Command and Control for Congestion Pricing of General Multimodal Transportation Networks

A. Nagurney[1] and P. Ramanujam[1]

Congestion on transportation networks is becoming a problem of increasing importance due not only to losses in productivity, but also to increased pollution. In this paper, a model is developed that allows the government policy maker, or "command", to impose "controls" on the links of the network in the form of desired levels of link loads for each mode of transportation. Deviations over and above the imposed controls are then subject to taxes that are functions of the size of the deviations. The behavior of the users of the transportation network remains that of user-optimization. The governing equilibrium conditions in the presence of such policies are shown to satisfy a variational inequality problem. A decomposition algorithm is then proposed to resolve the problem into series of traffic network equilibrium problems and a simpler subproblem. Convergence results are also given.

Finally, the algorithm is applied to several numerical examples. This work may be viewed as a contribution to transportation policy modelling.

INTRODUCTION

Congestion on transportation networks is a problem of growing concern in both developed and developing countries, leading to losses in productivity as well as environmental damage due to increased pollution. Social concerns related to such issues are motivating the development of rigorous theoretical frameworks for transportation policy modelling that can capture the effects of alternative regulations.

In this paper a "command and control" mechanism for alleviating congestion on transportation networks through the economics of congestion pricing is proposed. In particular, a model is developed that allows the governmental decision and policy maker, or "command", to impose controls on each of the links of the network, which may be distinct for each mode of transportation. Associated with the links are penalties for failure to satisfy the controls. The penalties are functions of the deviations above the imposed controls. Users of those links that have transportation loads below the controls are not subject to any penalties.

The basis for the model developed in the subsequent section is the well-known traffic network equilibrium model with fixed demands, introduced by Dafermos [1], which assumes that users of the transportation network behave in

1. Department of Finance and Operations Management, School of Management, University of Massachusetts, Amherst, MA, USA 01003.

a user-optimized fashion, that is, that travelers select the cost-minimizing route between their origin and destination pair. Here, however, the more general case where there are multiple modes of transportation is considered. Each user cost for a mode on a link in the network can, in general, depend on the entire link load pattern. In equilibrium, only those paths that have minimal costs are actually used (see also [2]). Overimposed on this model is a set of controls on the links that are idealized controls for the link loads for each mode of transportation, along with penalty functions that penalize travelers for congestion over and above the mandated controls. Congestion pricing is accomplished by setting taxes on the links equal to the penalties for deviation for each mode of transportation.

The equilibrium conditions in this new policy model reflect that users of each mode select routes of travel connecting each origin/destination pair so that the user cost of travel plus the associated taxes on the links comprising that path are minimal. The equilibrium conditions are then shown to satisfy a variational inequality problem.

This approach to congestion pricing is distinct from the form of tolls proposed by Dafermos [3] (see also [4]), wherein the tolls on the links were set so that the system-optimizing pattern was also user-optimizing. Here, the command has more flexibility in identifying the controls on the links and across modes and in determining the associated penalty functions.

A discussion regarding how to obtain certain qualitative properties of the equilibrium pattern consisting of the link loads, deviations, as well as the link tax rates for all modes, and a suggestion about an algorithm for the computation of the equilibrium pattern, with accompanying convergence results is given. The notable feature of the algorithm is that it resolves what is expected to be a large-scale problem into a series of traffic network equilibrium problems, for which numerous effective algorithms exist like projection and relaxation methods (cf. [5,6]), along with a simple subproblem. The algorithm handles the complicating constraints, those of the link controls for each mode, in a novel way, to make possible such a decomposition that enables one to exploit the network structure.

Finally, the algorithm for several numerical examples is applied.

THE TRANSPORTATION POLICY MODEL

In this section the transportation policy model on a general network is introduced. The policy model is one of command and control in that the government or policy decision maker, the "command", imposes the "controls" on the links of the network in the form of link load levels that represent the maximum desired level for each mode of transportation. Deviations from these transportation controls in terms of overflow are then penalized according to the imposed penalty functions, which are assumed to be functions of the deviations.

The notation is first introduced and subsequently, the equilibrium conditions are stated and the variational inequality formulation is derived.

A transportation network with a, b, c, etc., denoting the links and with p, q, etc., denoting the paths is considered. It is assumed that there are W origin/destination (O/D) pairs in the network, with a typical O/D pair denoted by w; and that there are J modes of transportation, with a typical mode denoted by i.

The flow on link a by mode i is denoted by f_a^i, and the user cost associated with traveling on link a by mode i is denoted by c_a^i. The link loads are grouped into a column vector $f \in R^{JL}$, and the link user travel costs into a row vector $c \in R^{JL}$, where L is the number of links in the network. It is assumed that, in general, the user travel cost on a link on each mode may depend upon the entire link load pattern, that is,

$$c = c(f) \; , \tag{1}$$

where c is a known smooth function.

A user of mode i traveling on path p incurs a personal travel cost C_p^i, where:

$$C_p^i(f) = \sum_a c_a^i(f)\delta_{ap} \ , \tag{2}$$

with $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise.

