Transaction B: Mechanical Engineering
Vol. 16, No. 2, pp. 189–195

Research Note

Traveling Wave Solutions of the Sine-Gordon and the Coupled Sine-Gordon Equations Using the Homotopy-Perturbation Method

A. Sadighi[1], D.D. Ganji[1,*] and B. Ganjavi[2]

Abstract. *In this research, the Homotopy-Perturbation Method (HPM) has been used for solving sine-Gordon and coupled sine-Gordon equations, which have a wide range of applications in physics. HPM deforms a difficult problem into a simple one which can be easily solved. The results obtained by HPM are then compared with those of the Adomian Decomposition Method (ADM). The method has been shown to effectively, easily and accurately solve a large class of nonlinear problems with approximations converging rapidly to accurate solutions.*

Keywords: *Sine-Gordon equation; Coupled sine-Gordon equation; Homotopy-perturbation method; Traveling wave solution.*

INTRODUCTION

Most of the problems and phenomena in different fields of science occur nonlinearly, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, thermo-elasticity and chemical physics. Except in a limited number of these problems, we encounter difficulties in finding their exact analytical solutions. Therefore, approximate analytical solutions, such as Backlund transformation [1], Darboux transformation [2], Hirota's bilinear method [3], the tanh method [4,5], the sine-cosine method [5,6], the homogeneous balance method [7,8], the exp-function method [9-11] the Adomian decomposition method [12-14], the variational iteration method [15-20] and the homotopy-perturbation method [21-29], are introduced, among which the homotopy perturbation method [21-29] is the most effective and convenient for both weakly and strongly nonlinear problems.

HPM was first proposed by He. The method does not depend on a small parameter in the equation. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter, $p \in [0, 1]$, which is considered as a "small parameter". HPM was successfully applied to different branches of science and engineering, such as: nonlinear oscillation equations [30,31], heat transfer equations [32-34], mechanics of fluids [35,36] and etc.

The main objective of this paper is to employ HPM for solving sine-Gordon and coupled sine-Gordon equations. The capability, effectiveness and convenience of the method are revealed by obtaining the analytical solutions of the models and comparing with ADM.

FUNDAMENTALS OF THE HOMOTOPY PERTURBATION METHOD

To illustrate the basic ideas of the method, we consider the following equation [21]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

1. *Department of Mechanical Engineering, Babol University of Technology, Babol, P.O. Box 484, Iran.*
2. *Department of Civil Engineering, Babol University of Technology, Babol, P.O. Box 484, Iran.*
*. *Corresponding author. E-mail: ddg_davood@yahoo.com*

Received 9 January 2006; received in revised form 23 July 2007; accepted 12 September 2007

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ the boundary of the domain, Ω.

A can be divided into two parts, which are L and N, where L is linear and N is nonlinear. Equation 1 can, therefore, be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{3}$$

The Homotopy perturbation structure is shown as follows:

$$H(\nu, p) = (1-p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)] = 0, \tag{4}$$

where:

$$\nu(r, p) : \Omega \times [0, 1] \to R. \tag{5}$$

In Equation 4, $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. It can be assumed that the solution of Equation 4 can be written as a power series in p, as follows:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots, \tag{6}$$

and the best approximation for the solution is:

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots. \tag{7}$$

The above convergence is discussed in [21,22].

IMPLEMENTATION OF THE METHOD

In order to illustrate the advantages and accuracy of HPM, we will consider the sine-Gordon nonlinear hyperbolic equation and the coupled sine-Gordon equation.

Sine-Gordon Equation

We first study the sine-Gordon equation in the form of [37]:

$$u_{tt} - u_{xx} + \sin u = 0, \quad -\infty < x < \infty, \tag{8}$$

subject to the initial conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 4 \sec h(x), \tag{9}$$

which has a wide range of applications in physics, not only in relativistic field theories but also in solid-state physics, nonlinear optics, etc. It also appears in a number of other physical applications, including the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of a rigid pendulum attached to a stretched wire and dislocations in crystals.

