Boundary Element Solution of Inhomogeneous Modified Helmholtz Equation

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A boundary element method (BEM) formulation is presented to obtain the solution of the inhomogeneous modified Helmholtz equation. The difficulty arising from the domain integral, due to the source function, is eliminated by obtaining an approximate particular solution to the equation. An approximate particular solution is obtained by a very simple special procedure which approximates both the source function and the particular solution as linear combinations of several radial basis functions simultaneously. BEM, with linear elements, has then been used to cast the homogeneous equation into the form of an integral equation over the boundary. Computations have been carried out for a two-dimensional scalar Helmholtz problem for several values of parameter λ in the equation. Selected graphs are given showing the accuracy of the methods used and the agreement with the exact solution.

INTRODUCTION

The boundary element method (BEM) is an alternative technique to domain methods, such as finite difference and finite element methods, for solving boundary value problems, since it simplifies the problem from one involving domain discretization and/or area integration to one involving line integration only. In general, the number of equations derived from such a formulation will be fewer than in the case of an interior method. Contrary to the sparse matrices encountered in other methods, the boundary element generated matrices are full. The main motivation behind the BEM is the reduction of the dimensionality of the problem. Unfortunately, this major advantage is lost when the partial differential equation is inhomogeneous, since the resulting integral equation will include the domain integral term. The domain must then be discretized to allow numerical evaluation of this domain integral [1]. Although this discretization does not introduce any further unknowns, the numerical integration process considerably increases the amount of computational time.

One way of removing the domain integral is to consider a particular solution to the inhomogeneous equation. The remainder will then satisfy a homogeneous differential equation, hence leading to a boundary integral equation only. For general cases, a closed form of particular solution is difficult or impossible to find. One can then proceed to find an approximate particular solution. The approximate particular solutions for potential problems can be obtained in several ways. Among others, Banerjee et al. [2] approximated the inhomogeneity as an infinite series, Tang [3] considered a Fourier transform method for evaluating the particular

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solution and Coleman et al. [4] and Zheng et al. [5,6] expressed the source function as a linear combination of radial basis functions. The idea in all these procedures is to represent the source function in terms of simpler functions for which the corresponding particular solution is known. For the Helmholtz equation, these procedures cannot be as easily applied as those applied to potential problems. Even if the source function is approximated in terms of simpler functions, it is still difficult to find the corresponding particular solution.

In this paper, both the source function and the corresponding particular solution for the modified Helmholtz equation are approximated as linear combinations of several radial basis functions simultaneously. The numerical implementation is presented for the case of a two-dimensional scalar Helmholtz problem, which was solved by Tsuchimoto et al. [1] using the BEM with constant elements. In their study, they treat the equation as a Laplace equation with an inhomogeneous term and use the fundamental solution of the Laplace equation. This, however, is not necessary, since it is possible to obtain more accurate results by solving the modified Helmholtz equation using BEM with its own fundamental solution [7,8]. Tsuchimito et al. [1] also divide the domain into cells and use numerical integration for the domain integral found in their formulation. This study uses BEM with linear elements to solve the problem in its original form by using the fundamental solution of the modified Helmholtz equation. This allows that computations to small values of parameter λ not be restricted. On the other hand, removing the domain integral with a very simple procedure, using several radial functions for both the source function and the particular solution, saves considerable computational time.

FORMULATION OF THE PROBLEM

Consider the following inhomogeneous modified Helmholtz equation

$$\nabla^2 u - k^2 u = g(x,y), \tag{1}$$

over a region Ω with boundary Γ, where $u(x,y)$ is the solution, g is an arbitrary source function and k is a constant. The boundary conditions are as follows

$$u = \bar{u} \quad \text{on} \quad \Gamma_1, \tag{2}$$

$$\frac{\partial u}{\partial n} = \bar{q} \quad \text{on} \quad \Gamma_2, \tag{3}$$

where the bar denotes prescribed values and n is the outward normal unit vector of the boundary.

The fundamental motivation behind the boundary element method [9,10] is the reduction of the dimensionality of the problem. Thus, we simplify the problem from one involving area integration (e.g., finite element method) to one involving line integration. In general, the number of equations derived from such a formulation will be fewer than in the case of an interior method.

