Scientia Iranica, Vol. 15, No. 2, pp 182–185

SCIENTIA
IRANICA

Research Note

On the Existence of Periodic Solutions for Nonlinear Ordinary Differential Equations

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In this paper, the existence of periodic solutions of autonomous ordinary differential equations of a 4th and 5th order is investigated. The method used is based on the Brower's degree theorem using the homotopy invariant a property of a topological degree.

INTRODUCTION

The existence of a nontrivial periodic solution of non autonomous nonlinear third order differential equations of the following form:

$$x''' + f(x, x', x'') = 0,$$

$$f(x, -x', x'') = -f(x, x', x''),$$

has been investigated in [1]. Here the authors generalize the results in [1-3] for 4th and 5th order autonomous equations.

It is interesting to note that the existence of periodic solutions of nonlinear autonomous differential equations has not been extensively investigated. The Poincare Bendixon theorem, which is a powerful tool for the investigation of periodic solutions of second order differential equations, is not applicable for third and higher order systems. In the following, the idea of Brower's degree theorem is used to prove the existence of periodic solutions of higher order systems. The numerical results obtained demonstrate the validity of the analytical method used.

Lemma 1

Consider the following second order system:

$$x'' = -w_1^2 x + f_1(x, y, x', y'),$$

$$y'' = -w_2^2 y + f_2(x, y, x', y'), \tag{1}$$

where f_1 and f_2 are smooth enough to ensure the uniqueness and existence of solutions. Without loss of generality one can assume $w_1 > w_2$. Suppose there exist $1 < k < 2$ and $c > 0$, such that:

i) $k < \frac{w_1}{w_2} < \frac{2}{k-1}$,

ii) $\delta w_2^2 c > (k+1)M$,

where:

$$\delta = \min_{i,j} |\sin w_i T_j|,$$

$$T_1 = \frac{\pi}{w_1 + w_2},$$

$$T_2 = \frac{(k+1)\pi}{w_1 + w_2},$$

$$\min_{i,j} |\sin w_i T_j| = \min\left\{ \begin{vmatrix} \sin w_1 T_1 & 0 \\ 0 & \sin w_2 T_1 \end{vmatrix}, \begin{vmatrix} \sin w_1 T_2 & 0 \\ 0 & \sin w_2 T_2 \end{vmatrix} \right\},$$

and:

$$M = \max\{|f_1|_D,\ |f_2|_D\},$$

$$D = \{x, y; x', y' : |x| < 2c, |y| < 2cw, |y'| < 2cw\},$$

then, there exist T, $T_2 < T < T_1$ and $c_1 \neq 0$, $c_2 \neq 0$, such that, if $x'(0) = c_1$ and $y'(0) = c_2$, then $x(T) = y(T) = 0$.

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Proof

For brevity, the following notation is introduced:

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(X, X') = \begin{pmatrix} f_1(x, y, x', y') \\ f_2(x, y, x', y') \end{pmatrix},$$

$$W = \begin{vmatrix} w_1 & 0 \\ 0 & w_2 \end{vmatrix}.$$

Next, the initial value problem is considered:

$$X'' + W^2 X = F(X, X'),$$

$$X(0) = 0, \qquad X'(0) = WC, \tag{2}$$

where C is the vector (c_1, c_2).

The solution of Equation 2 is given by:

$$X(t, C) = (\sin w_i t)C + G(t, X, X'),$$

$$G(t, X, X') = W^{-1} \int_0^t \sin w_i(t-s) F(X(s), X'(s)) ds. \tag{3}$$

Hence, it can be shown that:

$$d(|\sin w_i T_1|\underline{C}, \omega, 0) = 1, \quad d(|\sin w_i T_2|C, \omega, 0) = -1,$$

$$|\sin w_i T_j|C \geq \delta c > 0, \qquad |G(T_j, X, X')|_D < \delta c, \tag{4}$$

which implies:

$$d(X(T_j, C)\omega_c, 0) = (-1)^{j-1}, \qquad j = 1, 2. \tag{5}$$

By the homotopy invariant property of a topological degree [4,5] there exist $T, T_2 < T < T_1$, and $C, |C| = c$, such that $X(T, C) = 0$.

Next, Lemma 1 is used again to show the existence of periodic solutions of the following fourth order equation.

