Transaction B: Mechanical Engineering
Vol. 16, No. 1, pp. 22–39
© Sharif University of Technology, February 2009

SCIENTIA IRANICA

Vibration Analysis of Moderately Thick Rectangular Plates with Internal Line Support Using the Rayleigh-Ritz Approach

Sh. Hosseini-Hashemi[1], **K. Khorshidi**[1,*] **and H. Payandeh**[1]

Abstract. *In this study, the free vibration of moderately thick rectangular plates with several internal line supports was analyzed; the plates having twenty one possible boundary conditions (a combination of clamped, simply supported and free classical boundary conditions). The dimensionless equations of the strain (potential) and kinetic energy were derived, based on the Mindlin plate theory, to study the transverse vibration of moderately thick rectangular plates (in terms of the resultant stress, with consideration of transverse shear deformation and rotatory inertia). The Rayleigh-Ritz method, assuming two dimensional polynomial functions as admissible displacement functions, was applied. Numerical results were presented for a wide range of aspect ratios and thickness to length ratios. The influence of line support location and thickness to length ratio on the frequency parameters was shown graphically.*

Keywords: *Free vibration; Line support; Rectangular Mindlin plate; Rayleigh-Ritz.*

INTRODUCTION

Moderately thick plates are extensively used in modern structures. Analysis of these plates is of great importance for design engineers. The solution of the flexural vibration depends on the boundary conditions of the plate. Rectangular plates are commonly used as structural components in many branches of modern technology, namely, mechanical, aerospace, electronic, marine, optical, nuclear and structural engineering. Thus, the study of their free vibration behavior is very important to the structural designers. In recent years, many researchers have worked on the vibration of plates.

The published work concerning the vibration of such plates is abundant; however, the vast majority is based on the thin plate theory. An excellent reference source may be found in the well-known work of Leissa [1] and his subsequent articles [2-7] published in the Vibration Digest from time to time. His remarkable work on the free vibration of thin rectangular plates [8] also presents comprehensive and accurate analytical results for twenty one distinct cases, which involve all possible combinations of classical boundary conditions.

The thin plate theory neglects the effect of shear deformation and rotatory inertia, which result in the over-estimation of vibration frequencies. This error increases with increasing plate thickness. Improving on the thin plate theory, Mindlin and coworkers [9,10] proposed the so-called first order shear deformation theory for moderately thick plates and incorporated the effect of rotatory inertia. The first order shear deformation plate theory of Mindlin, however, requires a shear correction factor to compensate for the error resulting from the approximation made on the non-uniform shear strain distribution.

Over the years, solutions for eigenvalue problems of thick plates have been represented, using several different approximate methods. Dawe and Roufaeil [11] treated the free vibration of Mindlin rectangular plates using the Rayliegh-Ritz method. They used the Timoshenko beam functions as the admissible functions of the plate. Liew et al. [12,13] investigated the free vibration of Mindlin rectangular plates, respectively, by us-

1. *Impact Laboratory, Department of Mechanical Engineering, Iran University of Science and Technology, P.O. Box 16846-13114, Tehran, Iran.*
*. *Corresponding author. E-mail: k_khorshidi@iust.ac.ir*

Received 12 April 2006; received in revised form 29 September 2007; accepted 11 December 2007

ing two dimensional polynomials and one-dimensional Gram-Schmidt polynomials as the admissible functions of the plate in the Rayliegh-Ritz method. Cheung and Zhou [14] developed a set of static Timoshenko beam functions as the admissible functions to study the vibration of moderately thick rectangular plates by the Rayliegh-Ritz method. The finite element, finite strip, finite layer, collocation and superposition methods have also been used, respectively, by Al Janabi et al. [15], Dawe [16], Cheung and Chakrabarti [17], Mikami and Yoshimura [18] and Gorman [19], to study the eigenvalue problems of thick plates. Moreover, some investigations on the three-dimensional vibrations of rectangular plates have been reported by Srinvas et al. [20], Wittrick [21], Liew et al. [22-24] and Liew and Teo [25]. Zhou and Cheung [26] presented the free vibration of line supported thin rectangular plates using a set of static beam functions. In their work, the eigenfrequency equation for the plate is derived by minimizing the functional energy, by applying the Rayleigh-Ritz method. Xiang e al. obtained an exact solution for the vibration of multi-span moderately thick rectangular plates. They employed the Levy type solution method and the state-space technique to develop an analytical and exact approach for the free vibration of thin plates [27] and moderately thick plates [28]. In their work, the plates, having two opposite edges, have simply supported boundary conditions, namely SSSS, SCSS, SCSC, SSSF, SFSF and SCSF.

The study of the vibration behavior of multi span rectangular plates, having classical boundary conditions, can be found in [26] for thin plates and in [28] for six cases of moderately thick plates. No data concerning the vibration behavior of multi-span moderately thick rectangular plates for the rest of the fifteen boundary conditions are available in the literature. To fill this apparent void, the present work was carried out to provide the vibration analysis for all twenty-one possible classical boundary conditions. In the present work, a new set of two dimensional complete polynomial functions were developed for the vibration behavior of multi-span moderately thick rectangular plates, considering Mindlin's plate theory. The internal line support is set in one or two directions. The nondimensional equations of strain and kinetic energy were derived and the frequency parameters and mode shapes were obtained by applying the Rayleigh-Ritz method. Finally, some numerical results were given for moderately thick rectangular plates with a number of internal line supports in one or two directions. The influence of line support location and thickness to length ratio on the frequency parameters was studied graphically. Numerical results were compared with known values in the literature for thin plates [26] and a Levy type solution [28].

ASSUMPTIONS OF MINDLIN PLATE THEORY

Consider a flat, isotropic, rectangular Mindlin plate of uniform thickness, h, length, a, width, b, modulus of elasticity, E, Poisson's ratio, ν, shear modulus, $G = E/2(1+\nu)$, and density per unit volume, ρ, oriented so that its mid-plane surface contains the x_1 and x_2 axis of a Cartesian Co-ordinate system, (x_1, x_2, x_3), as shown in Figure 1.

The displacements along the x_1 and x_2 axes are denoted by W_1 and W_2, respectively, while the displacement in the direction perpendicular to the undeformed mid-plane surface is denoted by W_3. In the Mindlin plate theory, the displacement components are assumed to be given by:

$$W_1 = -x_3\psi_1(x_1, x_2, t), \tag{1}$$

$$W_2 = -x_3\psi_2(x_1, x_2, t), \tag{2}$$

$$W_3 = \psi_3(x_1, x_2, t), \tag{3}$$

where t is the time, ψ_3 is the transverse displacement, ψ_1 and ψ_2 are the slope, due to bending alone in the respective planes.

Using the displacement field given in Equations 1 to 3, the components of the strains may be expressed as:

$$\varepsilon_{11} = \frac{\partial W_1}{\partial x_1} = -x_3\psi_{1,1}, \tag{4}$$

$$\varepsilon_{22} = \frac{\partial W_2}{\partial x_2} = -x_3\psi_{2,2}, \tag{5}$$

$$\varepsilon_{33} = \frac{\partial W_3}{\partial x_3} = 0, \tag{6}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial W_1}{\partial x_2} + \frac{\partial W_2}{\partial x_1}\right) = -\frac{1}{2}(\psi_{1,2} + \psi_{2,1})x_3, \tag{7}$$

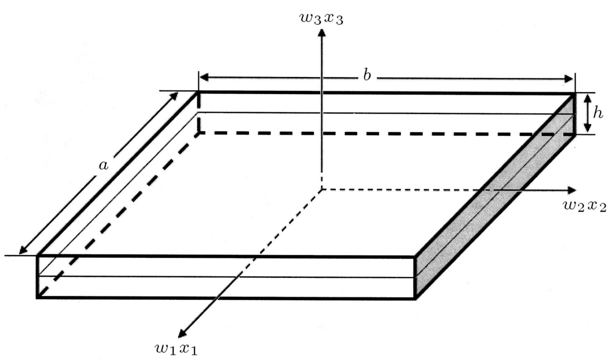


Figure 1. A Mindlin plate with co-ordinate convention.

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial W_1}{\partial x_3} + \frac{\partial W_3}{\partial x_1}\right) = -\frac{1}{2}(\psi_1 - \psi_{3,1}), \tag{8}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial W_3}{\partial x_2} + \frac{\partial W_2}{\partial x_3}\right) = \frac{1}{2}(\psi_2 - \psi_{3,2}). \tag{9}$$

