Transaction A: Civil Engineering
Vol. 16, No. 3, pp. 249–255

A Plate on Winkler Foundation with Variable Coefficient

M. Mofid[1,*] and M. Noroozi[2]

Abstract. *Plates on elastic foundations have attracted the attention of many researchers. Some elementary models have been introduced to consider interactions between the plate and its foundation. Other improved models have been proposed to develop basic models. In this work, a model based on the Winkler-foundation theory is proposed, while the constant parameter of Winkler is assumed to be variable; such as non-uniform springs with the functionality of the domain position, along with the plate and beam span in order to consider the non-uniform behavior of the foundation. The governing equation on the system is solved by using the Galerkin method and effects such as the presence of rigid points in the foundation are considered.*

Keywords: *Plate theory; Kirchhoff plate; Winkler elastic foundation; Galerkin's method.*

INTRODUCTION

Studies on plate structures and the consideration of effects such as elastic foundations on their behavior is among one of the most important research fields in applied mechanics, having attracted the attention of many researchers. Considering the effect of the foundation on the deflection, stability, response to static and dynamic loading, vibration and so on, plus presenting different models to describe the behavior of the plate, are some aspects of the numerous attempts in this area [1-4]. Various analytical and numerical methods have been employed to find effective factors regarding foundation-plate behavior for practical applications in civil/structural, mechanical, aerospace and marine engineering.

In the past, the model of a thin plate on an elastic foundation was mainly used in structural applications. Currently, thin films of metal, ceramic or synthetic materials deposited on the surface of the structural parts of electronic devices are used to improve their mechanical, thermal, electrical and tribological properties. These thin films of material are considered as thin plates and, in these applications, the substrate of thin film can be simulated as an elastic foundation [5,6].

The static response and dynamic behavior of the plates are considered commonly by researchers. Luura and Gutierrez [7] studied the vibration of rectangular plates by a non-homogenous elastic foundation using the Rayleigh-Ritz method. Biswas [8] considered the vibration of irregular shaped orthotropic plates resting on an elastic foundation subjected to in-plane forces. The vibration of rectangular plates resting on a non-uniform elastic Winkler foundation is considered by Lee and Lin [9], where the Levy solution method and the Green's function were employed in their study. Malekzadeh and Farid [10] considered a composite plate on a two parameter non-linear foundation using the differential quadrature method. Dutta and Roy [11] studied the interaction between the soil foundation and the structure in their article review.

In this work, an elastic thin plate on a modified Winkler foundation is considered. The Kirchhoff theory is assumed for the plate and the Winkler coefficient is assumed to have variations versus position with the functionality of the domain, along with the plate span. This type of foundation is also used to analyze a strip similar to an Euler-Bernoulli beam. The deflection of the beam and plate is considered using the static analysis by the Galerkin method and the effect of the non-uniform foundation has been considered.

1. *Department of Civil Engineering, Sharif University of Technology, Tehran, Iran.*
2. *Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran.*

Received 23 September 2007; received in revised form 30 August 2008; accepted 17 March 2009

REVIEW OF THE MODEL

In this section, a brief review regarding the models of the plate and elastic foundation is presented.

Plate

Different theories are used for the modeling of plates, wherein each theory tries to convert the 3-D problem of the elasticity of the plate to a lower degree. Reduction to a lower degree will reduce the number of attempts needed to obtain an approximated solution for the system. Two main types of plate modeling theory, for either thin or thick plates, can be classified as small and large deformations. A Kirchhoff plate is the simplest theory that represents a governing equation for a thin plate. However, other improved models such as the large deformation theory of the Von-Karman or Mindlin shear deformation plate, have been presented.

Elastic Foundation

The effect of a foundation can be modeled by various approaches to the plate [12]. The best realistic model is to represent the foundation as a continuum model, where the elasticity solution represents the behavior of the foundation. On the other hand, the elastic foundation can be modeled as a set of springs. The spring system can be multilayered where each layer has its special stiffness. Springs can be linear or nonlinear and the foundation can be divided into multi segments [13]. For a viscoelastic foundation, a system of springs and dampers can be used to consider damping effects [14-16].

The simplest model presented for the elastic foundation is the Winkler model, which assumes that the shear resistance of the foundation is ignorable compared to the shear capacity of the foundation, and models the foundation as a set of independent springs. Therefore, there is no lateral interaction between the springs. The governing equation for the system is;

$$D\nabla^4 w = q - N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} - kw. \qquad (1a)$$

In which $\nabla^4 = \nabla^2.\nabla^2$ and ∇^2 is the Laplacian operator as (in the Cartesian coordinate):

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad (1b)$$

Also, q is the lateral external force, N is the in-plane force (in x and y directions), k is the Winkler coefficient of the foundation, w is the deflection and D is the flexural rigidity of the plate.