In order to solve Equation 8, using HPM, we construct the following homotopy:

$$(1-p)\nu_{tt} + p(\nu_{tt} - \nu_{xx} + \sin \nu) = 0. \tag{10}$$

Computing the Taylor series expansion of $\sin u$ about the point zero and substituting into Equation 10, Equation 10 transforms to:

$$(1-p)\nu_{tt} + p(\nu_{tt} - \nu_{xx} + \nu - \frac{1}{3!}\nu^3 + \frac{1}{5!}\nu^5 - \cdots) = 0. \tag{11}$$

Substituting ν from Equation 6 into Equation 11 and rearranging, based on the powers of p-terms, we have:

$$p^0 : \frac{\partial^2 \nu_0}{\partial t^2} = 0, \tag{12}$$

$$p^1 : \frac{\partial^2 \nu_1}{\partial t^2} - \frac{\partial^2 \nu_0}{\partial x^2} + \nu_0 - \frac{1}{6}\nu_0^3 + \frac{1}{120}\nu_0^5 = 0, \tag{13}$$

$$p^2 : \frac{\partial^2 \nu_2}{\partial t^2} - \frac{\partial^2 \nu_1}{\partial x^2} + \nu_1 - \frac{1}{2}\nu_0^2\nu_1 + \frac{1}{24}\nu_0^4\nu_1 = 0. \tag{14}$$

Solving Equations 12 to 14, we obtain:

$$\nu_0 = \frac{4t}{\cosh(x)}, \tag{15}$$

$$\nu_1 = -\frac{64t^7}{315\cosh^5(x)} + \frac{8t^5}{15\cosh^3(x)} - \frac{4t^3}{3\cosh^3(x)}, \tag{16}$$

$$\begin{aligned}\nu_2 = {} & \frac{512t^{13}}{36855\cosh^9(x)} - \frac{128t^{11}}{1925\cosh^7(x)} \\ & + \frac{4t^9(-4290\cosh^4(x) + 143000\cosh^2(x))}{2027025\cosh^9(x)} \\ & - \frac{4t^7(51480\cosh^6(x) - 205920\cosh^4(x))}{2027025\cosh^9(x)} \\ & + \frac{4t^5(-270270\cosh^6(x) + 405405\cosh^4(x))}{2027025\cosh^9(x)}.\end{aligned} \tag{17}$$

The solution of the sine-Gordon equation (Example 1), Equation 8, when $p \to 1$, will be as follows:

$$\nu = \nu_0 + \nu_1 + \nu_2, \tag{18}$$

which is in good agreement with that obtained by ADM [37]. The behavior of $u(x, t)$ obtained by HPM, with different values of time, is shown in Figure 1. Figure 2 shows that the smaller the t is, the more accurate the numerical solution obtained by HPM is. As t becomes closer to 1, more terms must be considered for more accurate answers.

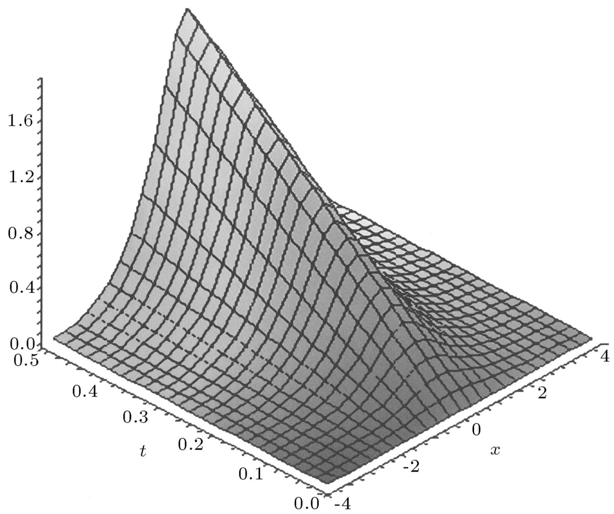


Figure 1. The behavior of $u(x,t)$ obtained by HPM for example 1.