Using Hook's law, the components of the stress may be expressed as:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}), \tag{10}$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}), \tag{11}$$

$$\sigma_{12} = 2G\varepsilon_{12}, \tag{12}$$

$$\sigma_{13} = 2G\varepsilon_{13}, \tag{13}$$

$$\sigma_{23} = 2G\varepsilon_{23}. \tag{14}$$

FORMULATION OF RAYLEIGH-RITZ APPROACH

From the vibration theory of moderately thick plates, the strain and kinetic energies of an elastic isotropic plate in the Cartesian coordinate are as follows:

$$T = \frac{1}{2}\rho \int_V \left(\left(\frac{\partial W_1}{\partial t}\right)^2 + \left(\frac{\partial W_2}{\partial t}\right)^2 + \left(\frac{\partial W_3}{\partial t}\right)^2\right) dV, \tag{15}$$

$$U = \frac{1}{2}\int_V (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13} + 2\sigma_{23}\varepsilon_{23})dV, \tag{16}$$

where V is the volume of the plate, T is the kinetic energy and U is the strain energy. For generality and convenience, the coordinates are normalized, with respect to the plate planar dimensions, and the following nondimensional terms are introduced:

$$X_1 = \frac{x_1}{a}, \qquad X_2 = \frac{x_2}{b}, \qquad \delta = \frac{h}{a}, \qquad \eta = \frac{a}{b},$$

$$\beta = \omega a^2\sqrt{\frac{\rho h}{D}}, \tag{17}$$

where β is the frequency parameter, ω is the natural frequency, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, δ is the thickness to length ratio and η is the aspect ratio. By assumption of the free harmonic vibration and using nondimensional parameters, the maximum strain and kinetic energy of the rectangular plate is given by:

$$T_{\max} = \frac{\eta}{2D}\beta^2 \int_{-0.5}^{0.5}\int_{-0.5}^{0.5} \left\{\frac{\delta^2}{12}(\tilde{\psi}_1^2 + \tilde{\psi}_2^2) + \tilde{\psi}_3^2\right\} dX_1 dX_2, \tag{18}$$

$$U_{\max} = \frac{\eta}{2D}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5} \Big\{\tilde{\psi}_{1,1}^2 + \tilde{\psi}_{2,2}^2\eta^2 + 2D\eta\tilde{\psi}_{1,1}\tilde{\psi}_{2,2} + \frac{1-D}{2}(\eta\tilde{\psi}_{1,2} + \tilde{\psi}_{2,1})^2 + \frac{6(1-\nu)\kappa^2}{\delta^2}\Big[(\tilde{\psi}_1 - \tilde{\psi}_{3,1})^2 + (\tilde{\psi}_2 - \eta\tilde{\psi}_{3,2})^2\Big]\Big\} dX_1 dX_2, \tag{19}$$

where κ is the shear correction factor to account for the fact that the transverse shear strains are not truly independent of the thickness coordinate. Also, $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$ are dimensionless parameters, chosen as below:

$$\tilde{\psi}_1(X_1, X_2) = \psi_1(x_1, x_2, t)e^{-i\omega t},$$

$$\tilde{\psi}_2(X_1, X_2) = \psi_2(x_1, x_2, t)e^{-i\omega t}, \tag{20}$$

$$\tilde{\psi}_3(X_1, X_2) = \psi_3(x_1, x_2, t)\frac{e^{-i\omega t}}{a}, \tag{21}$$

where $\imath = \sqrt{-1}$. In view of nondimensional terms, the energy functional for the Mindlin plate is given by:

$$L = U_{\max} - T_{\max} = \frac{\eta}{2D}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5} \Big\{\tilde{\psi}_{1,1}^2 + \tilde{\psi}_{2,2}^2\eta^2 + 2D\eta\tilde{\psi}_{1,1}\tilde{\psi}_{2,2} + \frac{1-D}{2}(\eta\tilde{\psi}_{1,2} + \tilde{\psi}_{2,1})^2 + \frac{6(1-\nu)\kappa^2}{\delta^2}\Big[(\tilde{\psi}_1 - \tilde{\psi}_{3,1})^2 + (\tilde{\psi}_2 - \eta\tilde{\psi}_{3,2}^2)^2\Big] - \beta^2\left\{\frac{\delta^2}{12}(\tilde{\psi}_1^2 + \tilde{\psi}_2^2) + \tilde{\psi}_3^2\right\}\Big\} dX_1 dX_2. \tag{22}$$

After choosing a set of appropriate admissible functions for $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$, the eigenvalue equation can be derived by applying the Rayleigh-Ritz method to minimizing Equation 22, with respect to the unknown coefficients in these admissible functions. For Mindlin plates, the transverse deflection and bending slopes may be parameterized by two dimensional complete polynomial functions:

$$\tilde{\psi}_1(X_1, X_2) = \sum_{i=0}^{N_1}\sum_{j=0}^{i} a_n X_1^j X_2^{i-j} G_1(X_1, X_2), \tag{23}$$

$$\tilde{\psi}_2(X_1, X_2) = \sum_{i=0}^{N_2} \sum_{j=0}^{i} b_n X_1^j X_2^{i-j} G_2(X_1, X_2), \qquad (24)$$

$$\tilde{\psi}_3(X_1, X_2) = \sum_{i=0}^{N_3} \sum_{j=0}^{i} c_n X_1^j X_2^{i-j} G_3(X_1, X_2), \qquad (25)$$

where $n = (i+1)(i+2)/2 - j$ and $N_i (i = 1, 2, 3)$ is the order of the two dimensional polynomial. a_n, b_n and c_n are unknown coefficients and G_1, G_2 and G_3 are fundamental functions, which satisfy the geometric boundary conditions of rectangular Mindlin plates. The fundamental functions are written as follows:

$$G_i(X_1, X_2) = (X_1 + 0.5)^{0 \text{ OR } 1}(X_1 - 0.5)^{0 \text{ OR } 1} (X_2 + 0.5)^{0 \text{ OR } 1}(X_2 - 0.5)^{0 \text{ OR } 1},$$

$$i = 1, 2, 3. \qquad (26)$$

The geometric boundary conditions of the Mindlin plates can be expressed as:

For simply supported edges:

$$\tilde{\psi}_1 = 1, \quad \tilde{\psi}_2 = 0, \quad \tilde{\psi}_3 = 0 \qquad \text{In } X_1 - \text{direction},$$

$$\tilde{\psi}_1 = 0, \quad \tilde{\psi}_2 = 1, \quad \tilde{\psi}_3 = 0 \qquad \text{In } X_2 - \text{direction}. \qquad (27)$$

For clamped edges:

$$\tilde{\psi}_2 = 0, \quad \tilde{\psi}_2 = 0, \quad \tilde{\psi}_3 = 0 \qquad \text{In } X_1 - \text{direction},$$

$$\tilde{\psi}_2 = 0, \quad \tilde{\psi}_2 = 0, \quad \tilde{\psi}_3 = 0 \qquad \text{In } X_2 - \text{direction}. \qquad (28)$$

For free edges:

$$\tilde{\psi}_1 = 1, \quad \tilde{\psi}_2 = 1, \quad \tilde{\psi}_3 = 1 \qquad \text{In } X_1 - \text{direction},$$

$$\tilde{\psi}_1 = 1, \quad \tilde{\psi}_2 = 1, \quad \tilde{\psi}_3 = 1 \qquad \text{In } X_2 - \text{direction}. \qquad (29)$$

For example, the fundamental functions for a SCFC Mindlin rectangular plate without internal line support (as shown in Figure 2, case 14), can be written as:

$$G_1(X_1, X_2) = (X_1 + 0.5)^0 (X_1 - 0.5)^0 (X_2 + 0.5)^1 (X_2 - 0.5)^1,$$

$$G_2(X_1, X_2) = (X_1 + 0.5)^1 (X_1 - 0.5)^0 (X_2 + 0.5)^0 (X_2 - 0.5)^1,$$

$$G_3(X_1, X_2) = (X_1 + 0.5)^1 (X_1 - 0.5)^0 (X_2 + 0.5)^1 (X_2 - 0.5)^1. \qquad (30)$$

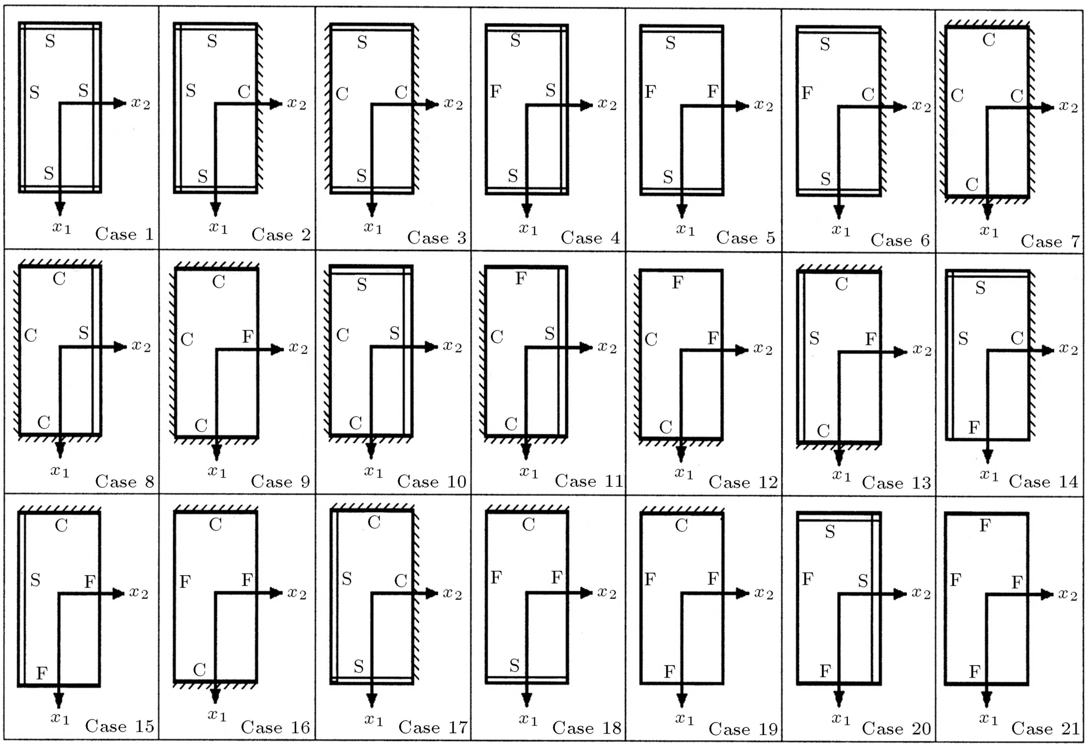


Figure 2. Boundary conditions of Mindlin plate analyzed.