To account for more realistic interaction between the springs, various models of the two-parameter elastic foundation have been presented [17]. The Filonenko-Borodich model [18], Pasternak [19] and Vlasor [20] can be mentioned as examples. The Pasternak foundation assumes a degree of shear interaction between adjacent springs by using the layers of tension and shear springs.

In this work, a Kirchhoff plate on an improved elastic foundation will be considered. In contrast with works in which the Winkler coefficient is assumed to be constant, in this work this parameter is assumed to be a function of the location. Therefore, $k = k(x, y)$ in Equation 1 is not constant and varies under the plate. This assumption allows for the consideration of interesting properties for the foundation. For example, to make a rigid location under the plate, it is sufficient to set k very large for the desirable location. Therefore, the deflection of the plate in this location gets very small because of the resistance of the foundation, and the problem changes to a plate on the locally rigid foundation. This approach can be used to solve many complicated problems for a plate on a multi rigid-point foundation.

On the other hand, if the deformation of the plate is assumed to be cylindrical and the variation in one direction is neglected, the plate changes to a strip and the governing equation reduces from PDE to ODE. In this manner, the strip acts as a beam with a similar equation. The combination of an elastic foundation with a variable coefficient with the beam theory leads to the solution of problems such as multi-segment beams.

SOLUTION FOR A STRIP

For a strip, derivation with respect to one variable vanishes in Equation 1 and the solution of the ODE can be represented directly. In the absence of the in-plane forces, N_x and N_y, and by the assumption of the linear-step variations of the Winkler coefficient (Figure 1):

$$\frac{d^4 w_{(x)}}{dx^4} + \frac{k_{(x)}}{D} w_{(x)} = \frac{q_{(x)}}{D}, \qquad (2)$$

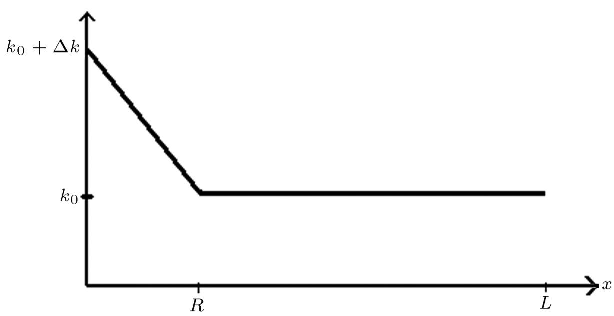

Figure 1. Variation of the Winkler coefficient of the foundation vs. position.

$$k = k_0 + \Delta k \left(1 - \frac{x}{R}\right)[1 - u_{(x-R)}], \tag{3}$$

in which k_0 is the fixed-average of the Winkler coefficient at points far away from $x = 0$. Also, Δk is the linear variation of the Winkler coefficient and $u_{(x)}$ is the step function. By using the Galerkin method, the deflection, w, could be represented as:

$$w = c_0 + \sum_{i=1}^{N} c_i \phi_{i(x)}. \tag{4}$$

And the error, due to substitution of this proposed answer in Equation 1, is:

$$\text{Error } = \frac{k}{D} c_0 - \frac{q_{(x)}}{D} + \sum_{i=1}^{N} c_i \left[\frac{d^4\phi_{i(x)}}{dx^4} + \frac{k}{D}\phi_{i(x)}\right]. \tag{5}$$

While unknown c_i's will be obtained by applying Equation 5 in Equation 6:

$$\int_0^L \text{Error } \phi_{j(x)} dx = 0, \qquad j = 1, \cdots, N, \tag{6}$$

$$\int_0^L \frac{k}{D} c_0 \phi_{j(x)} dx - \int_0^L \frac{q_{(x)}}{D} \phi_{j(x)} dx + \sum_{i=1}^{N} c_i M_{ij} = 0,$$

$$j = 1, \cdots, N, \tag{7}$$

where:

$$M_{ij} = \int_0^L \left[\frac{d^4\phi_{i(x)}}{dx^4} + \frac{k}{D}\phi_{i(x)}\right] \phi_{j(x)} dx, \tag{8}$$

in which ϕ_i and ϕ_j are perpendicular functions that satisfy the B.C.'s. If ϕ_i's are chosen so that:

$$\int_0^L \phi_{i(x)}\phi_{j(x)} dx = \|\phi_i\|^2 \delta_{ij},$$

$$\frac{d^4\phi_{i(x)}}{dx^4} = \lambda_i^4 \phi_{i(x)}, \qquad \frac{d^2\phi_{i(x)}}{dx^2} = -\lambda_i^2 \phi_{i(x)}, \tag{9}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