Coupled Sine-Gordon Equation

We next consider a system of a coupled sine-Gordon equation (Example 2) in the form of:

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \tag{19}$$

$$w_{tt} - c^2 w_{xx} = \sin(u - w), \tag{20}$$

which models one-dimensional nonlinear wave processes in two component media [38]. The coupled sine-Gordon equation generalizes the Frenkel-Kontorova dislocation model [39].

In order to solve the system of coupled sine-Gordon equations (Equations 19 and 20), using HPM, the following homotopy for these equations should be constructed as follows:

$$(1 - p)u_{tt} + p(u_{tt} - u_{xx} + \delta^2 \sin(u - w)) = 0, \tag{21}$$

$$(1 - p)w_{tt} + p(w_{tt} - c^2 w_{xx} - \sin(u - w)) = 0. \tag{22}$$

Computing the Taylor series expansion of $\sin(u - w)$ about the point zero and substituting into Equations 21 and 22, these equations transform into:

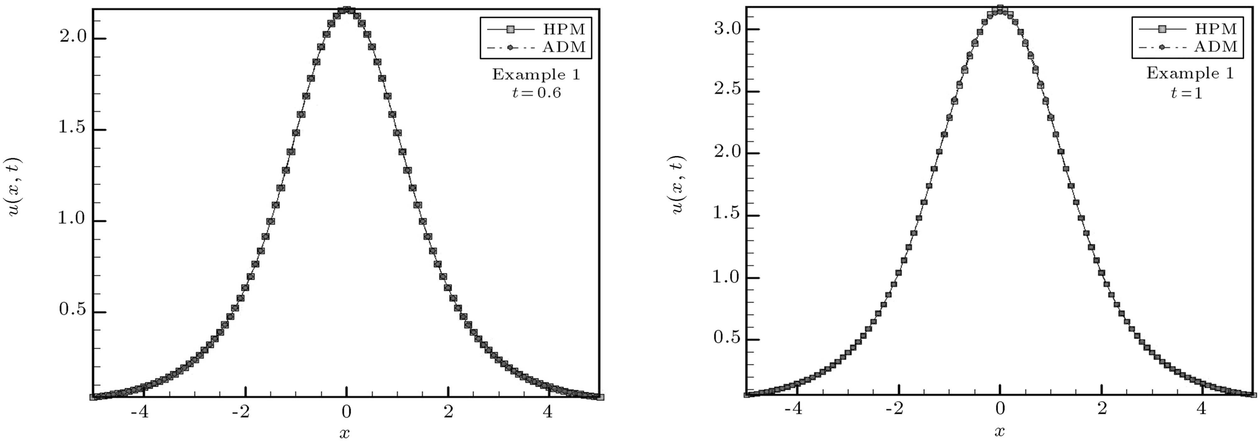
$$(1 - p)u_{tt} + p\left(u_{tt} - u_{xx} + \delta^2\left(u - w - \frac{1}{3!}(u - w)^3 + \frac{1}{5!}(u - w)^5\right)\right) = 0, \tag{23}$$

$$(1 - p)w_{tt} + p\Big(w_{tt} - c^2 w_{xx} - (u - w) + \frac{1}{3!}(u - w)^3 - \frac{1}{5!}(u - w)^5\Big) = 0. \tag{24}$$

Substituting u and w from Equation 6 into Equation 11 and rearranging, based on the powers of p-terms, we have:

$$p^0 : \begin{cases} \frac{\partial^2 u_0}{\partial t^2} = 0 \\ \frac{\partial^2 \nu_0}{\partial t^2} = 0 \end{cases} \tag{25}$$