The travel demand for mode i between each O/D pair w is assumed fixed and denoted by d_w^i.

The flow on path p of mode i, in turn, is denoted by x_p^i, with the path flows grouped into a column vector $x \in R^{JQ}$, where Q denotes the number of paths in the network. The flows in the network must satisfy the following conservation of flow equations:

$$d_w^i = \sum_{p \in P_w} x_p^i, \quad \forall i, \forall w \ , \tag{3}$$

where P_w denotes the set of paths connecting O/D pair w, and:

$$f_a^i = \sum_p x_p^i \delta_{ap}, \quad \forall i, \forall a \ . \tag{4}$$

Now the imposed controls are discussed. The column vector of transportation controls is denoted by $f^{\#} = \left[f_a^{i\#}\right]$, whereas the column vectors of nonnegative levels of overflow and underflow are denoted, respectively, by $\delta^+ = \left[\delta_a^{i^+}\right]$ and $\delta^- = \left[\delta_a^{i^-}\right]$. The following equation must be satisfied:

$$f_a^i - \delta_a^{i^+} + \delta_a^{i^-} = f_a^{i^\#}, \quad \forall i, \forall a \ , \tag{5}$$

which, by way of definition, states that the load on link a for mode i minus the possible overflow on link a of mode i plus the possible underflow on link a for mode i is equal to the control imposed on the load on the link and that mode. Let K denote the feasible set consisting of (f, δ^+, δ^-) such that there exists a vector of path flows x compatible with Equations 3 and 4.

In the absence of targets on the user flows and associated penalties levied on deviations from the targets, the well-known traffic network equilibrium conditions (cf. [2,7,8]) are stated as: for all modes i, for all paths $p \in P_w$ and all O/D pairs w, a link load pattern $f^\star$ induced by a path flow pattern $x^\star$ and compatible with Equations 3 and 4 is an equilibrium pattern if, and only if it satisfies the conditions:

$$C_p^i(f^\star) \begin{cases} = \lambda_w^i, & \text{if } x_p^{i^\star} > 0 \\ \geq \lambda_w^i, & \text{if } x_p^{i^\star} = 0 \ , \end{cases} \tag{6}$$

where λ_w^i denotes the travel disutility or minimal travel cost associated with traveling between O/D pair w by mode i. The minimal travel costs associated with the O/D pairs are grouped into a column vector $\lambda \in R^{JW}$.

If the command now intervenes in this user-optimized transportation network by imposing penalties for deviations from the controls in the form of excise transportation taxes, the actual user cost on a link a associated with mode i will then in general no longer be equal to c_a^i but, rather, it will have the unit tax t_a^i added to the cost, in the case of overflow. The equilibrium conditions in Statement 6, in the form of such policy interventions for all modes i, for all paths $p \in P_w$ and all O/D pairs w, would then take the form:

$$C_p^i(f^\star) + t_p^i \begin{cases} = \lambda_w^i, & \text{if } x_p^{i^\star} > 0 \\ \geq \lambda_w^i, & \text{if } x_p^{i^\star} = 0 \ , \end{cases} \tag{7}$$

where $t_p^i = \sum_a t_a^i \delta_{ap}$ denotes the total transportation tax levied on users of path p and mode i.

The command's role is to determine the transportation tax rates t_a^i on the links so that the constraint in Equation 5, reflecting the control, is satisfied as closely as possible, where the unit penalties on the links are given as follows. Here, it is assumed that a penalty function μ_a^i exists, where μ_a^i denotes the unit penalty for exceeding the transportation control on link a for mode i. Note that, in effect, subsidized travelers are not penalized in the case where the load on a link is below the imposed control. It is preassumed that the overflow penalty function is, in general, a function of the entire link overflow pattern and the overflow penalties are grouped into a row vector $\mu \in R^{JL}$, and hence, assume that:

$$\mu = \mu(\delta^+) \ . \tag{8}$$

The following condition must also be met at equilibrium:

$$0 \leq t_a^i \leq \mu_a^i(\delta^{+^*}), \quad \forall i, \forall a \ , \tag{9}$$

since the transportation tax can never exceed the corresponding unit penalty. Moreover, the following complementarity conditions exist:

$$\delta_a^{i+^*}(\mu_a^i(\delta^{+^*}) - t_a^i) = 0, \quad \text{and}$$

$$\delta_a^{i-^*}(t_a^i) = 0, \quad \forall i, \forall a \ . \tag{10}$$

In other words, if there is overflow on a link of a mode beyond the level of the control, the transportation tax will be equal to the penalty for that mode on that link. In addition, if there is underflow on a link of a mode, then the transportation tax on that link for that mode is set to zero.

Equivalently, at equilibrium for each mode i on each link a, the following conditions hold:

$$\mu_a^i(\delta^{+^*}) \begin{cases} = t_a^i, & \text{if } \delta_a^{i+^*} > 0 \\ \geq t_a^i, & \text{if } \delta_a^{i+^*} = 0 \ , \end{cases}$$

and:

$$t_a^i = \begin{cases} = 0, & \text{if } \delta_a^{i-^*} > 0 \\ \geq 0, & \text{if } \delta_a^{i-^*} = 0 \ . \end{cases} \tag{11}$$

The above transportation policy problem, defined by the conditions in Statements 7 and 11, can be formulated as a variational inequality problem.