Then, Equation 8 can be simplified by introducing the following new variables, $\Phi^{\mathrm{I}}_{ij(x)}$ and $\Phi^{\mathrm{II}}_{ij(x)}$:

$$\Phi^{\mathrm{I}}_{ij(x)} = \int_0^t \phi_{i(t)}\phi_{j(t)} dt,$$

$$\Phi^{\mathrm{II}}_{ij(x)} = \int_0^x \Phi^{\mathrm{I}}_{ij(t)} dt. \tag{11}$$

Using Equations 3, 7, 8, 9 and 11 and integrating by parts:

$$M_{ij} = \int_0^L \left[\frac{d^4\phi_{i(x)}}{dx^4} + \frac{k}{D}\phi_{i(x)}\right] \phi_{j(x)} dx, \tag{12a}$$

$$M_{ij} = \left[\lambda_i^4 + \frac{k_0}{D}\right] \|\phi_i\|^2 \delta_{ij}$$

$$+ \frac{\Delta k}{D} \int_0^R (1 - \frac{x}{R}) \phi_{i(x)} \phi_{j(X)} dx, \tag{12b}$$

$$M_{ij} = \left[\lambda_i^4 + \frac{k_0}{D}\right] \|\phi_i\|^2 \delta_{ij} + \frac{\Delta k}{RD} \Phi^{\mathrm{II}}_{ij(R)}. \tag{12c}$$

Introducing $\phi_{i(x)}$ leads to the calculation of M_{ij} and unknown c_i's from Equation 7, and the deflection of the strip at any location can be obtained by using Equation 4.

EXAMPLE 1: STRIP UNDER UNIFORM LATERAL LOADING ON THE SIMPLY SUPPORTED EDGES

For a strip under uniform loading with simply support B.C.'s, symmetrical assumption can be used to simplify the solution procedure (Figure 2). The plate rests on a Winkler foundation, but at the middle, there is a line of rigidity under the plate (or a rigid point under the beam). It can be shown easily that Equation 13 satisfies B.C.'s evidently:

$$\phi_i = \cos(\lambda_i x), \qquad \lambda_i = \frac{2i - 1}{2} \frac{\pi}{L}. \tag{13}$$

On the other hand, we can conclude from B.C.'s at $x = L$ that:

$$c_0 = 0. \tag{14}$$

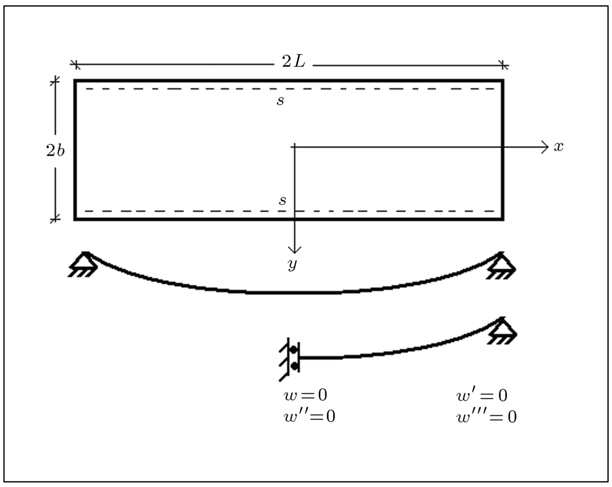

Figure 2. Strip and its B.C.'s.