$$p^1 : \begin{cases} \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} + \delta^2(u_0 - w_0) \\ \quad + \frac{1}{2}\delta^2(u_0^2 w_0 - u_0 w_0^2) + \frac{1}{6}\delta^2(w_0^3 - u_0^3) \\ \quad + \frac{1}{12}\delta^2(w_0^2 u_0^3 - w_0^3 u_0^2) \\ \quad + \frac{1}{24}\delta^2(w_0^4 u_0 - w_0 u_0^4) \\ \quad + \frac{1}{120}\delta^2(u_0^5 - w_0^5) = 0 \\ \frac{\partial^2 w_1}{\partial t^2} - c^2 \frac{\partial^2 w_0}{\partial x^2} + w_0 - u_0 \\ \quad + \frac{1}{2}(u_0 w_0^2 - u_0^2 w_0) + \frac{1}{6}(u_0^3 - w_0^3) \\ \quad + \frac{1}{12}(w_0^3 u_0^2 - w_0^2 u_0^3) + \frac{1}{24}(w_0 u_0^4 - w_0^4 u_0) \\ \quad + \frac{1}{120}(w_0^5 - u_0^5) = 0 \end{cases} \tag{26}$$



(a) (b)

Figure 2. The comparison of the results obtained by HPM and ADM for example 1, at: (a) $t = 0.6$, and (b) $t = 1$.

$$p^2:\begin{cases}\frac{\partial^2 u_2}{\partial t^2}-\frac{\partial^2 u_1}{\partial x^2}\\ \quad+\delta^2(u_1-w_1+u_0u_1w_0-u_0w_0w_1)\\ \quad+\frac{1}{2}\delta^2(u_0^2w_1-w_0^2u_1)+\frac{1}{2}\delta^2(w_0^2w_1-u_0^2u_1)\\ \quad+\frac{1}{4}\delta^2(w_0^2u_0^2u_1-w_0^2u_0^2w_1)\\ \quad+\frac{1}{6}\delta^2(w_0w_1u_0^3-u_0u_1w_0^3)\\ \quad+\frac{1}{6}\delta^2(w_0^3w_1u_0-w_0u_0^3u_1)\\ \quad+\frac{1}{24}\delta^2(u_0^4u_1-w_0^4w_1+w_0^4u_1-u_0^4w_1)=0\\ \frac{\partial^2 w_2}{\partial t^2}-c^2\frac{\partial^2 w_1}{\partial x^2}+u_0w_0w_1\\ \quad-u_0u_1w_0+1/2(u_0^2u_1-w_0^2w_1+w_0^2u_1-u_0^2w_1)\\ \quad+\frac{1}{4}(w_0^2u_0^2w_1-w_0^2u_0^2u_1)\\ \quad+\frac{1}{6}(u_0u_1w_0^3-w_0w_1u_0^3+w_0u_0^3u_1-w_0^3w_1u_0)\\ \quad+\frac{1}{24}(w_0^4w_1-u_0^4u_1+u_0^4w_1-w_0^4u_1)\\ \quad+w_1-u_1=0\end{cases} \tag{27}$$

Solving Equations 25 to 27, we obtain:

$$u_0=A\cos(kx), \tag{28}$$

$$w_0=0, \tag{29}$$

$$u_1=\frac{1}{2}t^2\Big(-A\delta^2\cos(kx)-Ak^2\cos(kx)$$
$$-\frac{1}{120}A^5\delta^2\cos^5(kx)+\frac{1}{6}A^3\delta^2\cos^3(kx)\Big), \tag{30}$$

$$w_1=\frac{1}{2}t^2\Big(A\cos(kx)-\frac{1}{6}A^3\cos^3(kx)+$$
$$\frac{1}{120}A^5\cos^5(kx)\Big), \tag{31}$$

$$u_2=\frac{1}{12}t^4A\cos(kx)\Big(\frac{1}{2}(\delta^2+\delta^4+k^4+k^2\delta^2A^2)$$
$$-\frac{1}{3}(\delta^2+\delta^4)A^2\cos^2(kx)$$
$$+\frac{1}{15}(\delta^2+\delta^4)A^4\cos^4(kx)$$
$$-\frac{1}{180}(\delta^2+\delta^4)A^6\cos^6(kx)$$
$$+\frac{1}{8}k^2\delta^2A^4\cos^4(kx)$$
$$+\frac{1}{5760}(\delta^2+\delta^4)A^8\cos^8(kx)$$
$$-\frac{1}{12}k^2\delta^2A^4\cos^2(kx)$$
$$+k^2\delta^2(1-A^2\cos^2(kx))\Big), \tag{32}$$