After minimizing Equation 22, the governing eigenvalue equation can be derived as:

$$([K]-\beta^2[M])\begin{Bmatrix}\{a_n\}\\ \{b_n\}\\ \{c_n\}\end{Bmatrix}=0, \tag{31}$$

where $[K]$ is the stiffness matrix and $[M]$ is the mass matrix. Because not all elements of $\left[\{a_n\} \quad \{b_n\} \quad \{c_n\}\right]^T$ are equal to zero, from Equation 31, one has:

$$\det([K]-\beta^2[M])=0. \tag{32}$$

The frequency parameter, β, is obtained by solving the generalized eigenvalue problem defined by Equation 32. The unknown coefficients vector in Equation 31 is the Null space of the $([K]-\beta^2[M])$ matrix for any mode sequence.

APPLYING THE RAYLEIGH-RITZ APPROACH FOR VIBRATIONS OF MULTI-SPAN MINDLIN RECTANGULAR PLATES

It is considered that the plate, as shown in Figure 3, consists of four spans that were divided at the locations of an internal line support in the X_1-direction and an internal line support in the X_2-direction. For analysis, this plate was separated into four plates. The boundary conditions of each span contain classical boundary conditions along the edges and special boundary conditions along the internal line supports.

Along the interface between the spans, for example span (I) and span (II), the following essential and natural boundary conditions must hold to ensure the continuity of the plate and satisfaction of the internal line support conditions:

Along the line support:

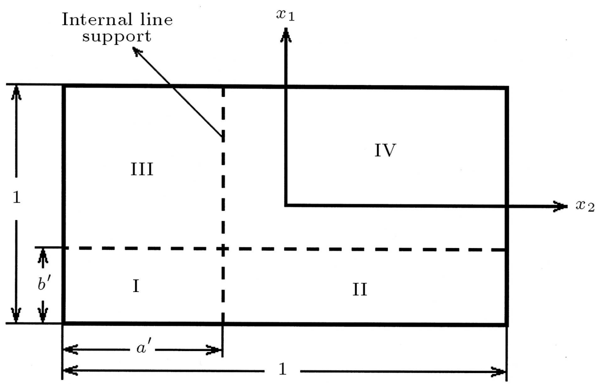


Figure 3. A Mindlin plate with four span in two directions.

$$\tilde{\psi}_3|^I=0, \tag{33}$$

$$\tilde{\psi}_3|^{II}=0, \tag{34}$$

$$\tilde{\psi}_{3,3}|^I=\tilde{\psi}_{3,1}|^{II}, \tag{35}$$

$$\tilde{\psi}_2|^I=\tilde{\psi}_2|^{II}, \tag{36}$$

$$\tilde{\psi}_1|^I=\tilde{\psi}_1|^{II}, \tag{37}$$

$$M_{11}|^I=M_{11}|^{II}. \tag{38}$$

For free vibration analysis of the plate, the domain of integration was separated. For example, for the four-span plate, as shown in Figure 3, the domain of integration is given by:

$$\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}()dX_1dX_2=\int_{-0.5}^{-0.5+a'}\int_{-0.5}^{-0.5+b'}()dX_1dX_2$$
$$+\int_{-0.5}^{-0.5+a'}\int_{-0.5+b'}^{0.5}()dX_1dX_2$$
$$+\int_{-0.5+a'}^{0.5}\int_{-0.5}^{-0.5+b'}()dX_1dX_2$$
$$+\int_{-0.5+a'}^{0.5}\int_{-0.5+b'}^{0.5}()dX_1dX_2. \tag{39}$$

All the results presented hereafter are for a rectangular Mindlin plate, having Poisson ratio $\nu=0.3$, the shear correction factor is $\kappa=\sqrt{5/6}$ and $N_1=N_2=N_3=N$.

Convergence Studies

After having developed a method of analysis in the preceding section, a primal question is: "How many terms of a two dimensional complete polynomial (Equations 23-25) were required to obtain reasonably convergent results?" This question was answered through several numerical studies, as shown in Tables 1 to 6, for some different boundary conditions.

Tables 1 to 6 illustrate the accuracy, convergency and usefulness of the approach described above. From the results presented in these tables, it is observed that, for the first ten frequency parameters of rectangular Mindlin plates with two equal spans in an X_1-direction, the size of terms of the two dimensiondal polynomial is $N=11$.

Table 1. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for SFSF boundary condition (case 5) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
1	1.1566	1.1566	1.1555	1.1555	1.1540	1.1529	1.1529
2	1.4000	1.2617	1.2617	1.2544	1.2544	1.2463	1.2463
3	3.0187	3.0187	2.6646	2.6646	2.6587	2.6528	2.6528
4	4.5556	3.8591	3.8591	3.2165	6.3265	3.1562	3.1562
5	4.7648	3.9095	3.9095	3.8859	3.8859	3.8838	3.8838
6	6.5902	4.5556	3.9774	3.9774	3.9424	3.9424	3.9324
7	10.6650	6.4946	6.4946	5.4951	5.4951	5.4106	5.4106
8	11.2676	9.5555	6.6596	6.6596	5.6880	5.6880	5.6629
9	11.4672	10.6650	7.2034	7.2034	5.9790	5.8337	5.8337
10	12.5240	11.2676	8.0827	8.0827	7.9561	6.8379	6.8378

Table 2. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for SSSF boundary condition (case 4) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
1	1.2340	1.2073	1.2036	1.2001	1.2000	1.1973	1.1957
2	2.3552	2.1925	2.1311	2.1111	2.0958	2.0900	2.0874
3	4.0878	3.1334	3.1194	2.9638	2.9579	2.9258	2.9247
4	4.4619	3.9732	3.9275	3.9122	3.9091	3.9076	3.9045
5	5.8913	5.5889	4.9408	4.7911	4.7653	4.7281	4.7212
6	9.6110	7.0022	5.7109	5.2212	5.0311	4.9620	4.9333
7	10.7125	7.6849	6.4289	5.8538	5.6695	5.6237	5.5888
8	10.9521	9.4502	8.1371	7.4678	6.9624	6.3506	6.3269
9	16.7501	11.5272	10.2920	8.0522	7.8457	7.5745	7.4613
10	18.5986	13.0911	10.5893	8.8053	7.9776	7.9707	7.9668

Table 3. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for SCSS boundary condition (case 2) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
1	2.3108	2.1379	2.0990	2.0653	2.0541	2.0477	2.0481
2	3.1974	2.9671	2.8310	2.7785	2.7531	2.7311	2.7287
3	5.4532	5.1853	4.7826	4.7501	4.6883	4.6800	4.6708
4	7.3194	5.5081	5.2884	5.0198	4.9389	4.8961	4.9037
5	7.6125	5.7188	5.3533	5.1257	5.0896	5.0328	5.0255
6	12.6581	10.1995	6.8593	6.7650	6.2865	6.2350	6.1570
7	13.1905	10.2871	8.3432	8.1001	7.5032	7.4395	7.3292
8	13.4555	10.7083	9.4893	8.8533	8.8151	8.3440	8.3318
9	14.2582	12.3057	10.1669	9.1221	8.8966	8.6523	8.6780
10	18.4549	14.8508	11.9872	9.3214	9.0184	8.9318	8.8754

Table 4. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for SSCC boundary condition (case 10) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	$N=11$
1	2.8679	2.7164	2.6327	2.5894	2.5637	2.5446	2.5424	2.5289
2	4.4599	3.5529	3.4209	3.2431	3.2041	3.1451	3.1313	3.1133
3	6.4961	5.7405	5.5955	5.5074	5.4704	5.4636	5.4527	5.4491
4	7.9128	6.7942	6.0899	5.9535	5.6737	5.5503	5.5375	5.4827
5	8.7816	7.3894	6.3197	5.9535	5.8085	5.7949	5.7501	5.7552
6	12.3833	10.5217	8.6831	7.6553	7.3615	6.8523	6.8485	6.7175
7	12.7345	10.8626	9.8252	8.8404	8.4895	8.3433	8.2450	8.2366
8	13.4848	11.5294	9.9600	9.7693	9.6106	9.5239	9.1348	9.1915
9	15.1323	12.2180	10.8022	10.0807	9.7124	9.5827	9.5805	9.5636
10	16.4597	14.4955	12.2997	10.7195	9.9395	9.7809	9.7838	9.7401

Table 5. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for CFSF boundary condition (case 18) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$
1	1.6307	1.6265	1.6249	1.6235	1.6221	1.6215	1.6206
2	1.7705	1.7236	1.7042	1.6972	1.6959	1.6907	1.6898
3	3.5757	3.1925	2.9911	2.9296	2.9258	2.9242	2.9205
4	4.7480	4.2740	3.9861	3.5418	3.4441	3.3902	3.3840
5	5.8437	4.7157	4.6819	4.6733	4.6701	4.6676	4.6660
6	6.4246	4.8571	4.7877	4.7320	4.7180	4.7115	4.7081
7	10.4410	7.8102	6.2957	6.1677	5.9410	5.8232	5.7969
8	11.4837	9.1497	7.0777	6.6838	6.0138	5.9890	5.9835
9	11.6112	10.3245	8.1198	7.1609	6.7004	6.4150	6.3732
10	12.8013	11.1113	9.0680	8.5869	8.3989	7.0742	6.9666

Table 6. Convergence studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for SCFS boundary condition (case 14) and geometry parameters ($\delta = 0.05$, $\eta = 2$).