Therefore, from Equation 13:

$$\|\phi\|^2 = \int_{\circ}^{L} \phi_{i(x)}^2 dx = \frac{L}{2}, \tag{15}$$

and:

$$\Phi_{ij(x)}^I = \frac{1}{2}\left[\frac{\sin(\lambda_i+\lambda_j)x}{(\lambda_i+\lambda_j)^2} + \frac{\sin(\lambda_i-\lambda_j)x}{(\lambda_i-\lambda_j)^2}\right],$$

if $i \neq j$,

$$\Phi_{ij(x)}^I = \frac{1}{2}\left[\frac{\sin(2\lambda_i x)}{2\lambda_i} + x\right],$$

if $i = j$, (16)

$$M_{ii} = \left[\lambda_i^4 + \frac{k_0}{D}\right]\frac{L}{2} + \frac{\Delta k}{4D}R\left[\frac{1-\cos(2\lambda_i R)}{2(\lambda_i R)^2} + 1\right],$$

$$M_{ij} = \frac{\Delta k}{2D}R\left[\frac{1-\cos(\lambda_i+\lambda_j)R}{(\lambda_i+\lambda_j)^2R^2} + \frac{1-\cos(\lambda_i-\lambda_j)R}{(\lambda_i-\lambda_j)^2R^2}\right],$$

$i \neq j$. (17)

On the other hand, for uniform distribution of loading $q_{(x)} = q_0$:

$$q_i = \int_0^L \frac{q_{(x)}}{D}\phi_{i(x)}dx = \frac{q_0 L}{D}\frac{\sin\lambda_i R}{\lambda_i R}, \quad i = 1,\cdots,N. \tag{18}$$

Solution $[M]\{c_i\} = \{q_i\}$ leads to calculating $\{c_i\}$ $i = 1,\cdots,N$. Deflection at any point of the strip can be obtained by using $w = \sum c_i\phi_{i(x)}$.

SOLUTION FOR A GENERAL PLATE

The solution for a strip in the previous section can be generalized to obtain a solution for a general plate. The governing equation for a general plate resting on a foundation is as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k}{D}w = \frac{q}{D}, \tag{19}$$

where $w = w_{(x,y)}$ is the deflection of the plate with dimensions a and b, $q_{(x,y)}$ is the distribution of lateral force, and $k = k_{(x,y)}$ is the coefficient of the elastic foundation assumed to vary linearly such as the step function as in Figure 3.

$$k_{(x,y)} = k_0 + \Delta k\left(1-\frac{x}{r_a}\right)\left(1-\frac{y}{r_b}\right)$$

$$[1-u_{(x-r_a)}][1-u_{(y-r_b)}]. \tag{20}$$

Obviously, increasing the value of Δk in Equation 20 leads to an increase in the rigidity of the foundation on domains r_a and r_b, which are the effective ranges of the variation of k in x and y directions, respectively. If these values are chosen very small, the foundation acts like a rigid point under the plate and k_0 is the average constant value of k far from the center. Because of the symmetrical conditions on the model, one fourth of the plate has been considered. In a general case, without symmetrical conditions, the model can be entirely considered with a similar method.

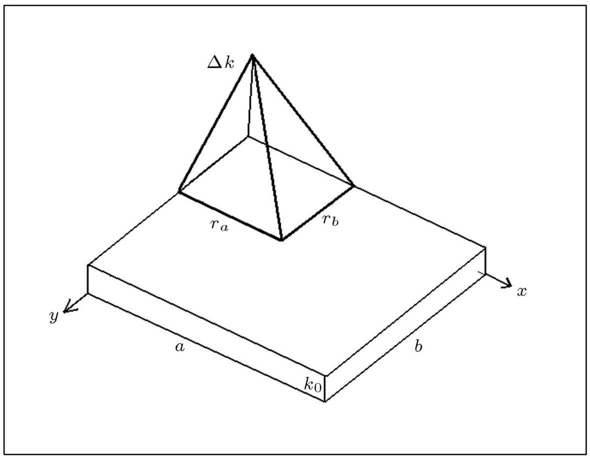

Figure 3. Variation of the coefficient of the foundation vs. position.

If the deflection of the plate is assumed to be as follows:

$$w = c_0 + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\phi_{ij(x,y)}$$

$$= c_0 + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\phi_{i(x)}\phi_{j(y)}, \tag{21}$$

then using the Galerkin method with weight functions $\psi_{m(x)}$ and $\psi_{n(x)}$, the procedure can be followed as:

$$\text{Error} = \frac{k}{D}c_0 - \frac{q}{D}$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\left[\nabla^4\phi_{ij(x,y)} + \frac{k}{D}\phi_{ij(x,y)}\right], \tag{22}$$

$$\iint_A \text{Error } \psi_{m(x)}\psi_{n(y)}dA = 0, \tag{23}$$

$$\iint_A \frac{k}{D}c_0\psi_{m(x)}\psi_{n(y)}dA - \iint_A \frac{q_{(x,y)}}{D}\psi_{m(x)}\psi_{n(y)}dA$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}M_{ijmn} = 0, \tag{24}$$

in which M_{ijmn} is represented as:

$$M_{ijmn}=\iint\limits_A\left[\nabla^4\phi_{ij(x,y)}+\frac{k}{D}\phi_{ij(x,y)}\right]\psi_{m(x)}\psi_{n(y)}dxdy. \tag{25}$$