$$w_2=-t^4A\cos(kx)\Big(-\frac{1}{2160}(1+\delta^2)A^6\cos^6(kx)$$
$$+\frac{1}{24}(1+k^2+\delta^2+c^2k^2(1+A^2))$$
$$-\frac{1}{36}(1+\delta^2)A^2\cos^2(kx)$$
$$+(1+\delta^2)A^8\cos^8(kx)$$
$$-\frac{1}{48}(1+3c^2)k^2A^2\cos^2(kx)$$
$$+\Big(\frac{1}{576}(1+5c^2)k^2$$
$$+\frac{1}{180}(1+\delta^2)\Big)A^4\cos^4(kx)$$
$$-\frac{1}{144}c^2k^2A^4\cos^2(kx)\Big). \tag{33}$$

The solution of the coupled sine-Gordon equation (Equations 19 and 20), when $p\rightarrow 1$, will be as follows:

$$u=u_0+u_1+u_2, \tag{34}$$

$$w=w_0+w_1+w_2, \tag{35}$$

which are in excellent agreement with those obtained by ADM [38]. The behavior of $u(x,t)$ and $w(x,t)$ obtained by HPM, with different values of time, are shown in Figure 3. The accuracy and efficiency of HPM at some fixed values of time, compared with ADM, are shown in Figures 4 and 5.

CONCLUSION

In this study, we have successfully developed HPM for solving sine-Gordon and coupled sine-Gordon equations. It is apparently seen that HPM is a very powerful and efficient technique for finding solutions for wide classes of nonlinear problems in the form of analytical expressions and presents a rapid convergence for the solutions.

Many of the results attained in this paper confirm the idea that HPM is a powerful mathematical tool for solving different kinds of nonlinear problem arising in various fields of science and engineering. It is worth pointing out that, in order to achieve more accurate solutions for the sine-Gordon and the coupled sine-Gordon, more components of Equation 6 must be taken into account.

In conclusion, HPM provides highly accurate numerical solutions for nonlinear problems, in comparison with other methods. It also does not require a large computer memory and discretization of variables t and x. As mentioned, this method avoids linearization and physically unrealistic assumptions.