Mode Sequence	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	$N=11$
1	1.7155	1.6830	1.6690	1.6621	1.6560	1.6519	1.6510	1.6508
2	2.9813	2.6525	2.5135	2.4543	1.4293	2.4125	2.4015	2.4007
3	4.5902	3.6581	3.4884	3.2598	3.2224	3.1866	3.1734	3.1674
4	4.9628	4.7857	4.7346	4.7073	4.6955	4.6887	4.6833	4.6788
5	7.6167	6.3851	5.6753	6.6198	5.2868	5.1559	5.1137	5.1025
6	8.8693	7.7765	6.0204	7.5434	5.4571	5.4174	5.3908	5.3823
7	9.4622	8.0651	6.9218	8.9564	6.3208	6.2261	6.1765	6.1681
8	10.7970	9.3896	9.0640	9.3443	7.2000	6.5273	6.4794	6.4052
9	15.7370	12.0571	10.5515	9.9701	8.5045	8.2273	7.9230	7.9174
10	16.7031	12.8721	10.8349	11.2708	8.9094	8.8923	8.8816	8.8770

Comparing Studies

Numerical results were compared with known results that are available in the literature. In order to validate the accuracy of the present prediction, a comparison has been carried out for both thin ($\delta = 0.001$) and moderately thick rectangular plates, with different boundary conditions (Figure 2) and different locations of internal line support.

The frequency parameters, $\lambda = \beta/(\eta^2\pi^2)$, given by the authors, were compared with the available results in Zhou's work [26], using the Rayleigh-Ritz method for thin rectangular plates and Xiang's work [27], using the Levy type solution for moderately thick rectangular plates.

From the results, as shown in Tables 7 to 15, there is excellent agreement between the present work and results in the literature.

NUMERICAL RESULTS

In this paper, the frequency parameters were obtained from the free vibration analysis of a multi-span moderately thick rectangular plate in dimensionless form, $\lambda = \beta/(\eta^2\pi^2)$, by using the Rayleigh-Ritz energy method. Numerical calculations have been performed for each of the twenty-one possible classical boundary conditions, with arrangement of the boundary conditions as shown in Figure 2.

The results were given in Tables 16 to 19 for the thickness to length ratio, $\delta = 0.1$, and the aspect ratio, $\eta = 2$. In each table, the frequency parameters were presented for the first ten mode sequences of a moderately thick rectangular plate with two equal spans in an X_1-direction.

As shown in Figures 4 and 5, it is observed that the behavior of the internal line support is close to a simply supported edge.

Table 7. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans in X_1-direction for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	SSSS Plate		SFSF Plate		SCSC Plate	
Sequence	Xiang [28]	Present	Xiang [28]	Present	Xiang [28]	Present
1	1.9317	1.9323	1.1523	1.1529	2.2684	2.6513
2	2.2663	2.2948	1.2406	1.2463	2.6992	2.8230
3	4.6084	4.6045	2.6500	2.6528	4.7726	5.0543
4	4.6084	4.6474	3.0780	3.1562	4.9693	5.0711
5	4.7671	4.8254	3.8792	3.8838	5.2839	5.9807
6	5.2781	5.5062	3.9134	3.9323	5.9928	6.6160
7	7.0716	7.1420	5.3950	5.4106	7.5084	8.5816
8	7.4914	7.8483	5.6358	5.6629	7.9604	8.6107
9	8.6162	8.6378	5.6448	5.8337	8.7010	8.8731
10	8.6162	8.7919	6.3488	6.8378	8.7906	8.9342

Table 8. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of rectangular Mindlin plate with two equal spans in X_2-direction for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	SSSS Plate		SFSF Plate		SCSC Plate	
Sequence	Xiang [28]	Present	Xiang [28]	Present	Xiang [28]	Present
1	1.1927	1.1957	1.1946	1.1931	2.0248	2.0481
2	2.0689	2.0874	2.3943	2.5976	2.5548	2.7287
3	2.8892	2.9247	2.9544	2.9770	4.6528	4.6708
4	3.8954	3.9045	3.8956	3.9031	4.8153	4.9037
5	4.6804	4.7212	4.8564	4.9926	4.9054	5.0255
6	4.8667	4.9333	5.4232	5.6005	5.7650	6.1570
7	5.5230	5.5888	5.5330	5.7243	7.1971	7.3292
8	6.0679	6.3269	6.2076	6.5187	7.8210	8.3318
9	7.2563	7.4613	7.6775	7.9669	8.6389	8.6780
10	7.9474	7.9668	7.9474	8.0729	8.7618	8.8754

Table 9. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SSSS boundary condition (case 1).

δ	a'	Method	Mode Sequence					
			1	2	3	4	5	6
0.01	0.1	Xiang [28]	2.6392	5.4604	6.7781	9.5650	10.3620	13.5290
		Present	2.7071	5.5960	6.9793	9.7684	10.4629	13.5769
	0.3	Xiang [28]	3.5328	6.3571	9.6807	11.2430	12.6200	14.8980
		Present	3.8739	6.4669	9.3245	11.6000	12.1222	14.2023
	0.5	Xiang [28]	4.9955	7.0108	7.9884	9.5607	12.9690	14.1610
		Present	4.9907	7.4812	8.6132	10.0759	13.4899	13.9791
0.05	0.1	Xiang [28]	2.5843	5.1324	6.5118	9.1169	9.9078	12.4250
		Present	2.5982	5.3745	6.5507	9.5783	10.1859	13.1340
	0.3	Xiang [28]	3.4630	6.1682	9.2836	10.7130	11.9640	13.8330
		Present	3.5113	6.2988	9.3556	11.2127	11.5998	13.8050
	0.5	Xiang [28]	4.8907	6.7106	7.7267	9.0653	12.3050	13.2290
		Present	4.9313	6.7857	7.7416	9.1842	12.2853	13.2692
0.10	0.1	Xiang [28]	2.4416	4.9380	5.8821	8.1036	8.8524	10.6750
		Present	2.4582	4.9807	5.9313	8.3533	9.0149	10.8019
	0.3	Xiang [28]	3.2732	5.6883	8.3308	9.4966	10.4850	11.6130
		Present	3.3045	5.6591	8.7895	9.9055	10.4761	11.2523
	0.5	Xiang [28]	4.6084	5.9863	7.0716	7.9475	10.8090	11.3000
		Present	4.5691	6.0985	6.9653	8.1135	10.691	11.5734

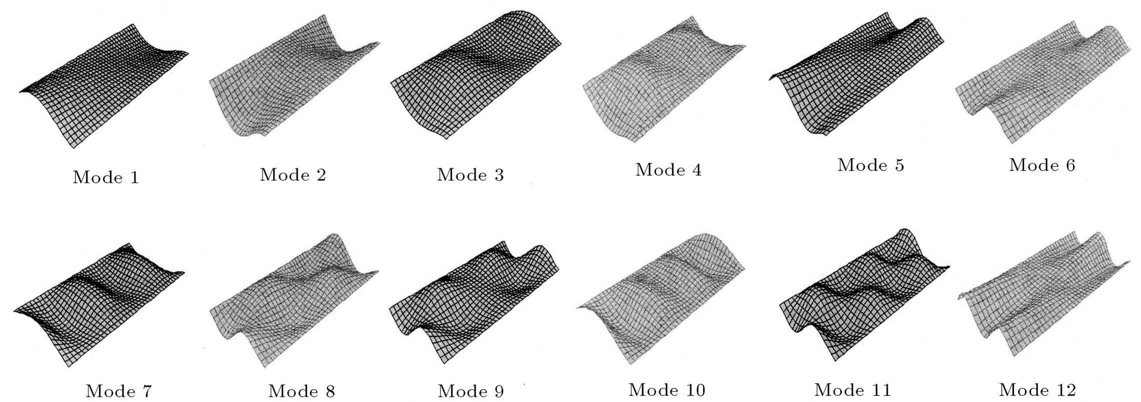


Figure 4. First nine mode shapes of CFSF rectangular Mindlin plates with two equal spans ($a' = 0.5$, $b' = 0$) for geometry parameters ($\delta = 0.1$, $\eta = 2$).

The results were given in Tables 20, and 21 for the thickness to length ratios, $\delta = 0.01$, 0.05 and 0.1, over a range of location of internal line supports in an X_1-direction, $a' = 0.1$, 0.3, 0.5 and 0.7. In each table, the frequency parameters are presented for the first six mode sequence for a moderately thick square plate.

The results are given in Table 22 for a moderately thick square plate with two equal spans in an X_1-direction for common shear correction factors, $\kappa = \sqrt{5/6}$, $\sqrt{\pi^2/12}$ and $\sqrt{0.86667}$.

In order to study the effect of line support location on the first four frequency parameters of the plates, consideration may now be paid to Figures 6 and 7. From the results in these figures, it is observed that,

Table 10. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SFSF boundary condition (case 5).