Also, ϕ and ψ are perpendicular functions that satisfy B.C.'s.

$$\int_0^a \phi_{i(x)}\psi_{m(x)}dx = \|\phi_i\|^2\,\delta_{im},$$

$$\int_0^a \phi_{j(y)}\psi_{n(y)}dx = \|\phi_j\|^2\,\delta_{jn}. \tag{26}$$

On the other hand, these functions are chosen so that they satisfy the following conditions:

$$\frac{d^4\phi_{i(x)}}{dx^4}=\lambda_i^4\phi_{i(x)}, \qquad \frac{d^2\phi_{i(x)}}{dx^2}=-\lambda_i^2\phi_{i(x)}. \tag{27}$$

Equations 27 also hold for ψ. By using Equations 20, 21, 26 and 27 and substituting them in Equation 25, it can be shown that:

$$M_{ijmn}=\iint\limits_A\left[(\lambda_i^2+\lambda_j^2)^2+\frac{k}{D}\right]$$

$$\phi_{i(x)}\phi_{j(y)}\psi_{m(x)}\psi_{n(y)}dxdy, \tag{28}$$

which is equal to:

$$M_{ijmn}=\left[(\lambda_i^2+\lambda_j^2)^2+\frac{k_0}{D}\right]\|\phi_i\|^2\|\phi_j\|^2\,\delta_{im}\delta_{jn}$$

$$+\cdots+\frac{\Delta k}{D}\int_0^{r_a}\int_0^{r_b}\left(1-\frac{x}{r_a}\right)\left(1-\frac{y}{r_b}\right)$$

$$\times\,\phi_{i(x)}\phi_{j(y)}\psi_{m(x)}\psi_{n(y)}dxdy. \tag{29}$$

The integral in Equation 29 can be split as two parts from x and y separately. By introducing new variables $\Phi^{\mathrm{I}}_{ij(x)}$ and $\Phi^{\mathrm{II}}_{ij(x)}$ so that:

$$\Phi^{\mathrm{I}}_{im(x)}=\int_0^x \phi_{i(t)}\psi_{m(t)}dt,$$

$$\Phi^{\mathrm{I}}_{jn(y)}=\int_0^y \phi_{j(t)}\psi_{n(t)}dt,$$

$$\Phi^{\mathrm{II}}_{im(x)}=\int_0^x \Phi^{\mathrm{I}}_{im(t)}dt,$$

$$\Phi^{\mathrm{II}}_{jn(y)}=\int_0^y \Phi^{\mathrm{I}}_{jn(t)}dt, \tag{30}$$

and integrating them by parts in Equation 28, it can be concluded that:

$$M_{ijmn}=\left[(\lambda_i^2+\lambda_j^2)^2+\frac{k_0}{D}\right]\|\phi_i\|^2\|\phi_j\|^2\,\delta_{im}\delta_{jn}$$

$$+\frac{\Delta k}{D}\Phi^{\mathrm{II}}_{im(r_a)}\Phi^{\mathrm{II}}_{jn(r_b)}. \tag{31}$$

EXAMPLE 2: RECTANGULAR PLATE UNDER UNIFORM LATERAL LOADING WITH SIMPLY SUPPORTED EDGES

By considering one fourth of the plate under uniform loading and symmetrical B.C.'s represented in Relations 32 and Figure 4, and by choosing ϕ and ψ functions, as given by Relations 33, the corresponding values of Φ^{II} and M_{ijmn} can be calculated.

The B.C.'s of the plate due to symmetrical conditions are presented as:

$$\frac{\partial w}{\partial x}\Big|_{x=0}=\frac{\partial w}{\partial y}\Big|_{y=0}=0\,,$$

$$\frac{\partial^3 w}{\partial x^3}\Big|_{x=0}=\frac{\partial^3 w}{\partial y^3}\Big|_{y=0}=0\,,$$

$$w(a,y)=w(x,b)=0,$$

$$\frac{\partial^2 w}{\partial x^2}\Big|_{x=a}=\frac{\partial^2 w}{\partial y^2}\Big|_{y=b}=0\,, \tag{32}$$