Unfortunately, the major advantage is generally lost when the partial differential equation is inhomogeneous as in Equation 1. The resulting integral equation will include the domain integral term. This integral is usually computed by numerical quadrature techniques which require a domain discretization [1]. Since these domain cells are created only for the purpose of numerical integration for the source function, they do not add new unknowns to the problem, however, the procedure is time-consuming and is inconvenient from the numerical point of view.

Thus, the first step in the formulation involves the generation of an integral equation over the boundary Ω only. One way of removing the domain integral is to find a particular solution to the inhomogeneous partial differential equation. Since a closed form of a particular solution is difficult to find, the next step is to proceed in finding an approximate particular solution by using radial basis functions [4,5,6]. The remainder of the solution will then satisfy the homogeneous modified Helmholtz equation and hence lead to a boundary integral equation only.

Knowing the particular solution to Equation 1, that is, for some function u_p obeying

$$\nabla^2 u_p - k^2 u_p = g, \tag{4}$$

but not necessarily satisfying the boundary conditions, Equation 1 can be transformed into

$$\nabla^2 u_H - k^2 u_H = 0 \quad \text{in} \quad \Omega, \tag{5}$$

with

$$u_H = \bar{u}_H = \bar{u} - u_p \quad \text{on} \quad \Gamma_1, \tag{6}$$

$$\frac{\partial u_H}{\partial n} = \bar{q}_H = \bar{q} - \frac{\partial u_p}{\partial n} \quad \text{on} \quad \Gamma_2. \tag{7}$$

Thus, the homogeneous modified Helmholtz Equation 5 will be solved with the boundary conditions, Equations 6 and 7, using the boundary element method. Then the solution to Equation 1 can be obtained by adding a particular solution to the solution of the homogeneous Equation 5 as

$$u(x, y) = u_p(x, y) + u_H(x, y). \tag{8}$$

APPROXIMATE PARTICULAR SOLUTION

Finding an approximate particular solution to Equation 1 is not as easy as in the case of Poisson's equation. For Poisson's equation, the source function can be expressed as a linear combination of radial basis functions for which the corresponding particular solutions are known [4,5,6], since corresponding particular solutions can be obtained by simply applying the inverse Laplace operator to those radial basis functions. For the modified Helmholtz operator, this procedure does not work. The present study continues to use radial basis functions, but starts with the assumption that a particular solution is a radial function and then expresses the source function as a linear combination of several radial basis functions.

Let us start with a radial function

$$\psi_n(r) = \left(1 + \frac{r^2}{\beta^2}\right)^n, \tag{9}$$

then

$$\nabla^2 \psi_n - \lambda \psi_n = \frac{4n}{\beta^2}[n\psi_{n-1} - (n-1)\psi_{n-2}] - \lambda \psi_n. \tag{10}$$

For $n = 1/2$,

$$\nabla^2 \psi_{1/2} - \lambda \psi_{1/2} = \frac{1}{\beta^2}[\psi_{-1/2} + \psi_{-3/2} - \beta^2 \lambda \psi_{1/2}]. \tag{11}$$

This means that, when the inhomogeneity is represented in terms of radial functions $\psi_{-1/2}$, $\psi_{-3/2}$ and $\psi_{1/2}$ as in Equation 11, the radial function $\beta^2\psi_{1/2}$ satisfies the modified Helmholtz equation with the inhomogeneity $\psi_{-1/2} + \psi_{-3/2} - \beta^2\lambda\psi_{1/2}$ as

$$\nabla^2 \beta^2 \psi_{1/2} - \lambda \beta^2 \psi_{1/2} = \psi_{-1/2} + \psi_{-3/2} - \beta^2 \lambda \psi_{1/2}. \tag{12}$$

We can now approximate the source function in Equation 1 in terms of radial functions $\psi_{-1/2}$, $\psi_{-3/2}$, and $\psi_{1/2}$:

$$g(r) \simeq \sum_{i=1}^{N} \alpha_i \psi^i_{-1/2}(r) + \sum_{i=1}^{N} \gamma_i \psi^i_{-3/2}(r) + \sum \zeta_i \psi^i_{1/2}(r), \tag{13}$$

where

$$\alpha_i : \gamma_i : \zeta_i = 1 : 1 : -\lambda \beta_i^2. \tag{14}$$

The functions ψ^i_n are chosen to have the form

$$\psi^i_n(r) = \psi_n\left(\frac{|\, r - r_i \,|}{\beta_i}\right) = \left(1 + \frac{|\, r - r_i \,|^2}{\beta_i^2}\right)^n, \tag{15}$$

where r_i and β_i are suitable constants and ψ^i_n is a function of a single variable. Such functions are known as radial basis functions [11].