δ	a'	Method	Mode Sequence					
			1	2	3	4	5	6
0.01	0.1	Xiang [28]	1.2456	3.2596	4.2351	6.5319	7.4051	9.1903
		Present	1.2470	3.2706	4.2402	6.5855	7.5085	9.2204
	0.3	Xiang [28]	1.4263	3.2920	4.4249	5.4786	6.4685	8.5657
		Present	1.4290	3.3070	4.4355	5.5920	6.6058	9.1578
	0.5	Xiang [28]	1.6309	2.3050	4.7253	5.1271	7.6042	9.7036
		Present	1.6332	2.3122	4.7297	5.1583	7.6132	9.7036
0.05	0.1	Xiang [28]	1.2324	3.1863	4.1481	6.3087	7.1414	8.8283
		Present	1.2357	3.1979	4.1554	6.3746	7.1530	8.9323
	0.3	Xiang [28]	1.4082	3.2051	4.3260	5.2780	6.2026	8.1429
		Present	1.4139	3.2232	4.3414	5.2114	6.1532	8.3649
	0.5	Xiang [28]	1.6067	2.2520	4.6094	4.9625	7.3277	9.2886
		Present	1.6078	2.2617	4.6166	4.9910	7.3318	9.3015
0.10	0.1	Xiang [28]	1.2046	3.0298	3.9283	5.8065	6.5417	7.9786
		Present	1.2072	3.0380	3.9354	5.8708	6.5570	8.0462
	0.3	Xiang [28]	1.3707	3.0266	4.0825	4.8245	5.6518	7.2601
		Present	1.3752	3.0281	3.8952	4.8007	5.0274	7.2050
	0.5	Xiang [28]	1.5593	2.1387	4.3358	4.5951	6.7071	8.1537
		Present	1.5593	2.1526	4.3386	4.6358	6.7074	8.3569

Table 11. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SCSC boundary condition (case 3).

δ	a'	Method	Mode Sequence					
			1	2	3	4	5	6
0.01	0.1	Xiang [28]	3.3169	5.8735	8.1316	10.6280	10.6670	15.2300
		Present	3.3396	5.9375	8.1986	10.3577	10.7797	15.4488
	0.3	Xiang [28]	4.6841	7.1453	11.7790	12.0230	14.6140	18.5410
		Present	4.8868	7.8217	12.1692	12.3386	14.9731	18.6906
	0.5	Xiang [28]	7.0108	9.5608	9.6209	11.6900	14.1510	15.7750
		Present	7.4218	9.6728	9.9063	11.7744	14.4464	15.9824
0.05	0.1	Xiang [28]	3.1958	5.6619	7.6536	9.9975	10.1140	13.9760
		Present	3.2094	5.6802	7.6894	9.1155	10.2124	13.9334
	0.3	Xiang [28]	4.5184	6.8505	11.1380	11.2380	13.5430	17.1470
		Present	4.8831	7.3487	11.1867	11.2991	13.7515	16.6390
	0.5	Xiang [28]	6.7130	8.9583	9.0738	10.7970	13.2440	14.4270
		Present	6.7618	9.0162	9.1234	10.8949	13.2788	14.5781
0.10	0.1	Xiang [28]	2.9489	5.1901	6.7054	8.6714	8.9777	11.6060
		Present	2.9696	5.2306	6.7635	8.7661	9.1216	11.5263
	0.3	Xiang [28]	4.1108	6.1636	9.5728	9.7472	11.3940	14.2020
		Present	4.1627	6.2729	9.8679	9.9255	12.1808	14.6765
	0.5	Xiang [28]	5.9992	7.5511	7.9854	9.0159	11.3540	11.9280
		Present	6.0244	7.6832	8.0103	9.1964	11.3708	12.1525

Table 12. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SSSF boundary condition (case 4).

δ	a'	Method	Mode Sequence					
			1	**2**	**3**	**4**	**5**	**6**
0.01	0.3	Xiang [28]	1.5211	4.4498	5.0600	8.2985	9.3796	11.6760
		Present	1.5988	4.5239	5.6179	9.1252	9.7252	11.4575
	0.5	Xiang [28]	1.9456	4.9069	5.7162	8.6236	9.1039	9.8289
		Present	2.0469	4.9813	5.9851	8.8918	9.5790	9.9238
	0.7	Xiang [28]	2.7881	4.2836	5.9685	7.0169	10.1550	11.0050
		Present	2.9095	4.3437	6.0560	7.1178	10.2627	11.0766
0.05	0.3	Xiang [28]	1.4993	4.3475	4.9037	7.9353	8.9944	11.1120
		Present	1.5139	4.3894	4.9884	8.3778	9.1991	11.7845
	0.5	Xiang [28]	1.9096	4.7706	5.5548	8.2757	8.6289	9.3885
		Present	1.9176	4.7864	5.5798	8.3149	8.7340	9.2446
	0.7	Xiang [28]	2.7372	4.1510	5.7939	6.7020	9.6786	10.4900
		Present	2.7483	4.1773	5.8055	6.7903	9.7329	10.5111
0.10	0.3	Xiang [28]	1.4513	4.0970	4.4535	7.1458	8.1058	9.8480
		Present	1.4618	4.1296	4.6117	7.1443	8.1027	9.8616
	0.5	Xiang [28]	1.8341	4.4564	5.1450	7.7497	7.5883	8.4083
		Present	1.8394	4.4704	5.1732	7.5049	7.7306	8.4389
	0.7	Xiang [28]	2.6251	3.8364	5.3805	6.0288	8.5772	9.3278
		Present	2.6283	3.8777	5.3867	6.1355	8.6594	9.3490

Table 13. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SCSF boundary condition (case 6).

δ	a'	Method	Mode Sequence					
			1	**2**	**3**	**4**	**5**	**6**
0.01	0.3	Xiang [28]	1.5363	4.4554	5.1510	8.3541	9.3817	12.3440
		Present	1.5616	4.4870	5.3321	8.8142	9.4487	13.1446
	0.5	Xiang [28]	1.9811	4.9168	7.2938	9.8321	9.8642	10.1340
		Present	2.0063	4.9407	7.4099	9.8719	9.9404	10.2143
	0.7	Xiang [28]	3.1117	5.1104	6.2351	7.5397	11.2270	12.1080
		Present	3.1531	5.1242	6.2817	7.5824	11.2761	12.9105
0.05	0.3	Xiang [28]	1.5116	4.3514	4.9743	7.9734	8.9956	11.6410
		Present	1.5220	4.3861	5.0326	8.2854	9.2099	12.0673
	0.5	Xiang [28]	1.9411	4.7782	6.9994	9.2334	9.3904	9.5773
		Present	1.9463	4.7918	7.0133	9.4242	9.4242	9.6064
	0.7	Xiang [28]	3.0339	4.9150	6.0097	7.1704	10.6420	11.3590
		Present	3.0420	4.9346	6.0271	7.2203	10.6751	11.4123
0.10	0.3	Xiang [28]	1.4578	4.0985	4.5769	7.1595	8.1061	10.1410
		Present	1.4680	4.1276	4.6395	7.3967	8.2519	9.9506
	0.5	Xiang [28]	1.8566	4.4603	6.3020	7.9088	8.3530	8.4089
		Present	1.8629	4.4743	6.3128	8.0886	8.3976	8.4388
	0.7	Xiang [28]	2.8680	4.4463	5.5194	6.3685	9.3958	9.6938
		Present	2.8749	4.4576	5.5340	6.4433	9.4324	9.7695

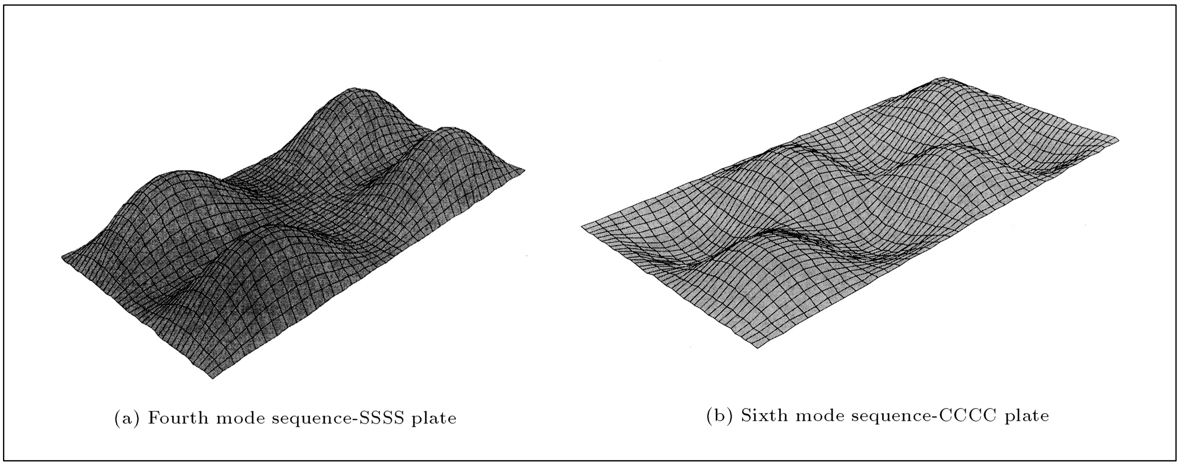


Figure 5. Mode shapes of SSSS and CCCC rectangular Mindlin plate with two equal spans in X_1-direction and two equal spans in X_2-direction.

Table 14. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ of square Mindlin plate with an internal line support for SSSF boundary condition (case 2).