$$\phi_{i(x)}=\cos(\lambda_i x), \qquad \phi_{j(y)}=\cos(\lambda_j y),$$

$$\psi_{m(x)}=\cos(\lambda_m x), \qquad \psi_{n(y)}=\cos(\lambda_n y),$$

$$\lambda_p=\frac{2p-1}{2}\,\frac{\pi}{(a \text{ or } b)}. \tag{33}$$

Therefore:

$$\|\phi_i\|^2=\frac{a}{2}, \qquad \|\phi_j\|^2=\frac{b}{2}. \tag{34}$$

And by using Equation 29:

$$\Phi^{\mathrm{II}}_{im(r_a)}=\frac{r_a^2}{2}\left[\frac{1-\cos(\lambda_i+\lambda_m)r_a}{(\lambda_i+\lambda_m)^2r_a^2}\right.$$

$$\left.+\frac{1-\cos(\lambda_i-\lambda_m)r_a}{(\lambda_i-\lambda_m)^2r_a^2}\right], \qquad \text{if } i\neq m,$$

$$\Phi^{\mathrm{II}}_{im(r_a)}=\frac{r_a^2}{2}\left[\frac{1-\cos(2\lambda_i r_a)}{2(\lambda_i r_a)^2}+1\right], \qquad \text{if } i=m. \tag{35}$$

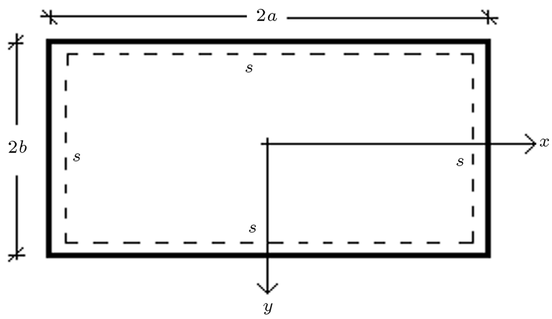

Figure 4. Plate and its B.C.'s.

Relations 35 hold similarly for $\Phi^{\text{II}}_{jn(x)}$. Therefore, M_{ijmn} can be obtained from Equation 31. Also, for the uniform distribution of lateral force $q_{(x,y)} = q_0$,

$$q_{mn} = \iint_A \frac{q_{(x,y)}}{D}\psi_{m(x)}\psi_{n(y)},$$

$$dA = \frac{abq_0}{D}\frac{\sin\lambda_m r_a}{\lambda_m r_a}\frac{\sin\lambda_n r_b}{\lambda_n r_b}. \tag{36}$$

Solving $[M_{ijmn}]\{c_{ij}\} = \{q_{mn}\}$ leads to find unknown constants $\{c_{ij}\}$ and a deflection from Equation 21.

RESULTS AND DISCUSSION

The equation was solved using the described Galerkin method for a numerical case and the results were compared with answers from MATLAB 7.1 software, while the answers agree with each other. Figure 5 shows a good approximation of the Galerkin solution in comparison with the MATLAB 7.1 numerical solution, for Example 1; while only 5 terms are used in the series for driving the Galerkin solution. The deflection of the strip (or equivalent beam) is depicted versus the position for the small rigidity, Δk, of the foundation at a small range, $\frac{R}{L} = \frac{1}{1000}$. The conditions at the center do not affect the response due to small values of range R and stiffness Δk. Furthermore, when the range, R, of the foundation stiffness increases (Figure 6), the deflection of the plate decreases as expected for Example 1. Also, Figure 7 illustrates the effect of the rigidity at a point. When the stiffness greatly increases at a point, that particular point will act as a rigid-point foundation and the deflection approaches zero at that point (as expected).

Similar results are also found for the general plate in Example 2. Figure 8 illustrates the plate behavior and has been non-scaled for clarity. The deflection is zero at the supported edges, $x = a$ and $y = b$, and has a small value at the rigid point, $x = y = 0$. Increasing the rigidity at this point leads to a lessening of the prescribed deflection toward zero.

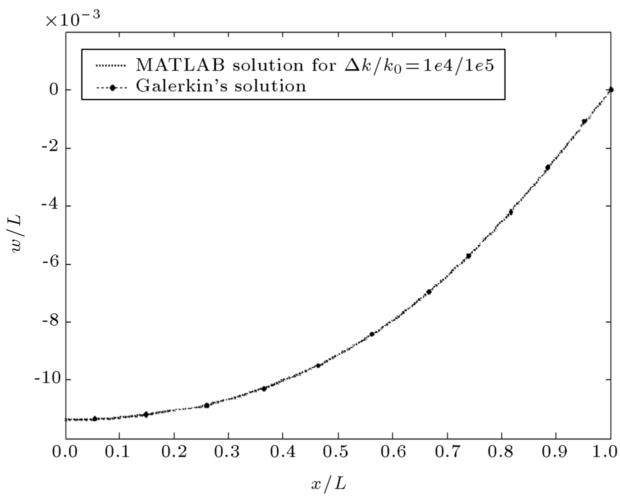

Figure 5. Comparison between Galerkin's solution and MATLAB answer of the strip in Example 1 for $R/L = 1/1000$.