The $r_i(i = 1, \ldots, N)$ values are chosen to be evenly distributed points within the domain Ω, and the optimal β_i values are found to be $h_i/6$, h_i being the minimum distance between two neighboring points. These choices for parameters lead to a good approximation for $g(r)$.

Collocating Equation 13 at these points r_i, yields a set of equations which can be written in matrix form as

$$\begin{bmatrix} \psi_{-1/2} & \psi_{-3/2} & \psi_{1/2} \\ I & -I & 0 \\ 0 & \lambda\beta^2 I & I \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\gamma} \\ \vec{\zeta} \end{bmatrix} = \begin{bmatrix} \vec{g(r)} \\ \vec{0} \\ \vec{0} \end{bmatrix}, \tag{16}$$

where, in the coefficient matrix above, the functions $\psi_{1/2}$, $\psi_{-3/2}$ and $\psi_{1/2}$ are evaluated at $\frac{|r_i - r_j|}{\beta_j}(i = 1, \ldots, N, j = 1, \ldots, N)$, giving $N \times N$ square matrices, I denotes the $N \times N$ identitiy matrix and 0 the $N \times N$ zero matrix.

Proving the non-singularity of such matrices as occur on the left hand side of Equation 16, for arbitrary grids with points in "general position", requires delicate reasoning [11].

However, if a particular grid is considered, it is enough to check for non-singularity numerically. (In fact, for the regular rectangular grids it is not difficult, but rather tedious, to demonstrate the non-singularity of the matrix in question, taking the different growth properties of the functions ψ_n, ψ_{n-1}, ψ_{n-2}.)

Remark: The importance of this method of obtaining approximate particular solutions lies in the fact that it can be applied to any inhomogeneous differential equation whose homogeneous part can be treated by means of the BEM.

These $3N$ well-conditioned algebraic equations represent the aproximation of the source function in Equation 1. Therefore, the system in Equation 16 determines the coefficients $\alpha_i, \gamma_i, \zeta_i (i = 1, \ldots, N)$ uniquely.

Once the parameters $\alpha_i = \gamma_i$ and $\zeta_i = -\lambda\alpha_i\beta_i^2$ are known, an approximate particular solution u_p will be given by

$$\begin{aligned} u_p &\simeq \sum_{i=1}^{N} \alpha_i \beta_i^2 \psi_{1/2}^i(r) \\ &= \sum_{i=1}^{N} \alpha_i \beta_i^2 \left(1 + \frac{|\, r - r_i \,|^2}{\beta_i^2}\right)^{1/2}. \end{aligned} \tag{17}$$

BOUNDARY ELEMENT SOLUTION OF HOMOGENEOUS MODIFIED HELMHOLTZ EQUATION

A weighted residual approach was used because of its inherent simplicity for obtaining the boundary integral equation for u_H. Introducing a weighting function W, which has continuous first derivatives and which satisfies governing Equation 5, the weighted residual statement can be written as

$$\int_\Omega (\nabla^2 u_H - k^2 u_H) W d\Omega = 0. \tag{18}$$

Employing Green's theorem in two steps to yield

$$\int_\Omega (\nabla^2 W - k^2 W) u_H d\Omega + \int_\Gamma \frac{\partial u_H}{\partial n} W dS - \int_\Gamma u_H \frac{\partial W}{\partial n} dS = 0, \tag{19}$$

results in an integral over the boundary Γ

$$\int_\Gamma \frac{\partial u_H}{\partial n} W dS - \int_\Gamma u_H \frac{\partial W}{\partial n} dS = 0, \tag{20}$$

since the weighting function W satisfies the Differential Equation 5.