Theorem 1

A link load, overflow and underflow pattern $(f^*, \delta^{+^*}, \delta^{-^*}) \in K$, with an attendant vector of tax rates $t \in R^L$ associated with the constraint in Equation 5 and minimal travel cost vector λ associated with the constraint in Equation 3, satisfies the network equilibrium condition in Statements 7 and 11, in the presence of policy interventions in the form of transportation controls and associated penalties if, and only if, it satisfies the variational inequality problem:

$$c(f^*) \cdot (f - f^*) + \mu(\delta^{+^*}) \cdot (\delta^+ - \delta^{+^*}) \geq 0 \ ,$$
$$\forall (f, \delta^+, \delta^-) \in K \ . \tag{12}$$

Proof

It is first shown that any solution that satisfies conditions in Statements 7 and 11, must also satisfy variational inequality in Statement 12. The equilibrium condition in Statement 7 implies that for a fixed mode i and a fixed path p:

$$\left[C_p^i(f^*) + t_p^i - \lambda_w^i\right] \cdot \left[x_p^i - x_p^{i^*}\right] \geq 0 \ . \tag{13}$$

Indeed, since if $x_p^{i^*} = 0$, then $x_p^i \geq x_p^{i^*}$, and the first term on the left-hand side of Statement 13 is also nonnegative. On the other hand, if $x_p^{i^*} > 0$, then the first term on the left-hand side of Statement 13 is zero and this expression also holds true.

Summing now Statement 13 over all modes i, paths p and O/D pairs w, and applying the definition of t_p^i, the following statement is obtained:

$$\sum_i \sum_w \sum_{p \in P_w} \left[C_p^i(f^*) + \sum_a t_a^i \delta_{ap} - \lambda_w^i\right]$$
$$\cdot \left[x_p^i - x_p^{i^*}\right] \geq 0 \ . \tag{14}$$

From the complementarity conditions in Statement 11 it follows, similarly, that for a fixed mode i and a fixed link a:

$$\left[\mu_a^i(\delta^{+^*}) - t_a^i\right] \cdot \left[\delta_a^{i+} - \delta_a^{i+^*}\right] \geq 0 \quad \text{and}$$
$$t_a^i \left[\delta_a^{i-} - \delta_a^{i+^*}\right] \geq 0 \ . \tag{15}$$

Summing now Statement 15 over all modes i and all links a, the following statement is obtained:

$$\sum_i \sum_a \left[\mu_a^i(\delta^{+^*}) - t_a^i\right] \cdot \left[\delta_a^{i+} - \delta_a^{i+^*}\right]$$
$$+ \sum_i \sum_a t_a^i \cdot \left[\delta_a^{i-} - \delta_a^{i^*}\right] \geq 0 \ . \tag{16}$$

Combining Statements 14 and 16, yields the following:

$$\sum_i \sum_w \sum_{p \in P_w} \left[C_p^i(f^*) + \sum_a t_a^i \delta_{ap}\right] \cdot \left[x_p^i - x_p^{i^*}\right]$$
$$+ \sum_i \sum_a \left[\mu_a^i(\delta^{+^*}) - t_a^i\right] \cdot \left[\delta_a^{i+} - \delta_a^{i+^*}\right]$$

$$+\sum_i \sum_a t_a^i \cdot \left[\delta_a^{i^-} - \delta_a^{i^{-*}}\right] \geq 0 \,. \tag{17}$$

Using then Equation 2, the following statement is given:

$$\sum_i \sum_a c_a^i(f^\star) \cdot (f_a^i - f_a^{i^*})$$
$$+\sum_i \sum_a \mu_a^i(\delta^{+^*}) \cdot (\delta_a^{i^+} - \delta_a^{i^{+^*}})$$
$$+\sum_i \sum_a t_a^i((f_a^i - \delta_a^{i^+} + \delta_a^{i^-})$$
$$-(f_a^{i^*} - \delta_a^{i^{+^*}} + \delta_a^{i^{-^*}})) \geq 0 \,. \tag{18}$$

However, in light of the goal constraint in Equation 5, the last term on the left-hand side of Statement 18 is zero. Hence, the above transportation problem satisfies the variational inequality problem in Statement 12.

Now the "if" part of the Theorem is considered.