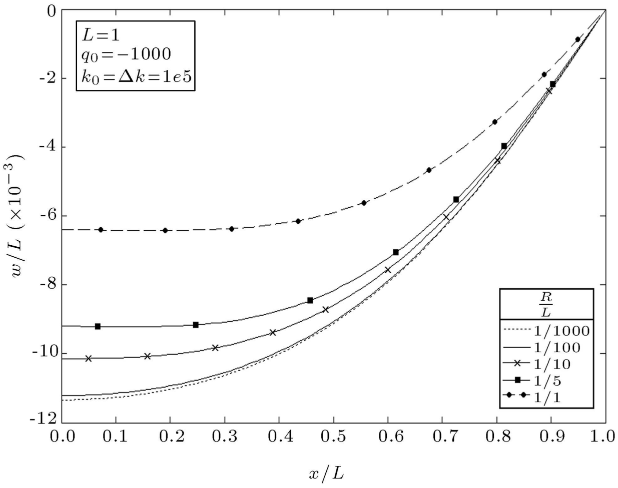

Figure 6. Effects of the various ranges on the deflection of the strip in Example 1.

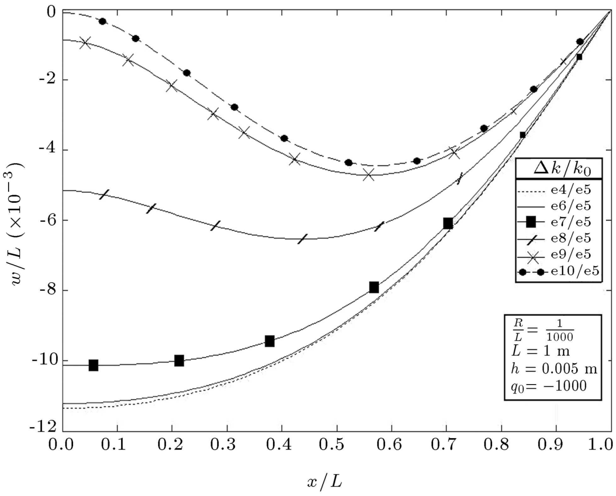

Figure 7. Effects of the various coefficients on the deflection for the strip in Example 1.

CONCLUSION

An improved version of the Winkler elastic foundation theory is proposed with the variable coefficients. Based on this proposed model, the foundation is assumed to be non-uniform (as in the Winkler foundation theory with a variable coefficient) as a function of the position. This model enables the foundation to act more interestingly by having rigid points in its domain or as multi-segmented with multiple stiffness constants to consider the non-uniform behavior of the foundation. Various functionalities of foundation stiffness versus

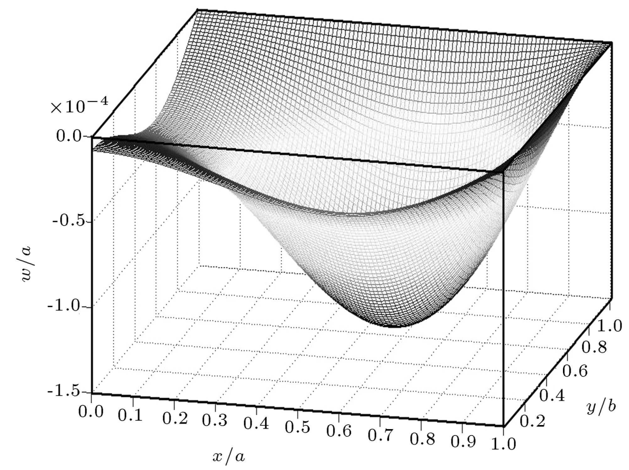

Figure 8. Deflection of a plate in Example 2 under uniform loading with a rigid-point foundation at the center for $k_0 = 10e6$, $\Delta k = 1e14$ and $r_a = r_b = a/1000$.

domain seem to be a very interesting aspect within the field of foundation-plate consideration.