Following the procedure in [12,13], the weighting function $W = K_0(Kr)$ (modified Bessel function of the second kind and of order zero) may be selected as the singular solution to the Helmholtz equation. The distance r is measured from an arbitrary point P to a point Q on the boundary. Substituting for W, Equation 10 becomes

$$\int_\Gamma \left(\frac{\partial u_H}{\partial n} K_0(kr) - u_H \frac{\partial K_0(kr)}{\partial n} \right) dS = 0. \tag{21}$$

The integrals in Equation 21 are readily evaluated, except in the vicinity of the singular point P. This point is excluded from the region by a small circle of radius r_0. The required integrations may now be written as

$$\int_\Gamma \left(\frac{\partial u_H}{\partial n}K_0(kr) - u_H\frac{\partial K_0(kr)}{\partial n}\right)dS + \int_0^{2\pi}\left(\frac{\partial u_H}{\partial n}K_0(kr_0) - u_H\frac{\partial K_0(kr_0)}{\partial n}\right)r_0 d\Theta = 0, \qquad (22)$$

where Θ increases counterclockwise when the integral along Γ is taken in a clockwise direction. By examining Equation 22 in the limit as $r_0 \to 0$ it is found that the second term reduces to

$$-\int_0^{2\pi} u_H\frac{\partial K_0(kr_0)}{\partial n}r_0 d\Theta = -2\pi u_H(P), \qquad (23)$$

since $K_0(kr_0)$ behaves like$-log(kr_0)$ for small arguments and $\lim_{r_0\to r_0}\log(kr_0) = 0$ implying

$$\int_0^{2\pi}\frac{\partial u_H}{\partial n}K_0(kr_0)r_0 d\Theta = 0, \qquad (24)$$

The boundary integral equation is now

$$u_H(P) = \frac{1}{2\pi}\int_\Gamma\left(\frac{\partial u_H}{\partial n}K_0(kr) - u_H\frac{\partial K_0(kr)}{\partial n}\right)dS. \qquad (25)$$

This equation provides a relationship between any point in the interior of Ω and information known only at the boundary. When a relationship is required at the boundary, P is located along the boundary Γ and Equation 21 is written as (for a smooth boundary)

$$\int_{\Gamma-\sigma}\left(\frac{\partial u_H}{\partial n}K_0(kr) - u_H\frac{\partial K_0(kr)}{\partial n}\right)dS + \int_0^{\pi}\left(\frac{\partial u_H}{\partial n}K_0(kr_0) - u_H\frac{\partial K_0(Kro)}{\partial n}\right)r_0 d\Theta = 0, \qquad (26)$$

where σ is the semi-circle around the point P with a radius r_0 which is colinear with the normal.

Taking the limit as $r_0 \to 0$ results in

$$u_H(P) = \frac{1}{\pi}\int_\Gamma\left(\frac{\partial u_H}{\partial n}K_0(kr) - u_H\frac{\partial K_0(kr)}{\partial n}\right)dS. \qquad (27)$$

This integral equation may be rewritten for the unknowns u_H and q_H as

$$u_H = \frac{1}{\pi}\Big(\int_{\Gamma_1} q_H K_0(kr)dS + \int_{\Gamma_2}\bar{q}_H K_0(kr)dS - \int_{\Gamma_1}\bar{u}_H\frac{\partial K_0(kr)}{\partial n}dS - \int_{\Gamma_2}u_H\frac{\partial K_0(kr)}{\partial n}dS\Big), \qquad (28)$$

where $\bar{q}_H$ and $\bar{u}_H$ are known, therefore second and third integrals can be taken to the right-hand side of the equation as known values.

DEVELOPMENT OF A SET OF SIMULTANEOUS EQUATIONS

The boundary Γ is discretized into M elements and the values of u_H and its normal derivative are assumed to vary linearly within each element as

$$u_H(\xi) = N_1(\xi)u_{H,1} + N_2(\xi)u_{H,2}, \qquad (29)$$

$$\frac{\partial u_H}{\partial n}(\xi) = N_1(\xi)\frac{\partial u_{H,1}}{\partial n} + N_2\frac{\partial u_{H,2}}{\partial n}, \qquad (30)$$

where ξ is the dimensionless coordinate $\xi = 2X/l$ (Figure 1), l is the length of the element and N_1, N_2 are shape functions given by

$$N_1 = \frac{1-\xi}{2} \quad , \quad N_2 = \frac{1+\xi}{2}. \qquad (31)$$

Thus, Equation 27 becomes in discretized form

$$\pi u_H^i = -\sum_{j=1}^{M}\int_{\Gamma_j}\frac{\partial K_0(kr_i)}{\partial n}(N_1^j u_{H,1}^j + N_2^j u_{H,2}^j)dS + \sum_{j=1}^{M}\int_{\Gamma_j}K_0(kr_i)\left(N_1^j\frac{\partial u_{H,1}^j}{\partial n} + N_2^j\frac{\partial u_{H,2}^j}{\partial n}\right)dS, \qquad (32)$$