δ	a'	Method	Mode Sequence					
			1	2	3	4	5	6
0.01	0.3	Xiang [28]	3.5944	6.3860	10.0140	11.2560	12.8600	17.6970
		Present	3.7437	6.5575	10.077	11.2788	12.8711	17.6341
	0.5	Xiang [28]	5.5739	8.2407	8.7081	10.9560	13.2920	15.2300
		Present	6.1077	8.8694	9.1853	11.4361	13.6450	15.2626
	0.7	Xiang [28]	4.5939	7.1024	11.4080	11.7610	14.1560	15.6420
		Present	4.8438	7.3323	12.1342	12.2010	15.1674	16.8852
0.05	0.3	Xiang [28]	3.5119	6.1887	9.5403	10.7200	12.1330	16.4230
		Present	3.5365	6.2490	9.7451	10.7702	12.2389	16.4423
	0.5	Xiang [28]	5.4171	8.0994	8.1958	10.2190	12.5600	14.0370
		Present	5.6595	8.1475	8.2554	10.3180	12.6104	14.2651
	0.7	Xiang [28]	4.4487	6.8211	10.7800	11.1280	13.2330	14.3430
		Present	4.4668	6.8436	10.8232	11.1550	13.2812	14.5194
0.10	0.3	Xiang [28]	3.2973	5.6959	8.4551	9.4984	10.5510	13.8620
		Present	3.3242	5.7513	8.4564	9.4756	10.7739	13.9399
	0.5	Xiang [28]	5.0135	7.0519	7.3241	8.6832	10.9540	11.7410
		Present	5.0377	7.1502	7.3550	8.8243	10.9825	11.9049
	0.7	Xiang [28]	4.0783	6.1534	9.3656	9.7448	11.2830	11.8090
		Present	4.0946	6.1798	9.3847	9.7805	11.3208	12.0894

Table 15. Comparison studies of frequency parameters $\lambda = \beta/(\eta^2)$ of square Mindlin plate with an internal line support for SSSS boundary condition (case 1) with an internal line support in X_1-direction and an internal line support in X_2-direction.

$a' = b'$	Method	δ	Mode Sequence					
			1	2	3	4	5	6
0.5	Zhou [26]	—	78.956	94.590	94.590	108.24	197.39	197.39
	Present	0.001	78.9547	94.5882	94.5882	108.2375	197.3860	197.3861

Table 16. Frequency parameters of the SSSS, SFSF, SCSC, SSSF, SCSF and SCSS rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	Boundary Condition					
Sequence	SSSS	SFSF	SCSC	SSSF	SCSF	SCSS
1	1.9323	1.1529	2.6513	1.1957	1.1931	2.0481
2	2.2948	1.2563	2.8230	2.0874	2.5976	2.7287
3	4.6045	2.6528	5.0543	2.9247	2.9770	4.6708
4	4.6474	3.1562	5.0711	3.9045	3.9031	4.9037
5	4.8254	3.8838	5.9807	4.7212	4.9926	5.0255
6	5.5062	3.9323	6.6160	4.9333	5.6005	6.1570
7	7.1420	5.4106	8.5816	5.5888	5.7243	7.3292
8	7.8483	5.6629	8.6107	6.3269	6.5187	8.3318
9	8.6378	5.8337	8.8731	7.4613	7.9669	8.6780
10	8.7919	6.8378	8.9342	7.9668	8.0729	8.8754

Table 17. Frequency parameters of the CCCC, CSCC, CFCC, CCSS, FSCC and rectangular Mindlin plates with two equal spans ($a' = 0.5$, $b' = 0$) for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	Boundary Condition				
Sequence	CCCC	CSCC	CFCC	CCSS	FSCC
1	3.3124	2.7951	2.2156	2.5289	1.6509
2	3.3713	3.3218	3.2114	3.1133	2.8633
3	6.2768	5.2668	3.5282	5.4491	3.2188
4	6.3641	6.0573	5.4911	5.4827	4.6848
5	6.5377	6.3655	6.0338	5.7552	5.6353
6	6.7758	6.5225	6.3049	6.7175	5.8808
7	8.6143	8.2725	6.7627	8.2366	6.1702
8	8.6143	9.2605	6.8369	9.1915	6.6955
9	8.6885	9.5185	8.8632	9.5636	8.4797
10	9.3670	10.3611	10.4389	9.7401	8.8808

Table 18. Frequency parameters of the FFCC, CFCS, SCFS, CFFS, CFCF, rectangular Mindlin plates with two equal spans ($a' = 0.5$, $b' = 0$) for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	Boundary Condition				
Sequence	FFCC	CFCS	SCFS	CFFS	CFCF
1	0.6002	2.2176	1.6510	0.5968	2.1932
2	1.9409	2.8176	2.4015	1.1474	2.2444
3	2.3071	3.4825	3.1734	2.1366	3.2550
4	2.4468	5.2998	4.6833	2.4074	3.6780
5	3.6232	5.4917	5.1137	3.2157	5.4768
6	4.1819	6.0856	5.3908	4.1404	5.5081
7	5.0660	6.6137	6.1765	4.2478	5.9532
8	5.6836	6.7519	6.4794	5.5458	6.6031
9	5.9873	8.3103	7.9230	5.7728	6.9437
10	6.6796	9.5331	8.8816	6.0177	7.1050

Table 19. Frequency parameters of the CFSF, CFFF, SSFF, SFFF and FFFF rectangular Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for geometry parameters ($\delta = 0.1$, $\eta = 2$).

Mode	Boundary Condition				
Sequence	CFSF	CFFF	SSFF	SFFF	FFFF
1	1.6206	0.4812	0.4349	0.2274	0.3498
2	1.6898	0.6464	1.1766	0.7275	0.6457
3	2.9205	1.2193	1.4178	1.1301	0.8245
4	3.3840	1.8743	2.0988	1.7439	1.4671
5	4.6660	2.2774	2.6687	1.8693	2.0726
6	4.7081	2.5244	2.9468	2.5438	2.4103
7	5.7969	2.2576	3.5601	2.6921	2.4929
8	5.9835	3.5371	4.3677	3.5928	2.6137
9	6.3732	4.3720	5.4012	4.0054	2.9279
10	6.9666	5.0284	5.5982	4.7070	4.4603

Table 20. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ for CFSF square Mindlin plate with an internal line support.

δ	a'	Mode Sequence					
		1	2	3	4	5	6
0.01	0.1	1.7531	3.5948	5.2641	7.3755	7.6998	10.7398
	0.3	1.9043	3.6234	5.4386	5.8555	7.4098	9.7045
	0.5	2.0847	2.6673	5.6966	6.0792	7.8217	10.5219
0.05	0.1	1.7272	3.4937	5.1058	7.0729	7.3358	10.1886
	0.3	1.8747	3.5357	5.2755	5.6948	7.1667	9.7902
	0.5	2.0419	2.5967	5.5007	5.8285	7.5037	9.7889
0.10	0.1	1.6620	3.2791	4.7099	5.2708	6.6723	8.9120
	0.3	1.7973	3.3074	4.8514	5.1641	6.4245	8.3814
	0.5	1.9495	2.4516	5.0417	5.3008	6.8157	8.5180

Table 21. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ for SSCC square Mindlin plate with an internal line support.

δ	a'	Mode Sequence					
		1	2	3	4	5	6
0.01	0.3	5.2217	8.2761	12.8703	13.4343	13.9534	16.1474
	0.5	6.0277	8.8868	9.5415	11.5396	14.9569	15.4820
	0.7	3.9734	7.2688	10.5730	12.7203	13.7753	19.5108
0.05	0.3	4.6984	7.6108	10.1882	12.2748	12.9447	14.5447
	0.5	5.6446	8.3722	8.7444	10.7574	13.6007	15.1016
	0.7	3.7772	6.9303	9.7069	11.8450	12.5638	13.7899
0.10	0.3	4.2796	5.8378	6.7317	9.6316	10.7632	11.8953
	0.5	5.1868	7.2577	7.7783	9.1934	11.5940	12.5396
	0.7	3.5287	6.2691	8.5768	10.2609	10.8898	12.7524

for SSSS plates, the frequency parameters for the first and second modes increase monotonically as the location parameter, a', moves from the plate edge to the plate center ($a' = 0.5$). As the results are shown in figure 6, the frequency parameters for the third and fourth modes of the SSSS and SCSC plates increase monotonically as the location parameter, a', changes from $a' = 0.1$ to $a' = 0.3$ and, after that, decreases monotonically as the location parameter a' changes from $a' = 0.3$ to $a' = 0.5$ (center of the plate). From the results presented in these figures, it shows that the SCSC boundary conditions of the moderately thick rectangular plate have a similar tendency to the SSSS boundary conditions of the moderately thick rectangular plate. As shown in Figure 6, it is observed that the behaviors of the SSCC, SCSS and CFSF are similar, and that the behaviors of the SSSS, SCSC and SCSF are also similar.

Table 22. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ for square Mindlin plate with two equal spans ($a' = 0.5$, $b' = 0$) for common shear correction factors.

κ	Boundary Conditions	Mode Sequence					
		1	2	3	4	5	6
$\sqrt{\frac{5}{6}}$	SSSS	1.9323	2.2948	4.6045	4.6474	4.8254	5.5062
	SFSF	1.1529	1.2463	2.6528	3.1562	3.8838	3.9323
	SCSS	2.0481	2.7287	4.6708	4.9037	5.0255	6.1570
$\sqrt{\frac{\pi^2}{12}}$	SSSS	1.9311	2.2935	4.6005	4.6428	4.8210	5.4996
	SFSF	1.1526	1.2460	2.6515	3.1540	3.8812	3.9296
	SCSS	2.0464	2.7259	4.6664	4.8975	5.0209	6.1478
$\sqrt{0.86667}$	SSSS	1.9355	2.2987	4.6164	4.6613	4.8381	5.5254
	SFSF	1.1537	1.2475	2.6566	3.1627	3.8914	3.9404
	SCSS	2.0533	2.7369	4.6838	4.9222	5.0427	6.1888

As shown in Figures 6 and 7 and Tables 16 and 21, it can be seen that, between two plates having identical geometry parameters and location of internal line support, the frequency parameters of the one having more clamp boundary conditions is greater than the one having more simply supported or free boundary conditions, and the one which has more simply supported boundary conditions is greater than the one which has more free edge boundary conditions.

