mathbf{u}) = \mathbf{u}, \qquad \mathbf{Q}(\mathbf{u}) = \mathbf{0}.$$

The exact solution of the problem is defined as follows [37]:

$$\mathbf{u}(x, y, t) = \frac{1}{1 + e^{(x+y-t-0.25)/(2k)}}. \tag{31}$$

From which the initial and boundary conditions can be defined.

The computational parameters of $\theta = 0.5$, $\Delta t = 0.01$ and $k = 0.03$ were used on three meshes of irregularity indexes 0.0, 0.02 and 0.057, respectively, to get the solution at time $t = 0.5$ seconds. Figures 11 to 13 shows the contours of the solutions obtained on three

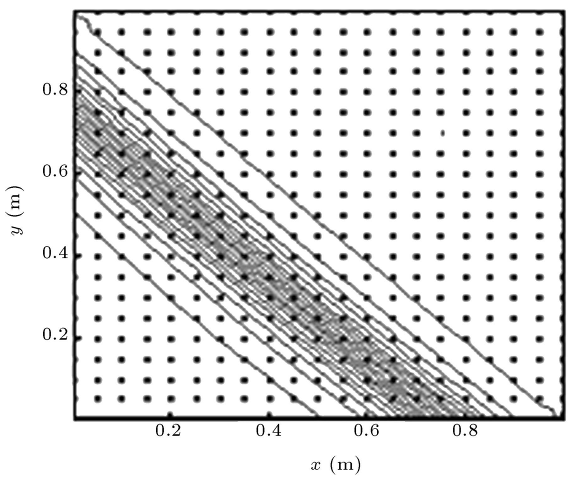


**Figure 11.** Solution of 2-D Burgers problem at $t = 0.5$ s on a regular mesh.

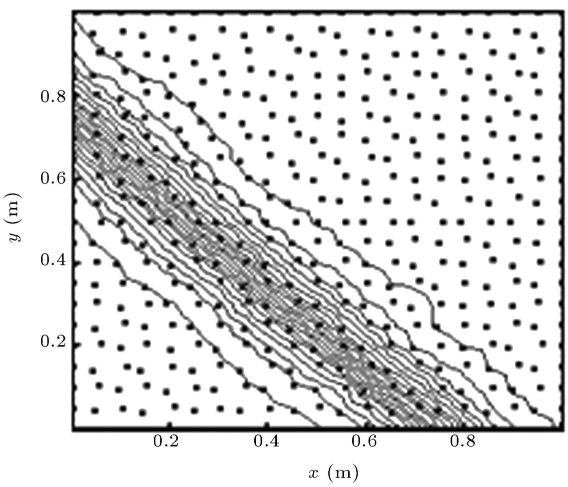


**Figure 12.** Solution of 2-D Burgers problem at $t = 0.5$ s on a mesh of 0.02 irregularity.

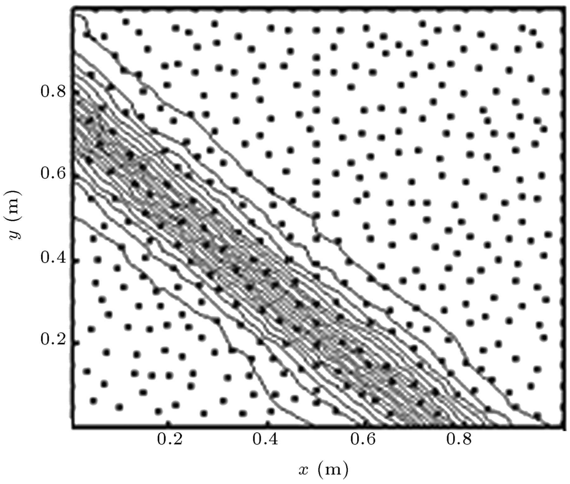


**Figure 13.** Solution of 2-D Burgers problem at $t = 0.5$ s on a mesh of 0.057 irregularity.

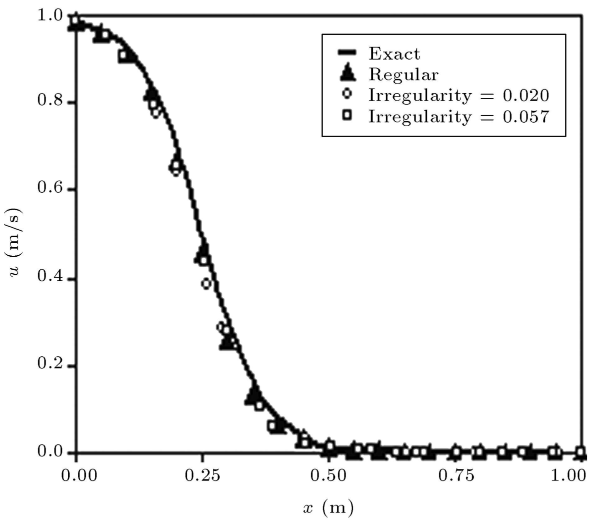


**Figure 14.** Comparison of exact and numerical solutions obtained on different meshes at $t = 0.5$ s.

meshes of 441 nodal points using 841 collocation points. Once again 441 of the collocation points coincided with the nodal points while the 400 remaining points was considered to be uniformly distributed on the domain. A comparison of the solutions obtained along $y = 0.5$ with that of exact solution is shown in Figure 14, indicating the ability of the method to produce nearly exact solution irrespective of the irregularity of the meshes used.

## CONCLUDING REMARKS

A fully least squares approach named Collocated Discrete Least Squares Mesh-less (CDLSM) method was used in this paper for the solution of convection-dominated problems on irregular meshes. In this method a fully least squares approach is used in both function approximation and the discretization of the governing differential equations. The meshless shape functions were derived using the Moving Least Squares (MLS) method of function approximation. The problem domain was discretized by nodal points which are used to construct the trial function. The least square functional is constructed using collocation points that are basically independent of the nodal points. A study is performed on the effect of irregularity of domain discretization on the performance of the proposed Collocated Discreet Least Square Mesh-less (CDLSM) method. Four benchmark examples of hyperbolic nature, namely steady nonlinear 1-D Burgers equation, 1-D dam break problem, 2-D pure convection problem and 2-D viscous Burgers equation were solved on three meshes of different irregularity, and the results were presented and compared with the exact solutions where available. The results clearly indicated that the proposed CDLSM method is able to produce highly accurate results for hyperbolic problems even on highly irregular meshes of nodes.

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