Let $(f^\star, \delta^{+^*}, \delta^{-^*}) \in K$ solve Statement 12. Then the same point also solves the linear programming problem:

$$\text{Minimize}_{(f', \delta^{+'}, \delta^{-'}) \in K} c(f^\star) \cdot f' + \mu(\delta^{+^*}) \cdot \delta^{+'} \,. \tag{19}$$

Letting the dual variables to the corresponding dual problem of Statement 19 be λ and t, and using complementary slackness, the relationships in Statements 7 and 11 are found.■

The variational inequality in Statement 12 is distinct from the variational inequality governing the well-known traffic network equilibrium problem with fixed demand (cf. [1]) by the addition of the penalty expression on the left-hand side of the variational inequality in Statement 12 and in the definition of the feasible set K which in the case of this transportation policy model includes the goal constraints for all the modes of transportation. Nevertheless, similarity in the structures of the variational inequality is exploited in Statement 12 and the fixed demand multimodal traffic network equilibrium model in the suggestion of a decomposition algorithm in the subsequent section. The decomposition algorithm will allow one to apply the numerous algorithms, which have been developed to the latter problem for the solution of the major subproblem, such as projection and relaxation methods (cf. [5]). It accomplishes this, in part, by splitting the goal constraints in Equation 5 so that the first subproblem is characterized by constraints with a network structure.

In Nagurney, Thore and Pan [9], a spatial market policy model was proposed with supply and demand targets and transportation targets on links of the underlying bipartite network and a variational inequality problem governing the equilibrium conditions derived. In that model the penalties were assumed fixed and the network had a special structure. In Nagurney and Ramanujam [10], on the other hand, a single modal transportation network policy model was proposed, which allowed for generalized penalty functions.

A discussion of the qualitative properties of the transportation policy model, as developed above, is now given; in particular a uniqueness result is presented.

Theorem 2

Assume that the user link cost functions c and the penalty functions μ are strongly monotone, i.e. for all (f^1, δ^{+^1}), $(f^2, \delta^{+^2}) \in K$,

$$(c(f^1) - c(f^2)) \cdot (f^1 - f^2) \geq \alpha \|f^1 - f^2\|^2 \,, \tag{20}$$

$$(\mu(\delta^{+^1}) - \mu(\delta^{+^2})) \cdot (\delta^{+^1} - \delta^{+^2}) \geq$$
$$\beta \|\delta^{+^1} - \delta^{+^2}\|^2 \,, \tag{21}$$

for some $\alpha, \beta > 0$.

Then there exists at most one solution $(f^\star, \delta^{+^*}, \delta^{-^*}) \in K$ to the variational inequality in Statement 12.

Proof

Assume that there are two solutions to the variational inequality in Statement 12 given by:

$$(f^1, \delta^{+^1}, \delta^{-^1}) \text{ and } (f^2, \delta^{+^2}, \delta^{-^2}) \,.$$

It follows then that:

$$c(f^1)\cdot(f^2-f^1)+\mu(\delta^{+^1})\cdot(\delta^{+^2}-\delta^{+^1})\geq 0\ , \tag{22}$$

and:

$$c(f^2)\cdot(f^1-f^2)+\mu(\delta^{+^2})\cdot(\delta^{+^1}-\delta^{+^2})\geq 0\ . \tag{23}$$

Adding inequalities in Statements 22 and 23 yields:

$$[c(f^1)-c(f^2)]\cdot[f^2-f^1] + \left[\mu(\delta^{+^1})-\mu(\delta^{+^2})\right]\cdot\left[\delta^{+^2}-\delta^{+^1}\right]\geq 0\ . \tag{24}$$

But Statement 24 is in contradiction to the strong monotonicity assumptions that Statements 20 and 21 hold. Hence, it is concluded that $f^1=f^2$ and $\delta^{+^1}=\delta^{+^2}$. Finally, in view of the control constraints in Equation 5, it can also be concluded that $\delta^{-^1}=\delta^{-^2}$.■

THE ALGORITHM AND NUMERICAL EXAMPLES

In this section an algorithm is proposed for the computation of the equilibrium link load, overflow and underflow pattern for the multimodal model introduced in the preceding section. The algorithm converges under the assumptions that the link cost functions c are strongly monotone and the penalty functions μ are monotone.

The algorithm, which is a version of the method of multipliers (cf. [11,12]), was proposed earlier for the spatial policy market model with goal targets and fixed penalties developed in [9]. Its principal advantage in the framework of the multimodal transportation policy model on a general network is its splitting feature. In particular, it resolves the variational inequality problem in Statement 12 into a series of two simpler variational inequality subproblems. The first subproblem takes the form of the variational inequality governing the well-known traffic network equilibrium problem with fixed demand, for which numerous efficient algorithms exist (see e.g. [1,5], and the references therein). The second variational inequality subproblem is a very simple subproblem in the (δ^+,δ^-) variables only and is further decomposable into the δ^+ and δ^- variables respectively. These subproblems, in turn, can be solved via a Gauss-Seidel decomposition method described in [13].

This algorithm is a specialization of an algorithm developed for a class of variational inequalities to which this transportation policy model belongs. Additional theoretical results and applications can be found in [9] and the references therein.

The Algorithm for the Transportation Policy Model

Given a sequence of parameters $0<r_1\leq r_2\leq r_3\leq\ldots,$

Step 0: Initialization

Set the iteration count $t=0$.
Initialize $(\delta^{+^0},\delta^{-^0})$ and (λ^0).