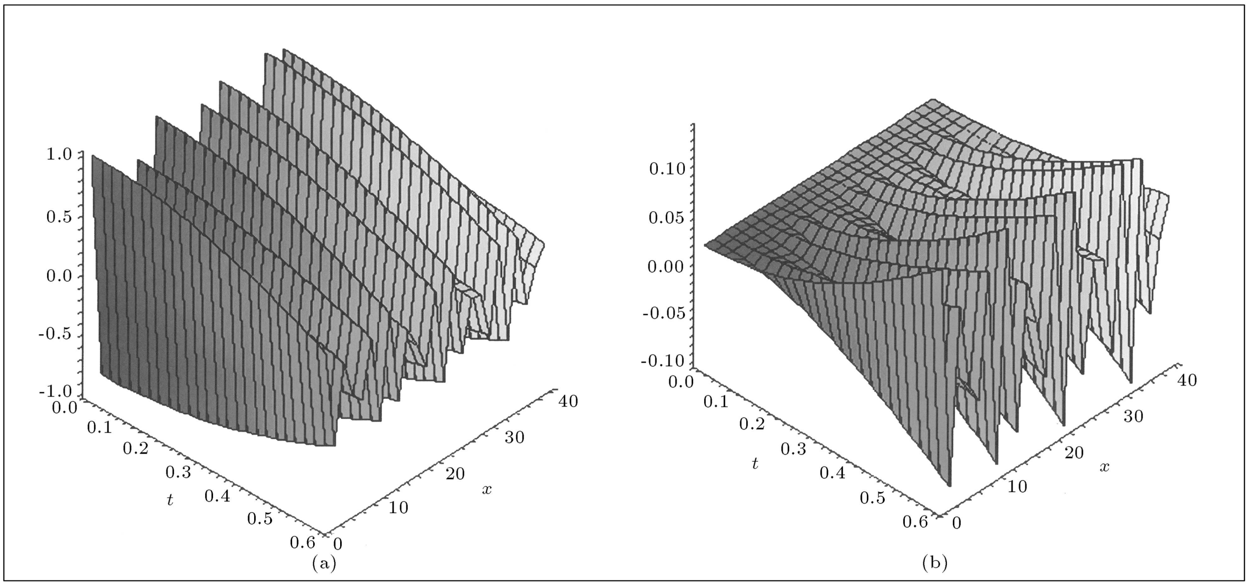


Figure 3. The behavior of: (a) $u(x,t)$ and (b) $w(x,t)$ obtained by HPM at $A = 1$, $c = 1$, $k = 1$ and $\delta = 1$ for example 2.

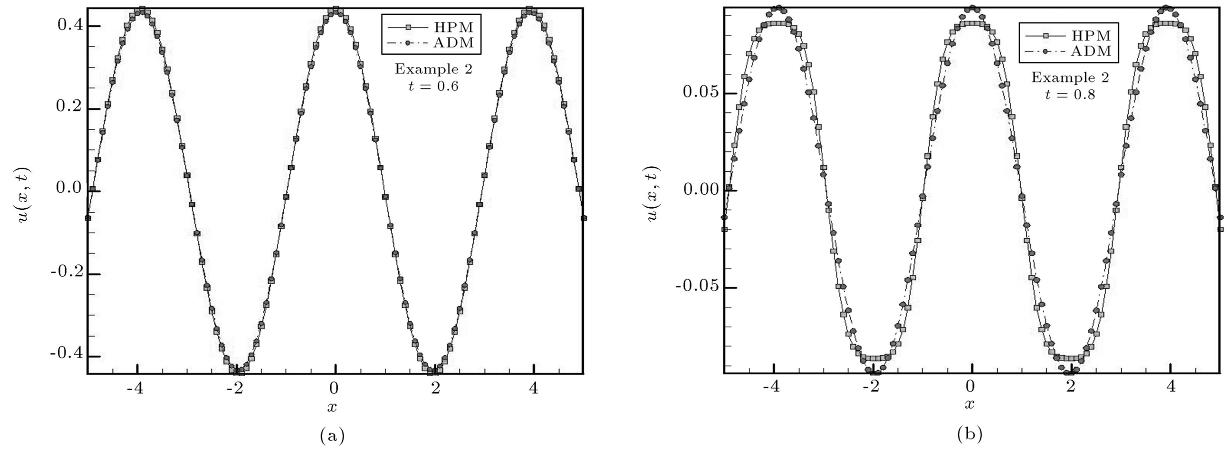


Figure 4. The comparison of the $u(x,t)$ obtained by HPM and ADM at $A = 1$, $c = 1$, $k = 1$ and $\delta = 1$ for example 2, at: (a) $t = 0.6$ and (b) $t = 0.8$.