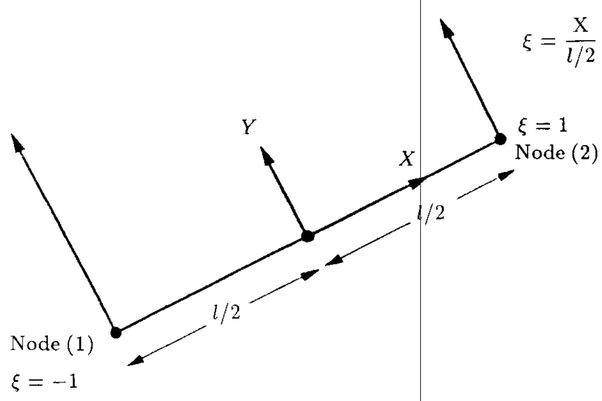

Figure 1. Linear elements.

where r_i is the distance from node i to element j and can be written in the following form

$$\pi u_H^i = \sum_{j=1}^{M}(h_{ij,1}u_{H,1}^j + h_{ij,2}u_{H,2}^j) + \sum_{j=1}^{M}\left(g_{ij,1}\frac{\partial u_{H,1}^j}{\partial n} + g_{ij,2}\frac{\partial u_{H,2}^j}{\partial n}\right), \tag{33}$$

where

$$h_{ij,1} = -\int_{\Gamma_j} N_1^j \frac{\partial K_0(kr_i)}{\partial n}dS, \tag{34}$$

$$h_{ij,2} = -\int_{\Gamma_j} N_2^j \frac{\partial K_0(kr_i)}{\partial n}dS, \tag{35}$$

$$g_{ij,1} = -\int_{\Gamma_j} N_1^j K_0(kr_i)dS, \tag{36}$$

$$g_{ij,2} = -\int_{\Gamma_j} N_2^j K_0(kr_i)dS. \tag{37}$$

Substituting Equations 34–37 into Equation 33 for all j elements, one obtains the following equation for node i:

$$\pi u_H^i = \sum_{j=1}^{M} H_{ij}u_H^j + \sum_{j=1}^{M} G_{ij}\frac{\partial u_H^j}{\partial n} \quad , \quad i = 1, \ldots, M, \tag{38}$$

where H_{ij} is equal to the $h_{ij,1}$ term of element j plus $h_{ij,2}$ term of element $j-1$ and similarly for G_{ij}. Hence Equation 38 represents the assembled equation for node i. This equation can be written as

$$[H]u_H + [G]\frac{\partial u_H}{\partial n} = 0, \tag{39}$$

and the elements of the matrices $[H]$ and $[G]$ are the assembled form of $h_{ij,1}$ and $h_{ij,2}$, $g_{ij,1}$ and $g_{ij,2}$ from Equations 34–37. Recall, however, that some of the values of u_H and $\frac{\partial u_H}{\partial n}$ are given from boundary conditions. Thus, this information can be transferred to the right-hand side of the system, Equation 39, and the remaining equations are re-ordered to make it a square system of algebraic equations,

$$[A]\{U\} = \{F\}, \tag{40}$$

where the vector $\{U\}$ contains unknown u_H and $\frac{\partial u_H}{\partial n}$ values at the nodes and $\{F\}$ is known from the boundary information. One can now solve $\{U\}$ using direct methods. Note that when u_H and $\frac{\partial u_H}{\partial n}$ are known at each point (node) along the boundary, one can compute u_H anywhere in the interior. This is achieved by discretizing the general expression, Equation 25:

$$2\pi u_H^i = \sum_{j=1}^{M} H_{ij}u_H^j + \sum_{j=1}^{M} G_{ij}\frac{\partial u_H^j}{\partial n}. \tag{41}$$

r_i is now measured from the interior point P.