Step 1: Computation of Decomposed Traffic Network Equilibrium Subproblem

Compute $(f^{t+1})\geq 0$, where $f_a^{i^{t+1}}=\sum_p x_p^{i^{t+1}}\delta_{ap}$, for all a, i, and $d_w^i=\sum_{p\in P_w}x_p^{i^{t+1}}$, for all w, i, and satisfying:

$$\sum_i\sum_a\left[c_a^i(f^{t+1})+r_t(f_a^{i^{t+1}}-(\delta_a^{i^+})^t +(\delta_a^{i^-})^t-f_a^{i^\#})-\lambda_a^{i^t}\right]\cdot\left[f_a^{i'}-f_a^{i^{t+1}}\right]\geq 0\ ,$$
$$\forall f'\geq 0,\in K\ . \tag{25}$$

Step 2: Computation of Target Deviations Subproblem

Compute $((\delta^+)^{t+1},(\delta^-)^{t+1})\geq 0$ satisfying:

$$\sum_i\sum_a(\mu_a^i(\delta^+)^{t+1}-r_t(f_a^{i^{t+1}}-(\delta_a^{i^+})^{t+1} + (\delta_a^{i^-})^{t+1}-f_a^{i^\#})+\lambda_a^{i^t})\cdot(\delta_a^{i^{+'}}-(\delta_a^{i^+})^{t+1})$$
$$+\sum_i\sum_a(r_t(f_a^{i^{t+1}}-(\delta_a^{i^+})^{t+1}+(\delta_a^{i^-})^{t+1} - f_a^{i^\#})-\lambda_a^{i^t})\cdot(\delta_a^{i^{-'}}-(\delta_a^{i^-})^{t+1})\geq 0\ ,$$

$$\forall \delta_a^{+'}, \delta_a^{i^{-'}} \geq 0, \quad \forall a, i \; . \tag{26}$$

Update: Set

$$\lambda_a^{i^{t+1}} = \lambda_a^{i^t} + r_t(f_a^{i^\#} - f_a^{i^{t+1}} + (\delta_a^{i^+})^{t+1} - (\delta_a^{i^-})^{t+1}), \;\; \forall a, i \; . \tag{27}$$

Step 3: Convergence Verification

If convergence has been reached within a prespecified tolerance ε, then stop; otherwise, set $t = t+1$, and go to Step 1.

In Steps 1 and 2 the subproblems are strongly monotone, and hence, the sequence of iterates f^{t+1}, $(\delta^+)^{t+1}$, $(\delta^-)^{t+1}$ is well-defined.

Now numerical results are provided for several examples. Here, for simplicity, a single mode of transportation is assumed.

The transportation networks that are considered had user link cost functions that were nonlinear and asymmetric, of the form:

$$c_a(f) = g_{aa} f_a^4 + \sum_b g_{ab} f_b + h_a, \quad \forall a \; . \tag{28}$$

The penalty functions, on the other hand, were linear and separable, that is, of the form:

$$\mu_a(\delta_a^+) = m_a \delta_a^+ + o_a \quad \forall a \; . \tag{29}$$

Separable functions were considered since it seems reasonable that in practice one may wish to primarily penalize according to the deviation on the particular link.

These functions were strongly monotone, thereby guaranteeing convergence of the decomposition algorithm.

The algorithm was coded in FORTRAN and implemented on the IBM ES/9000. All the examples utilized the projection method to solve the traffic network equilibrium subproblem in Statement 25. In addition, the parameter $\rho = 0.1$ and the $\boldsymbol{G}$ matrix in the projection method were set equal to the diagonal of the Jacobian matrix of the user link cost functions evaluated at the initial link flow pattern (cf. [1]). The equilibration algorithm of Dafermos and Sparrow [8] (see also e.g. [14]) was used here to solve the embedded quadratic programming network problems. In view of the structure of the penalty functions given by Equation 29 the subproblem in Statement 26 was solved explicitly and in closed form.

The sequence $\{r\}$ (see the Initialization Step) was set as follows: $r_1 = 0.5$, $r_i = 2^i$, for $i > 1$. The convergence criterion was $|f_a^{t+1} - f_a^t| \leq \epsilon$, $|(\delta_a^+)^{t+1} - (\delta_a^+)^t| \leq \epsilon$, $|(\delta_a^-)^{t+1} - (\delta_a^-)^t| \leq \epsilon$, for all a, with ϵ set to 0.001.

For all the examples, the CPU time, exclusive of input and output, the number of iterations, the computed link load, overflow and underflow pattern, along with the transportation taxes are reported. For the first two and the smallest, network examples, the computed path flows and user path costs (after taxes) are also given.