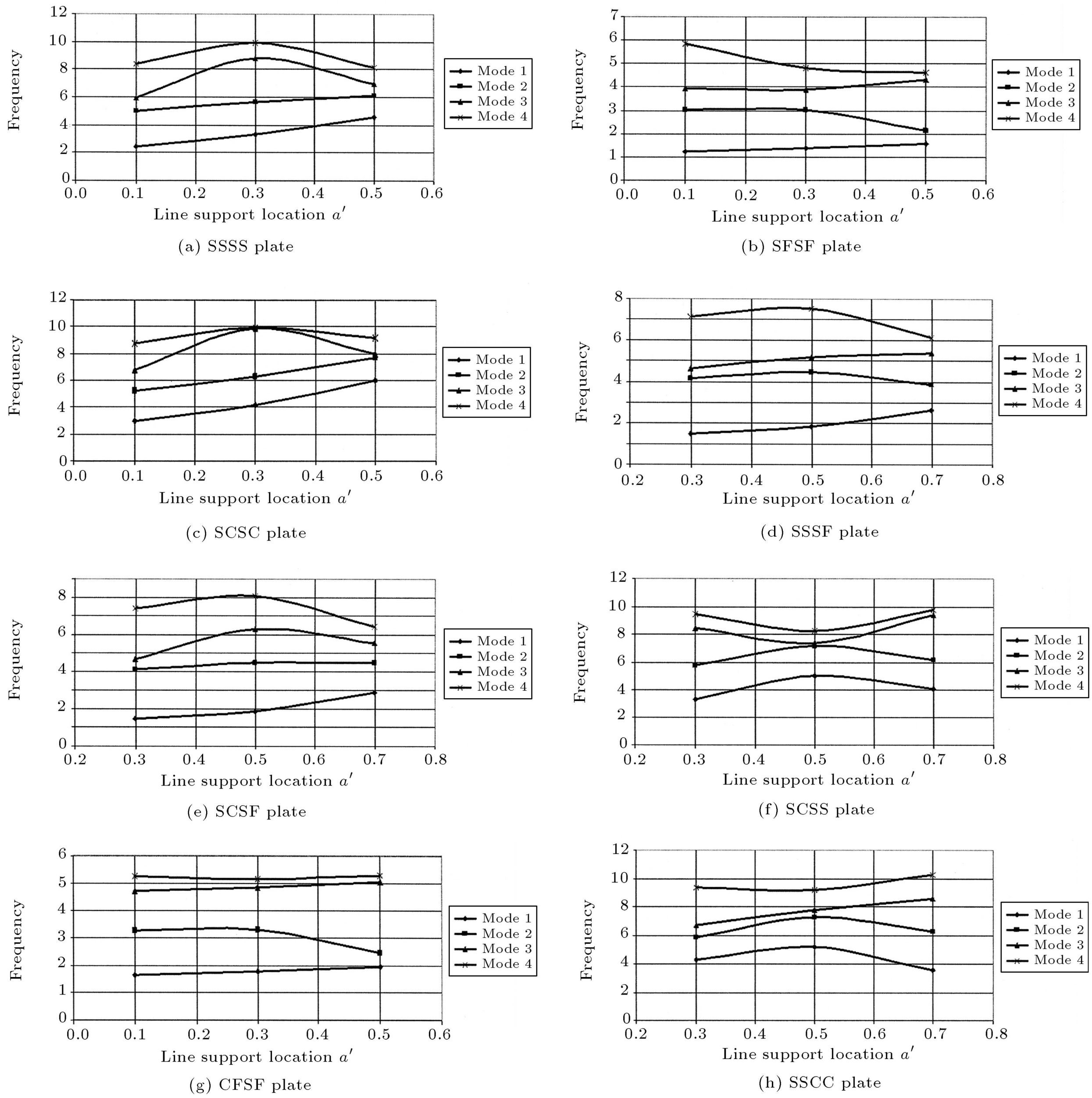
In order to study the effect of thickness to length ratios on the frequency parameters of the plates, consider Figure 8 and Tables 20-21. From the results in these illustrations, it is observed that, with increasing thickness to length ratios, the frequency parameters are decreasing.

The results is shown in Table 23 for a moderately thick rectangular plate, for the thickness to length ratio, $\delta = 0.1$, and for the aspect ratio, $\eta = 2$, with two equal spans in an X_1-direction and two equal spans in an X_2-direction. In this table, the boundary conditions of the plate are SSSS and CCCC.

In Table 23, the effect of the locations of the internal line supports on the first eight frequency parameters of simply supported and fully clamped square Mindlin plates, with an internal line support in an X_1-direction and an internal line support in an X_2, were studied for geometry parameters thickness to length ratio: $\delta = 0.05$ and aspect ratio $\eta = 2$. It can be seen that, for all cases, the fundamental frequency parameters increase when the internal line support approaches the respective center line of the plate ($a' = b' = 0.5$). However, for simply supported plates, mode sequence 4 starts to decrease when $a' > 0.3$. For fully clamped plates, mode sequence 1 monotonically increases when the internal line support approaches the respective center line of the plate. Also, Figure 5 illustrated the fourth mode shape of the SSSS plate and the sixth mode shape of the CCCC plate.

Table 23. Vibration frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ for rectangular Mindlin plate with an internal line support in X_1-direction and an internal line support in X_2-direction for geometry parameters ($\delta = 0.05$, $\eta = 2$).

Boundary Conditions	$a' = b'$	Mode Sequence							
		1	2	3	4	5	6	7	8
SSSS	0.1	1.9403	2.7796	4.2916	5.3677	6.0211	6.1352	7.9037	8.6797
	0.3	2.9701	4.2861	5.2843	6.6484	8.0510	9.4053	10.1796	10.9368
	0.5	4.6387	4.8446	6.1018	6.2982	7.1666	7.8359	8.2475	9.2885
CCCC	0.1	2.6838	3.5432	3.9319	4.8857	6.2760	6.5175	6.9689	8.0161
	0.3	2.2926	2.8158	3.9688	5.1509	5.4860	6.6053	8.7806	9.4201
	0.5	6.1620	6.3442	7.7915	8.0274	8.1135	8.4663	9.4377	9.9418



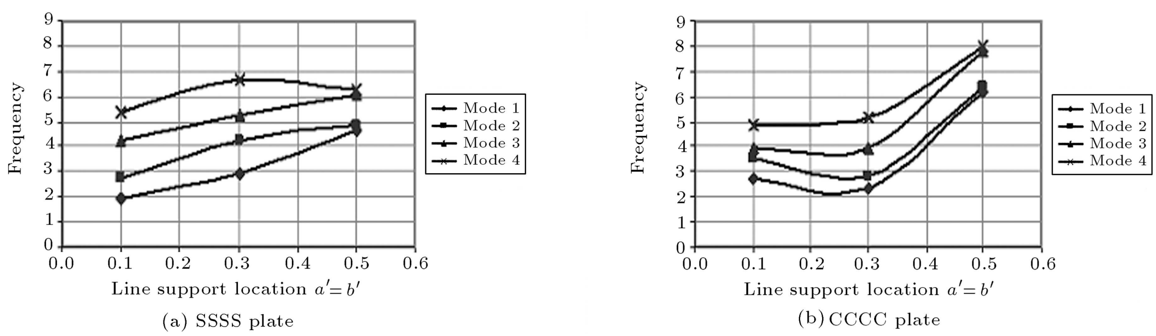
(a) SSSS plate (b) SFSF plate

(c) SCSC plate (d) SSSF plate

(e) SCSF plate (f) SCSS plate

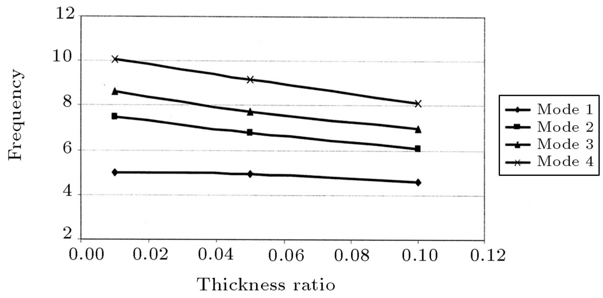
(g) CFSF plate (h) SSCC plate

Figure 6. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ versus line support location a' for square Mindlin plate with an internal line support ($\delta = 0.1$).

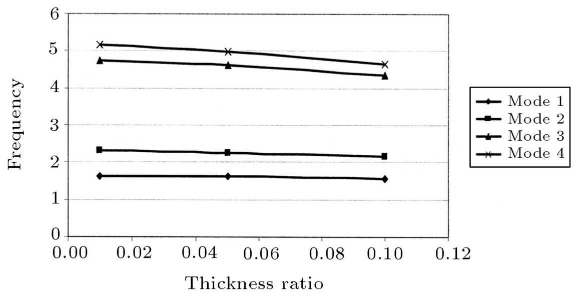


(a) SSSS plate (b) CCCC plate

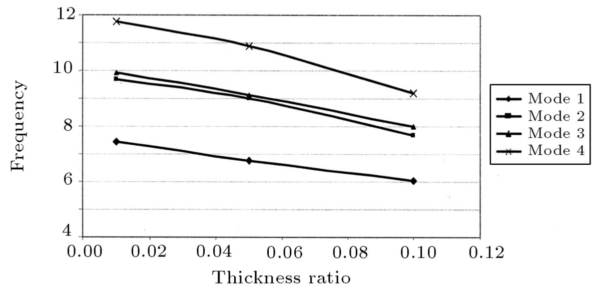
Figure 7. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ versus line support location $a' = b'$ for square Mindlin plate with one internal line support in X_1-direction and one internal line support in X_2-direction, for geometry parameters ($\delta = 0.05$, $\eta = 2$).