NOMENCLATURE

$E = 70$	(GPa) Young's modulus
$\nu = 0.3$	Poisson ratio
$h = 5e-3$	(m) Plate thickness
$L = 1$	(m) half length of the strip
$a = 1$	(m) half length of the plate in x span
$b = 1$	(m) half length of the plate in y span
$r_a = a/1000$	effective range of the variation of k in x span
$r_b = b/1000$	effective range of the variation of k in y span
k_0	(N/m^3) average Winkler coefficient at infinity
Δk	(N/m^3) variations of Winkler coefficient
$q = 1000$	(N/m^2) distributed lateral load

REFERENCES

1. Fryba, L. "History of winkler foundation", *Vehicle Sys. Dyn.*, **24**(S1), pp. 7-12 (1995).
2. Timoshenko, S. and Winowsky-Krieger, S., *Theory of Plates and Shells*, 2nd Ed., McGraw-Hill, New York (1959).
3. Voyiadjis, G.Z. and Kattan, P.I. "Thick rectangular plates on an elastic foundation", *Trans ASCE, J. Eng. Mech.*, **112**, pp. 1218-1240 (1986).
4. Kobayashi, H. and Sonoda, K. "Rectangular Mindlin plates on elastic foundations", *Int. J. Mech. Sci.*, **31**, pp. 679-92 (1989).
5. Utku, M. and Citipitioglu, E. and Inceleme, I. "Circular plate on elastic foundation modeled with annular plates", *Comput. Struct.*, **78**, pp. 365-374 (2000).
6. Chadrashekhara, K. and Antony, J. "Elastic analysis of an annular slab-soil interaction problem using the hybrid method", *Compt. Geotech.*, **20**, pp. 161-76 (1997).
7. Laura, P. and Gutierrez, R. "Vibrating non uniform plates on elastic foundation", *J. Eng. Mech. ASCE.*, **111**, pp. 1185-96 (1985).
8. Biswas, P. "Vibration of irregular-shaped orthotropic plates resting on elastic foundation under in-plane forces", *J. Indian. Inst. Sci.*, **60**, pp. 73-8 (1978).
9. Lee, S. and Lin, S. "Levy-type solution for the analysis of nonuniform plates", *Comput. Struct.*, **49**, pp. 931-9 (1993).
10. Malekzadeh, P. and Farid, M. "A DQ large deformation analysis of composite plates on nonlinear elastic foundation", *Compos. Struct.*, **79**(2), pp. 251-60 (2007).
11. Dutta, S.C. and Roy, R. "A critical review on idealization and modeling for interaction among soil-foundation structure system", *Comput. Struct.*, **80**, pp. 1579-94 (2002).
12. Khazanovich, L. "Finite element analysis of curling of slabs on Pasternak foundation", *16th ASCE Eng. Mech. Conference, Univ. of Washington*, Seattle (2003).
13. Xiang, Y. "Vibration of rectangular Mindlin plates resting on non homogenous elastic foundations", *Int. J. Mech. Sci.*, **45**, pp. 1229-1244 (2003).
14. Kerr, A.B. "Elastic and viscoelastic foundation models", *J. Appl. Mech.*, **31**, pp. 491-498 (1964).
15. Sun, L. "A closed form solution of Bernoulli-Euler beam on a viscoelastic foundation under harmonic line loads", *J. Sound Vib.*, **242**(4), pp. 619-30 (2001).
16. Chen, Y.H., Huang, Y.H. and Shih, C.T. "Response of infinite Timoshenko beam on a viscoelastic foundation to a harmonic moving load", *J. Sound Vib.*, **241**(5), pp. 809-824 (2001).
17. Lam, K.Y., Wang, C.M. and He, X.Q. "Canonical exact solutions for Levy-plates on two-parameter foundation using Green's functions", *Eng. Struc.*, **22**, pp. 364-378 (2000).
18. Filonenko-Borodich, M.M. "Some approximate theories of the elastic foundation", *Uchenyie Zapiski Moskogo Gosudarstvennogo Universiteta*, Mekhanica, **46**, pp. 3-18 (1940).
19. Pasternak, P.L. "On a new method of analysis of an elastic foundation by means of two foundation constants", *Gosudastrennoe Izdatelstvo Literaturi po Stroitelstvu i Arkhitekture*, Moscow, USSR (1954).
20. Vlasov, V.Z., *Structural Mechanics of Thin Walled Three Dimensional System*, Stroizdat, Moscow (1949).