Example 1

The first example, depicted in Figure 1, consisted of four nodes, five links and two origin/destination pairs. The user link cost functions were:

$$c_1(f) = 0.00005 f_1^4 + 7f_1 + 2f_2 + 3 \; ,$$
$$c_2(f) = 0.00003 f_2^4 + 11f_2 + f_1 + 8 \; ,$$
$$c_3(f) = 0.00005 f_3^4 + 2f_3 + f_5 + 1 \; ,$$
$$c_4(f) = 0.00003 f_4^4 + 2.5f_4 + f_2 + 10 \; ,$$
$$c_5(f) = 0.00004 f_5^4 + f_5 + 0.5f_1 + 6 \; .$$

The O/D pairs were: $w_1 = (1,4)$ and $w_2 =$

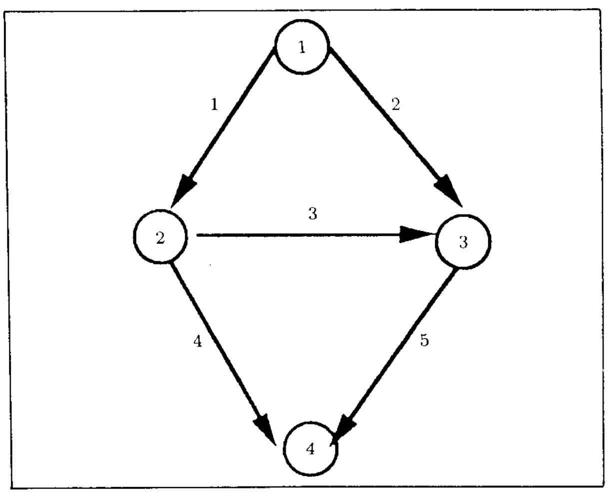

Figure 1. Network 1.

Table 1. Computed flows $(f^\star, \delta^{+^\star}, \delta^{-^\star})$ and transportation taxes (t).

Link	f_a	δ_a^+	δ_a^-	t_a
1	38.24	28.24	0.00	58.47
2	36.76	26.76	0.00	55.53
3	13.70	0.00	6.30	0.00
4	24.54	14.54	0.00	31.08
5	25.46	15.46	0.00	32.92

$(1,3)$, and the travel demands:

$$d_{w_1} = 50, \quad d_{w_2} = 25 \ .$$

The goal targets, $f_a^{\#}$ were set to 10 for all links a, except that $f_3^{\#} = 20$, and the penalty function terms for the overflow penalties were set to $m_a = 2$ and $o_a = 2$.

The initial path assignment was an equal distribution of the demand over the paths connecting each O/D pair. The initial overflow and underflow pattern was set to zero.

The algorithm converged in seven iterations and utilized 0.007 seconds of CPU time.

In Table 1, the user link load and overflow and underflow pattern, along with the transportation taxes are reported.

The first path for O/D pair w_1, p_1, consisted of links 1, 3 and 5; the second path, p_2, of links 1 and 4, and the third path, p_3, of links 2 and 5. The first path for O/D pair w_2, p_4, consisted of links 1 and 3 and the second path for O/D pair w_2, p_5, of link 2.

The equilibrium path flows and associated path costs, inclusive of the taxes, were as follows.

Equilibrium Path Flows and Costs

O/D Pair w_1:

$$x^\star_{p_1} = 1.3, \quad x^\star_{p_2} = 24.5, \quad x^\star_{p_3} = 24.1 \ ,$$

$$C_{p_1}(f^\star) + t_{p_1} = 665.43 \ ,$$

$$C_{p_2}(f^\star) + t_{p_2} = 659.56 \ ,$$

$$C_{p_3}(f^\star) + t_{p_3} = 661.26 \ .$$

O/D Pair w_2:

$$x^\star_{p_4} = 12.4, \quad x^\star_{p_5} = 12.6 \ ,$$

Table 2. Computed flows $(f^\star, \delta^{+^\star}, \delta^{-^\star})$ and transportation taxes (t).

Link	f_a	δ_a^+	δ_a^-	t_a
1	37.95	27.95	0.00	579.01
2	37.05	27.05	0.00	560.95
3	13.20	0.00	6.80	0.00
4	24.75	14.75	0.00	315.10
5	25.24	15.24	0.00	324.87

$$C_{p_4}(f^\star) + t_{p_4} = 565.12 \ ,$$

$$C_{p_5}(f^\star) + t_{p_5} = 560.95 \ .$$

Example 2

Example 2 is constructed from Example 1 as follows. The network topology of Figure 1, the user link cost functions and the travel demands were retained. The goal targets were as in Example 1, except that the penalty function terms $m_a = 20$ and $o_a = 20$ for all links were increased. The algorithm converged in ten iterations and 0.008 CPU seconds.

The new equilibrium link load, overflow and underflow pattern and transportation taxes are reported in Table 2.

The new computed equilibrium path flows and costs are as follows.