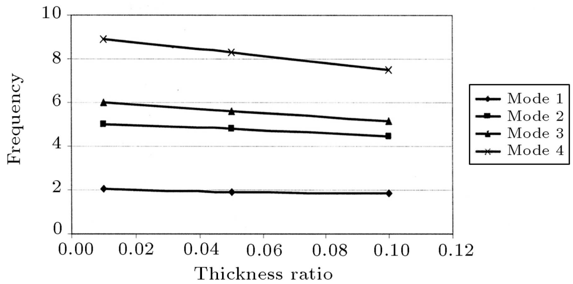
(a) SSSS plate



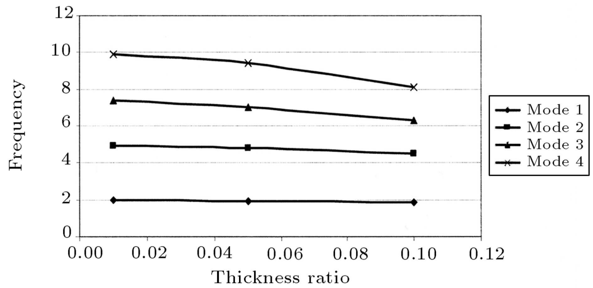
(b) SFSF plate



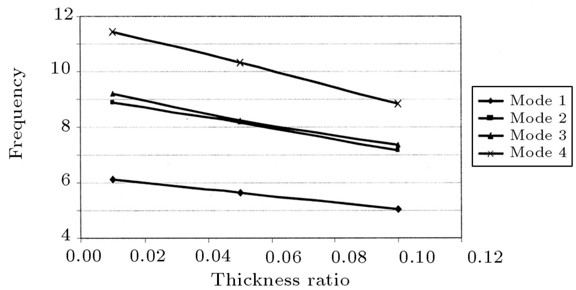
(c) SCSC plate



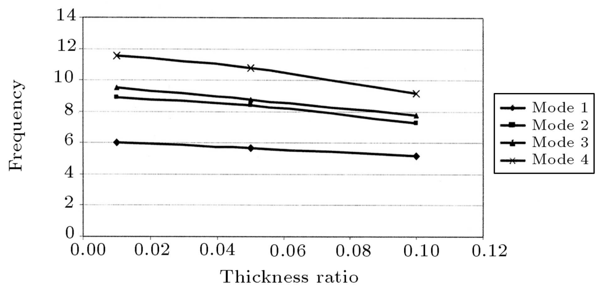
(d) SSSF plate



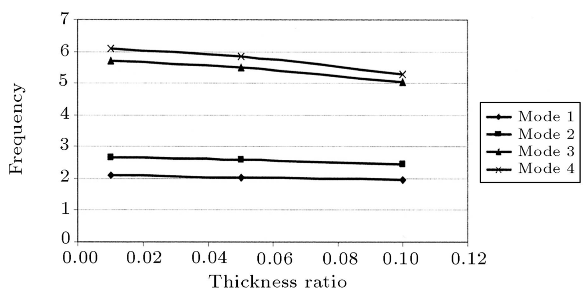
(e) SCSF plate



(f) SCSS plate



(g) CFSF plate



(h) SSCC plate

Figure 8. Frequency parameters $\lambda = \beta/(\eta^2\pi^2)$ versus thickness to length ratio δ for square Mindlin plate with an internal line support ($a' = 0.5$).

CONCLUSION

This paper presents an energy method for the vibration analysis of multi-span moderately thick rectangular plates. A rectangular plate was divided into two spans in an X_1-direction and two spans in an X_2-direction. The Rayleigh-Ritz method was employed to obtain the frequency parameters and mode shapes of the plates. The conclusions of this approach are follows:

- The frequency parameters decrease as the plate thickness to length ratios increase, due to the influence of transverse shear deformation and rotary inertia.

- It can be seen that rapid convergency and good accuracy is achieved with a small number of terms of the two dimensional complete polynomial functions.
- The present method is especially suitable for the moderately thick rectangular plate problem, with a large amount of internal line support in both X_1 and X_2-directions.
- Finally, based on comparison with results available in the literature, the validity of the present results were established.

REFERENCES

1. Leissa, A.W. "Vibration of plates", *NASA SP-160* (1969).
2. Leissa, A.W. "Recent research in plate vibrations: classical theory", *The Shock and Vibration Digest*, **9**(10), pp. 13-24 (1977).
3. Leissa, A.W. "Recent research in plate vibrations, 1973-1976: complicating effects", *The Shock and Vibration Digest*, **10**(12), pp. 21-35 (1977).
4. Leissa, A.W. "Plate vibration research, 1976-1980: classical theory", *The Shock and Vibration Digest*, **13**(9), pp. 11-22 (1981).
5. Leissa, A.W. "Plate vibration research, 1976-1980: complicating effects", *The Shock and Vibration Digest*, **13**(10), pp. 17-36 (1981).
6. Leissa, A.W. "Recent studies in plate vibrations, 1981-1985. Part I: Classical theory", *The Shock and Vibration Digest*, **19**(2), pp. 11-18 (1987).
7. Leissa, A.W. "Recent studies in plate vibrations, 1981-1985. Part II: Complicating effects", *The Shock and Vibration Digest*, **19**(3), pp. 10-24 (1987).
8. Leissa, A.W. "The free vibration of rectangular plates", *J. of Sound and Vibration*, **31**(3), pp. 257-293 (1973).
9. Mindlin, R.D. "Influence rotatory inertia and shear in flexural motion of isotropic, elastic plates", *ASME J. of Applied Mechanics*, **18**(1), pp. 31-38 (1951).
10. Mindlin, R.D. et al. "Flexural vibration of rectangular plates", *ASME J. of Applied Mechanics*, **23**, pp. 430-436 (1956).
11. Dawe, D.J. and Roufaeil, O.L. "Rayleigh-Ritz vibration analysis of Mindlin plates", *J. of Sound and Vibration*, **69**(3), pp. 345-359 (1980).
12. Liew, K.M. et al. "Transverse vibration of thick rectangular plates - I. Comprehensive sets of boundary conditions", *Computers and Structures*, **49**(1), pp. 1-29 (1993).
13. Liew, K.M. et al. "Vibration of Mindlin plates using boundary characteristic orthogonal polynomials", *J. of Sound and Vibration*, **182**(1), pp. 77-90 (1995).
14. Cheung, Y.K. and Zhou, D. "Vibrations of moderately thick rectangular plates in terms of a set of static Timoshenko beam functions", *Computers and Structures*, **78**(6), pp. 757-768 (2000).
15. Al Janabi, B.S. et al. "Free vibration of Mindlin plates using the finite element methods, Part I: Square plates with various edge conditions", *Engineering Computers*, **6**(2), pp. 90-96 (1978).
16. Dawe, D.J. "Finite strip models for vibration of Mindlin plates", *J. of Sound and Vibration*, **59**, pp. 441-452 (1987).
17. Cheung, Y.K. and Chakrabarti, S. "Free vibration of thick layered rectangular plates by a finite layer method", *J. of Sound and Vibration*, **21**(3), pp. 277-284 (1972).
18. Mikami, T. and Yoshimura, J. "Application of the collocation method to vibration analysis of rectangular Mindlin plates", *Computers and Structures*, **18**(3), pp. 425-431 (1984).
19. Gorman, D.J. "Accurate free vibration analysis of clamped Mindlin plates using the method of superposition", *J. of Sound and Vibration*, **189**(3), pp. 341-353 (1996).
20. Srinvas, S. "An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates", *J. of Sound and Vibration*, **12**(2), pp. 187-199 (1970).
21. Wittrick, W.H. "Analytical three-dimensional elasticity solutions to some plate problems, and some observations on Mindlin's plate theory", *Int. J. of Solids and Structures*, **23**(4), pp. 441-464 (1987).
22. Liew, K.M. et al. "A continuum three-dimensional vibration analysis of thick rectangular plate", *Int. J. of Solids and Structures*, **30**(24), pp. 3357-3379 (1993).
23. Liew, K.M. et al. "Three-dimensional vibration of rectangular plates: Variance of simply support conditions and influence of in-plane inertia", *Int. J. of Solids and Structures*, **31**(23), pp. 3233-3247 (1994).
24. Liew, K.M. et al. "Three-dimensional vibration of rectangular plates: Effects of thickness and edge constraints", *J. of Sound and Vibration*, **182**(5), pp. 709-727 (1995).
25. Liew, K.M. and Teo, T.M. "Three-dimensional vibration analysis of rectangular plates based on differential quadrature method", *J. of Sound and Vibration*, **220**(4), pp. 577-599 (1999).
26. Zhou, D. and Cheung, Y.K. "Free vibration analysis of line supported rectangular plates using a set of static beam functions", *J. of Sound and Vibration*, **223**(2), pp. 231-245 (1999).
27. Xiang, Y. et al. "Levy solutions for vibration of multi-span rectangular plates", *Int. J. of Mechanical Sciences*, **44**(6), pp. 1195-1218 (2002).
28. Xiang, Y. and Wei, G.W. "Exact solutions for vibration of multi-span rectangular Mindlin plates", *J. of Vibration and Acoustics*, **124**(4), pp. 545-551 (2002).