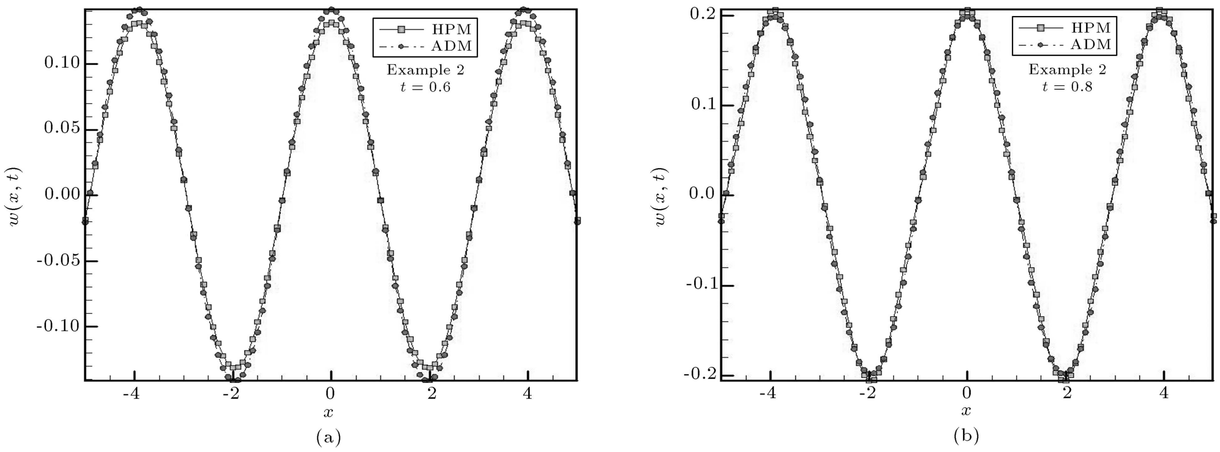


Figure 5. The comparison of the $w(x,t)$ obtained by HPM and ADM at $A = 1$, $c = 1$, $k = 1$ and $\delta = 1$ for example 2, at: (a) $t = 0.6$ and (b) $t = 0.8$.

REFERENCES

1. Denes, P. and Finley, J.D. "Backlund transformations for general PDE's", *Physica D: Nonlinear Phenomena*, **9**, pp. 236-250 (1983).
2. Coely, A. "Backlund and Darboux Transformations", American Mathematical Society, *Providence*, Rhode Island (2001).
3. Xing Biao Hu and Yong-Tang Wu "Application of the Hirota bilinear formalism to a new integrable differential-difference equation", *Physics Letters A*, **246**, pp. 523-529 (1998).
4. Engui, F. "Extended tanh-function method and its applications to nonlinear equations", *Physics Letters A*, **277**, pp. 212-218 (2000).
5. Wazwaz, A.-M. "The tanh and the sine-cosine methods for the complex modified K-dV and the generalized K-dV equations", *Computers & Mathematics with Applications*, **49**, pp. 1101-1112 (2005)
6. Wazwaz, A.-M. "A sine-cosine method for handling nonlinear wave equations", *Mathematical and Computer Modeling*, **40**, pp. 499-508 (2004).
7. Mingliang Wang, Yubin Zhou and Zhibin Li "Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics", *Physics Letters A*, **216**, pp. 67-75 (1996).
8. Engui, F. "Two new applications of the homogeneous balance method", *Physics Letters A*, **265**, pp. 353-357 (2000).
9. He, J.H. and Wu, X.H. "Exp-function method for nonlinear wave equations", *Chaos, Solitons & Fractals*, **30**(3), pp. 700-708 (2006).
10. Zhu, S.D. "Exp-function method for the hybrid-lattice system", *International Journal of Nonlinear Science and Numerical Simulation*, **8**(3), pp. 461-464 (2007).
11. Zhu, S.D. "Exp-function method for the discrete mKdV lattice", *International Journal of Nonlinear Science and Numerical Simulation*, **8**(3), pp. 465-469 (2007).
12. Shawagfeh, N. and Kaya, D. "Comparing numerical methods for the solutions of systems of ordinary differential equations", *Applied Mathematics Letters*, **17**, pp. 323-328 (2004).
13. Adomian, G. "Solution of physical problems by decomposition", *Computers & Mathematics with Applications*, **27**, pp. 145-154 (1994).
14. Wazwaz, A.M. "Construction of solitary wave solutions and rational solutions for the KdV equation by Adomian decomposition method", *Chaos, Solitons & Fractals*, **12**, pp. 2283-2293 (2001).
15. He, J.H. "Variational iteration method - a kind of non-linear analytical technique: Some examples", *International Journal of Nonlinear Mechanics*, **34**(4), pp. 699-708 (1999).
16. He, J.H. "Variational iteration method - Some recent results and new interpretations", *Journal of Computational and Applied Mathematics*, **207**(1), pp. 3-17 (2007).
17. Sadighi, A. and Ganji, D.D. "Exact solutions of nonlinear diffusion equations by variational iteration method", *Computers and Mathematics with Application*, **54**(7,8), pp. 1112-1121 (2007).
18. He, J.H. and Wu, X.H. "Construction of solitary solution and compaction-like solution by variational iteration method", *Chaos, Solitons & Fractals*, **29**, pp. 108-113 (2006).
19. Biazar, J. and Ghazvini, H. "He's variational iteration method for solving hyperbolic differential equations", *International Journal of Nonlinear Science and Numerical Simulation*, **8**(3), pp. 311-314 (2007).
20. Khaleghi, H., Ganji, D.D. and Sadighi, A. "Application of variational iteration and homotopy-perturbation methods to nonlinear heat transfer equations with variable coefficients", *Numerical Heat Transfer*, **Part A**(52), pp. 25-42 (2007).
21. He, J.H. "A coupling method of a homotopy technique and a perturbation technique for non-linear problems", *International Journal of Nonlinear Mechanics*, **35**(1), pp. 37-43 (2000).
22. He, J.H. "Some asymptotic methods for strongly nonlinear equations", *International Journal of Modern Physics B*, **20**(10), pp. 1141-1199 (2006).
23. He, J.H. "Homotopy perturbation method for solving boundary value problems", *Physics Letters A*, **350**(1-2), pp. 87-88 (2006).
24. He, J.H. "New interpretation of homotopy perturbation method", *International Journal of Modern Physics B*, **20**(18), pp. 2561-2568 (2006).
25. Sadighi, A. and Ganji, D.D. "Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods", *International Journal of Nonlinear Science and Numerical Simulation*, **8**(3), pp. 435-444 (2007).
26. Ganji, D.D. and Sadighi, A. "Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations", *International Journal of Nonlinear Science and Numerical Simulation*, **7**(4), pp. 411-418 (2006).
27. Rafei, M. and Ganji, D.D. "Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method", *International Journal of Nonlinear Science and Numerical Simulation*, **7**(3), pp. 321-328 (2006).
28. Sadighi, A. and Ganji, D.D. "Exact solutions of Laplace equations by homotopy-perturbation and variational iteration methods", *Physics Letters A*, **367**(1-2), pp. 83-87 (2007).
29. Abbasbandy, S. "Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method", *Applied Mathematics and Computation*, **172**(1), pp. 485-490 (2006).
30. Beléndez, A. and Hernández, T. "Application of He's homotopy perturbation method to the duffing-