The integrals in the coefficients H_{ij} and G_{ij} will be evaluated numerically. The integrals in H_{ii} and G_{ii} contain the singularities, therefore, if the integral contains the point P, a different algorithm is required than when the integral does not contain P. The integrals H_{ij} and G_{ij} can be evaluated using an 8-point Gauss Legendre quadrature [14] for all elements except the one including the node under consideration. When the element contains the node under consideration, the diagonal entries H_{ii} are calculated from

$$H_{ii} = \pi - \sum_{j=1}^{M} H_{ij} \quad , \quad i \neq j \tag{42}$$

and G_{ii} terms can be computed analytically as follows.

Since $K_0(z)$ behaves like $-(log(z/2) + \gamma)$ for small arguments,

$$K_0(kr) \simeq log\left(\frac{2}{k}\right) - \gamma - log(r), \tag{43}$$

after the assembly of $g_{ii,1}$ and $g_{ii,2}$ produces

$$G_{ii} = l\left(log\left(\frac{2}{k}\right) - \gamma\right) + l\left(\frac{3}{2} - log(l)\right), \quad (44)$$

or

$$G_{ii} = l\left(\frac{3}{2} - \gamma + log(\frac{2}{kl})\right). \quad (45)$$

Thus, when the system in Equation 40 is solved for the unknown vector $\{U\}$, which contains unknown homogeneous solution u_H and $\frac{\partial u_H}{\partial n}$ values on the boundary, one can easily compute u_H anywhere inside the region through Equation 41. To find the solution u to Equation 1 with the boundary conditions of Equations 2 and 3, the particular solution u_p obtained from Equation 17 is added to u_H (solution to homogeneous modified Helmholtz equation) as

$$u(x, y) = u_H(x, y) + u_p(x, y). \quad (46)$$

NUMERICAL RESULTS AND DISCUSSION

Consider the following scalar Helmholtz equation [1,15]:

$$\nabla^2 u - \lambda u = f \qquad \text{in} \quad \Omega, \quad (47)$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \Gamma, \quad (48)$$

where

$$f = 2(x - y) + \lambda\left[\frac{x^2 - y^2}{2} - \frac{x^3 - y^3}{3}\right], \quad (49)$$

$\Omega = 0 \leq x \leq 1 \cup 0 \leq y \leq 1$ is a square region and Γ is the boundary of Ω. The exact solution is independent of λ and is given by

$$u(x, y) = \frac{y^2 - x^2}{2} - \frac{y^3 - x^3}{3}. \quad (50)$$

This problem was solved by Tsuchimoto et al. [1] by using the BEM with constant elements. The authors treated the Helmholtz equation as the Laplace equation with an inhomogeneous term and used the fundamental solution of the Laplace equation. By doing so, they were restricted to small values of λ, since the fundamental solution of the modified Helmholtz equation $K_0(kr)$ behaves like $log(kr)$ (fundamental solution of Laplace equation) only for small values of k, which is $\sqrt{\lambda}$ in this case. Furthermore, results for $\lambda = 10^{-6}$ were produced in the computations. The domain was also divided into cells and numerical integration was performed for the domain integral, which requires enormous computational time.

In the present study, the same problem, Equation 47–49, was solved by using BEM with linear elements, taking the inhomogeneous modified Helmholtz equation as it is in Equation 47 and using its fundamental solution $K_0(kr)$ in the formulation. This allows for consideration of a wide range of values for $\lambda(10^{-6}$ to $10)$. On the other hand, by removing the domain integral with the help of radial basis functions and solving a discretized boundary integral equation only, a considerable amount of computational time is saved.

For the approximation of the source function $g(x, y)$ in terms of radial functions $\psi_{-1/2}, \psi_{-3/2}$ and $\psi_{1/2}$, the collocation method was used on a regular grid with step size $h = 0.1(N = 121)$ on both directions x and y. The best value for parameters β_i's, is found to be $h/6$. For solving the system of Equation 16, the Gauss elimination with complete pivoting (L2ARG matrix solver from IMSL library) was used. The absolute error on the agreement with $g(x, y)$ inside the region was 10^{-8}. Then, with those coefficients α_i's, an approximate particular solution was obtained from Equation 17 with the use of radial function $\psi_{1/2}$. The normal derivative of the particular solution, which is required in the solution of the homogeneous Helmholtz equation by the BEM, was also obtained from Equation 17 by differentiation with respect to the normal.