Equilibrium Path Flows and Costs

O/D Pair w_1:

$$x^\star_{p_1} = 1.0, \quad x^\star_{p_2} = 24.8 \ , \quad x^\star_{p_3} = 24.3 \ ,$$

$$C_{p_1}(f^\star) + t_{p_1} = 1470.95 \ ,$$

$$C_{p_2}(f^\star) + t_{p_2} = 1460.76 \ ,$$

$$C_{p_3}(f^\star) + t_{p_3} = 1462.27 \ .$$

O/D Pair w_2:

$$x^\star_{p_4} = 12.2, \quad x^\star_{p_5} = 12.8 \ ,$$

$$C_{p_4}(f^\star) + t_{p_4} = 1079.61,$$

$$C_{p_5}(f^\star) + t_{p_5} = 1070.94 \ .$$

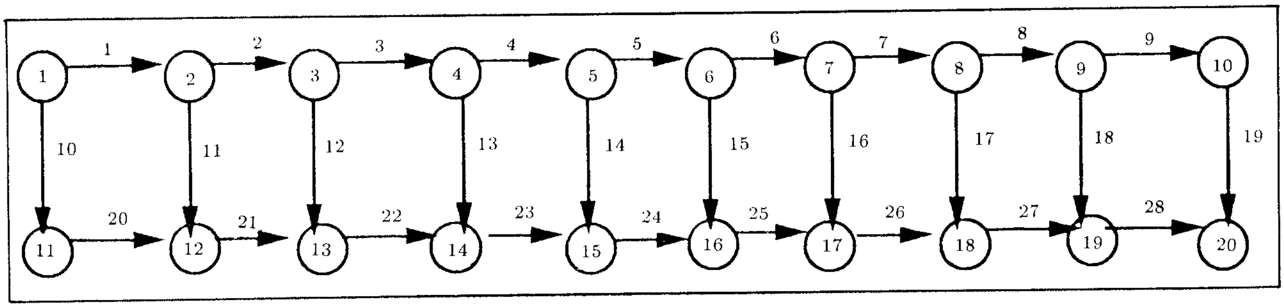

Figure 2. Network 2.

Example 3

The third example consisted of 20 nodes, 28 links and 8 O/D pairs, and is depicted in Figure 2. The network had been used previously (without penalty functions) in Nagurney [6] where it is referred to as Network 20. The user link cost functions are reproduced here for the convenience of the reader.

$$
\begin{aligned}
c_1(f) &= 0.00005 f_1^4 + 5f_1 + 2f_2 + 500 ,\\
c_2(f) &= 0.00003 f_2^4 + 4f_2 + 4f_1 + 200 ,\\
c_3(f) &= 0.00005 f_3^4 + 3f_3 + f_4 + 350 ,\\
c_4(f) &= 0.00003 f_4^4 + 6f_4 + 3f_5 + 400 ,\\
c_5(f) &= 0.00006 f_5^4 + 6f_5 + 4f_6 + 600 ,\\
c_6(f) &= 7f_6 + 3f_7 + 500 ,\\
c_7(f) &= 0.00008 f_7^4 + 8f_7 + 2f_8 + 400 ,\\
c_8(f) &= 0.00004 f_8^4 + 5f_8 + 2f_9 + 650 ,\\
c_9(f) &= 0.00001 f_9^4 + 6f_9 + 2f_{10} + 700 ,\\
c_{10}(f) &= 4f_{10} + f_{12} + 800 ,\\
c_{11}(f) &= 0.00007 f_{11}^4 + 7f_{11} + 4f_{12} + 650 ,\\
c_{12}(f) &= 8f_{12} + 2f_{13} + 700 ,\\
c_{13}(f) &= 0.00001 f_{13}^4 + 7f_{13} + 3f_{18} + 600 ,\\
c_{14}(f) &= 8f_{14} + 3f_{15} + 500 ,\\
c_{15}(f) &= 0.00003 f_{15}^4 + 9f_{15} + 2f_{14} + 200 ,\\
c_{16}(f) &= 8f_{16} + 5f_{12} + 300 ,\\
c_{17}(f) &= 0.00003 f_{17}^4 + 7f_{17} + 2f_{15} + 450 ,\\
c_{18}(f) &= 5f_{18} + f_{16} + 300 ,\\
c_{19}(f) &= 8f_{19} + 3f_{17} + 600 ,\\
c_{20}(f) &= 0.00003 f_{20}^4 + 6f_{20} + f_{21} + 300 ,\\
c_{21}(f) &= 0.00004 f_{21}^4 + 4f_{21} + f_{22} + 400 ,\\
c_{22}(f) &= 0.00002 f_{22}^4 + 6f_{22} + f_{23} + 500 ,\\
c_{23}(f) &= 0.00003 f_{23}^4 + 9f_{23} + 2f_{24} + 350 ,\\
c_{24}(f) &= 0.00002 f_{24}^4 + 8f_{24} + f_{25} + 400 ,\\
c_{25}(f) &= 0.00003 f_{25}^4 + 9f_{25} + 3f_{26} + 450 ,\\
c_{26}(f) &= 0.00006 f_{26}^4 + 7f_{26} + 8f_{27} + 300 ,\\
c_{27}(f) &= 0.00003 f_{27}^4 + 8f_{27} + 3f_{28} + 500 ,\\
c_{28}(f) &= 0.00003 f_{28}^4 + 7f_{28} + 3f_{29} + 650 .
\end{aligned}
$$

The O/D pairs were: $w_1 = (1,20)$, $w_2 = (1,19)$, $w_3 = (2,17)$, $w_4 = (4,20)$, $w_5 = (6,19)$, $w_6 = (2,20)$, $w_7 = (2,13)$ and $w_8 = (3,14)$ and the travel demands:

$$d_{w_1} = 50,\quad d_{w_2} = 60,\quad d_{w_3} = 100,\ d_{w_4} = 100,$$

$$d_{w_5} = 100,\ d_{w_6} = 100,\ d_{w_7} = 100,\ d_{w_8} = 50.$$

The penalty functions for this example were set as follows. All the m_a and o_a terms in the overflow penalty functions were set equal to ten. The goal target for this and the subsequent example were set to $f_a^{\#} = 10$ for all links. In this and the next example, the algorithm was initialized with all the demand for each O/D pair allocated to the minimal uncongested cost path. The overflow and underflow were initialized to zero for both examples.