harmonic oscillator", *International Journal of Nonlinear Science and Numerical Simulation*, **8**(1), pp. 79-88 (2007).

31. He, J.H. "The homotopy perturbation method for nonlinear oscillators with discontinuities", *Applied Mathematics and Computation*, **151**, pp. 287-292 (2004).

32. Ganji, D.D. and Sadighi, A. "Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations", *Journal of Computational and Applied Mathematics*, **207**, pp. 24-34 (2007).

33. Ganji, D.D. and Rajabi, A. "Assessment of homotopy-perturbation and perturbation methods in heat radiation equations", *International Communications in Heat and Mass Transfer*, **33**(3), pp. 391-400 (2006).

34. Ganji, D.D. "The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer", *Physics Letters A*, **355**(4-5), pp. 337-341 (2006).

35. Siddiqui, A.M., Mahmood, R. and Ghori, Q.K. "Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method", *International Journal of Nonlinear Science and Numerical Simulation*, **7**(1), pp. 7-14 (2006).

36. Siddiqui, A.M., Ahmed, M. and Ghori, Q.K. "Couette and Poiseuille flows for non-Newtonian fluids", *International Journal of Nonlinear Science and Numerical Simulation*, **7**(1), pp. 15-26 (2006).

37. Kaya, D. "A numerical solution of the sine-Gordon equation using the modified decomposition method", *Applied Mathematics and Computation*, **143**, pp. 309-317 (2003).

38. Saha Ray, S. "A numerical solution of the coupled sine-Gordon equation using the modified decomposition method", *Applied Mathematics and Computation*, **175**, pp. 1046-1054 (2006).

39. Braun, O.M. and Kivshar, Yu, S. "Nonlinear dynamics of the Frenkel-Kontorova model", *Phys. Rep.*, **306**, pp. 1-108 (1998).