The boundary of the region Ω was divided into 40 linear elements ($M = 40$) which resulted in a 40×40 system of algebraic equations for $\{U\}$. Throughout the computations, double precision was used and the Bessel functions K_0 and K_1 were computed by using the subroutines from IMSL library in double precision. For solving this 40 × 40 system of equations also, the solver L2ARG was used. All the curves have been drawn using the MATLAB package, which uses linear interpolation.

Figures 2, 3, 4 and 5 show the solution u at $y = 0.5$ $(0 \leq x \leq 1)$, for several values of parameter λ and the agreement with exact solution. One can easily notice that the fundamental solution $K_0(kr)$ is not a problem for both small and large values of λ. In Figures 6 and 7, similar behaviors are shown for u at $x = 0.5$ $(0 \leq y \leq 1)$ and again for several values of λ. Finally, the effect of parameters β_i is tested and is shown in Figure 8. The best value is found to be $h/6$ as mentioned above.

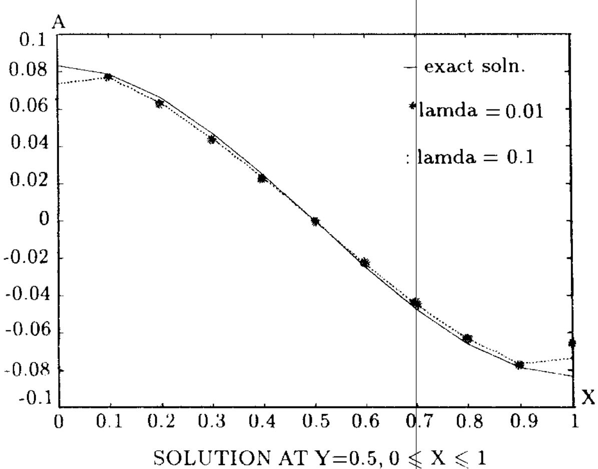

Figure 2. Solution at $y = 0.5, 0 \leq x \leq 1$, for $\lambda = 0.01$ and $\lambda = 0.1$.

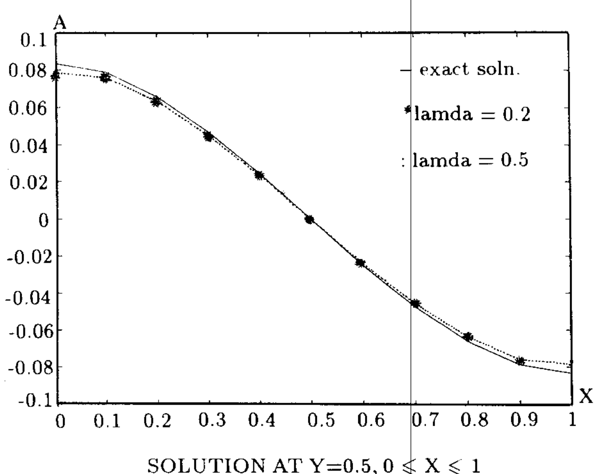

Figure 3. Solution at $y = 0.5, 0 \leq x \leq 1$, for $\lambda = 0.2$ and $\lambda = 0.5$

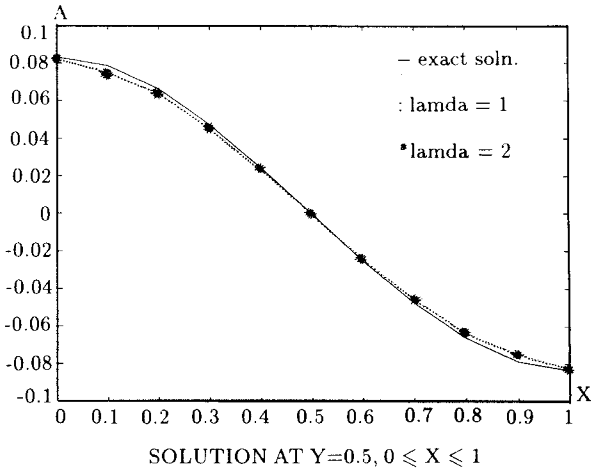

Figure 4. Solution at $y = 0.5, 0 \leq x \leq 1$, for $\lambda = 1$ and $\lambda = 2$.