The algorithm converged in ten iterations and required 0.730 CPU seconds for convergence. The computed link load, overflow and underflow values, as well as the transportation taxes are reported in Table 3.

Example 4

The final example is constructed from Example 3 as follows. The network topology and all the functions were retained except that the penalty

Table 3. Computed flows ($f^\star, \delta^{+^\star}, \delta^{-^\star}$) and transportation taxes (t).

Link	f_a	δ_a^+	δ_a^-	t_a
1	13.12	3.12	0.00	41.20
2	215.28	205.28	0.00	2062.80
3	160.80	150.80	0.00	1518.02
4	205.52	195.52	0.00	1965.22
5	199.75	189.75	0.00	1097.54
6	299.30	289.30	0.00	2903.02
7	198.08	188.08	0.00	1890.81
8	197.49	187.49	0.00	1884.86
9	125.01	115.01	0.00	1160.09
10	96.86	86.86	0.00	878.57
11	97.78	87.78	0.00	887.81
12	104.49	94.49	0.00	954.91
13	55.25	45.25	0.00	462.54
14	5.77	0.00	4.23	0.00
15	0.00	0.00	10.00	0.00
16	101.66	91.66	0.00	926.62
17	0.59	0.00	9.41	0.00
18	72.48	62.48	0.00	634.77
19	125.01	115.01	0.00	1160.09
20	96.86	86.86	0.00	878.57
21	194.64	184.64	0.00	1856.19
22	199.14	189.14	0.00	1901.42
23	204.41	194.41	0.00	1954.09
24	210.18	200.18	0.00	2011.84
25	210.16	200.16	0.00	2011.61
26	211.86	201.86	0.00	2028.60
27	212.40	202.40	0.00	2034.02
28	124.95	114.95	0.00	1159.52

Table 4. Computed flows ($f^\star, \delta^{+^\star}, \delta^{-^\star}$) and transportation taxes (t).

Link	f_a	δ_a^+	δ_a^-	t_a
1	14.96	4.96	0.00	595.53
2	214.77	204.77	0.00	20576.55
3	165.35	155.35	0.00	15635.33
4	207.14	197.14	0.00	19814.00
5	195.93	185.93	0.00	18693.07
6	295.47	285.47	0.00	28646.58
7	200.18	190.18	0.00	19117.70
8	196.07	186.07	0.00	18707.42
9	124.32	114.32	0.00	11532.21
10	95.02	85.02	0.00	8602.14
11	100.13	90.13	0.00	9113.07
12	99.42	89.42	0.00	9042.29
13	58.19	48.19	0.00	4918.72
14	11.22	1.22	0.00	221.58
15	0.00	0.00	10.00	0.00
16	95.74	85.74	0.00	8674.29
17	4.10	0.00	5.90	0.00
18	71.75	61.75	0.00	6275.17
19	124.32	114.32	0.00	11532.21
20	95.02	85.02	0.00	8602.14
21	195.15	185.15	0.00	18615.23
22	194.59	184.59	0.00	18558.75
23	202.79	192.79	0.00	19379.00
24	214.01	204.01	0.00	20500.61
25	213.98	203.98	0.00	20498.33
26	209.76	199.76	0.00	20075.34
27	213.81	203.81	0.00	20481.33
28	125.64	115.64	0.00	11663.84

function terms m_a and o_a were increased to 100 for all the links. The algorithm required sixteen iterations for convergence and 0.829 seconds of CPU time.

The computed link load, overflow and underflow values, as well as the transportation taxes are reported in Table 4.

SUMMARY AND CONCLUSIONS

In this paper a new model has been developed that allows the decision maker to impose transportation controls on the links and modes of a general transportation network. Associated with the goals are penalty functions that penalize the users of the transportation network if the controls are not met.

The model is first developed and the equilibrium conditions derived. The governing conditions are then shown to satisfy a variational inequality problem. A uniqueness result is also given.

An algorithm is then proposed to compute the equilibrium link load, overflow and underflow pattern, along with the transportation taxes to be assessed on the users. The algorithm resolves the large-scale problem into a series of traffic network equilibrium problems and a simpler subproblem. For the latter, one may avail oneself of numerous algorithms in the literature.

Finally, numerical examples are provided to illustrate the approach.

This work may be viewed as a contribution to the growing literature on transportation policy modelling (see also [14]).

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