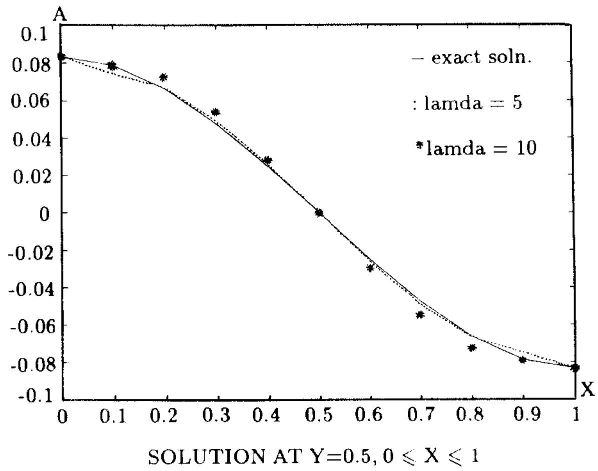

Figure 5. Solution at $y = 0.5, 0 \leq x \leq 1$, for $\lambda = 5$ and $\lambda = 10$.

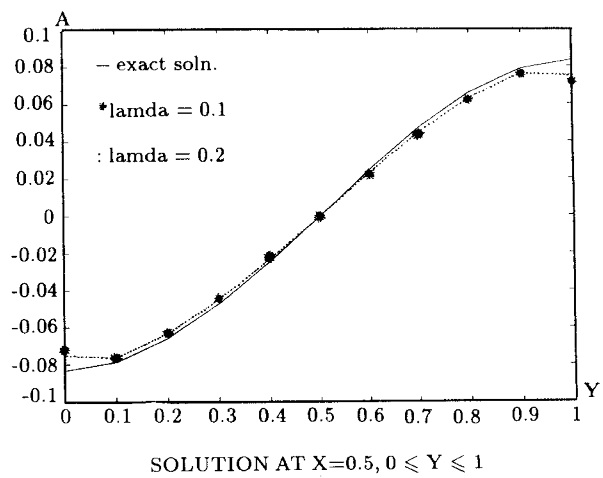

Figure 6. Solution at $x = 0.5, 0 \leq y \leq 1$, for $\lambda = 0.1$ and $\lambda = 0.2$.

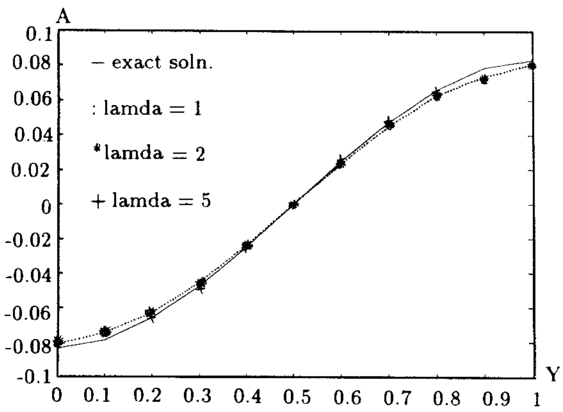

Figure 7. Solution at $x = 0.5, 0 \leq y \leq 1$, for $\lambda = 1, \lambda = 2$ and $\lambda = 5$.

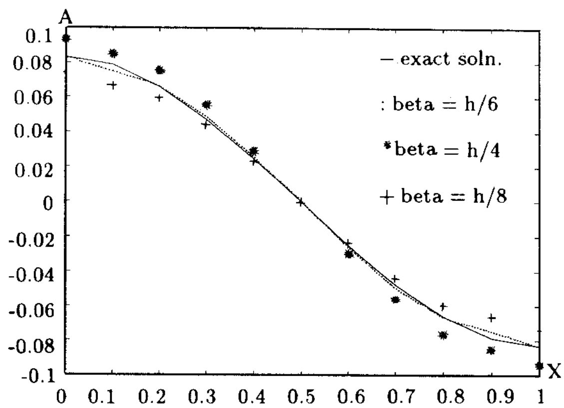

Figure 8. Solution at $y = 0.5, 0 \leq x \leq 1$, for $\beta = h/6, \beta = h/4$ and $\beta = h/8$.

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