Theorm 1

Consider the following fourth order equation:

$$X^{(4)} + f(x, x', x'' x''') = 0, \tag{6}$$

where f is continuous and,

$$f(-x, x', -x'', x''') = -f(x, x', x'', x'''). \tag{7}$$

Assuming there exist $a, b > 0$ and $1 < k < 2$, such that:

$$k + \frac{1}{k} < \frac{a}{\sqrt{b}} < \frac{k-1}{2} + \frac{2}{k-1}, \tag{8}$$

and for a domain $D \in R^4$:

$$M = |ax'' + bx + f|_D,$$

is sufficiently small, then Equation 6 has a periodic solution.

Proof

Assume the following:

$$w_1{}^2 = \frac{a + \triangle}{2}, \qquad w_2{}^2 = \frac{a - \triangle}{2},$$

where $\triangle = \sqrt{a^2 - 4b}$. Now, Equation 6 is written in the equivalent form:

$$x'' = -w_1{}^2 x + \varepsilon y,$$

$$y'' = -w_2{}^2 y + \frac{1}{\varepsilon} g(x, y, x', y'). \tag{9}$$

Choosing M and ε small compared to w_2, by Lemma 1, a non zero solution is obtained with the condition $(x'(0),\ y'(0)) = 0$, such that $x(0) = y(0) = x(T) = y(T) = 0$.

The solution obtained on $[0, T]$ can be extended, periodically, to $[0, 2T]$, by defining $x(t)$, $y(t)$ as:

$$\begin{cases} x = x(t) \\ \qquad\qquad t \in\ [0, T]\ , \\ y = y(t) \end{cases}$$

$$\begin{cases} x = -x(2T - T) \\ \qquad\qquad t \in [T, 2T]\ . \\ y = -y(2T - t) \end{cases}$$

Furthermore, instead of Equation 7 one assumes:

$$f(-x, x', -x'', x''') = f(x, x', x'', x'''). \tag{10}$$

Again, a periodic solution is obtained by choosing the initial conditions:

$$x(0) = c, \qquad x'(0) = 0,$$

which in turn results in the following boundary conditions:

$$x'(0) = x'(T) = 0.$$

Thus, if f, in Equation 6, is independent of x' and x''', the parity condition Equation 10 can be ignored.

Next, the above results are generalized to the third order systems.

Lemma 2

Here, the third order system is considered, as follows:

$$x''' = -w_1^2 x' + f_1(x, y, x', y', x'', y''),$$

$$y''' = -w_2^2 y' + f_2(x, y, x', y', x'', y''). \tag{11}$$

Using the same assumptions as in Lemma 1, with:

$$D = \left\{ |x| \leq \frac{2c}{w_2},\ |x'| < 2c,\ |x''| \leq 2cw_1. \right.$$

Then, there exist T, $T_1 < T < T_2$, constants C and non-zero numbers c_1 and c_2 such that if:

$$x(0) = -\frac{c_1}{w_1}, \qquad y(0) = \frac{c_2}{w_2},$$

$$x''(0) = w_1 c_1, \qquad y''(0) = w_2 c_2,$$

then, $x'(T) = y'(T) = 0$.

Proof

Consider the following system:

$$\begin{cases} x''' + w^2 x' = F(x, x', x'') \\ x(0) = -W^{-1} c \\ x'(0) = 0 \\ x''(0) = WC \end{cases} \tag{12}$$

The solution of the above system is given by:

$$X'(t, C) = (\sin w_i t)C + G'(t, X, X', X''),$$

$$G'(t, x, x', x'') =$$

$$W^{-1} \int_0^t \sin w_i (t - s) F(x(s), x'(s), x''(s)) ds.$$

Again, one can write:

$$d[(\sin w_i T_1), \Omega_c, 0] = 1, \qquad d[(\sin w_i T_2), \Omega_c, 0] = -1,$$

$$|\sin w_i T_j| \geq \delta c > 0, \qquad |G'(T_j, x, x', x'')|_D < \delta c.$$

This implies:

$$d[X'(T_j, C), \Omega_c, 0] = (-1)^{j-1}, \qquad j = 1, 2.$$

Again by hemotopy invariance property, it is concluded that there exist T, $T_1 < T < T_2$ and C, $|C| = c$, such that $X'(T, C) = 0$.

Now, using Lemma 2, the following result on the existence of periodic solutions of the 5th order equation in the following theorem can be shown.

Theorem 2

Here, the authors consider the 5th order equation:

$$x^{(5)} + f(x, x', x'', x''', x^{(4)}) = 0, \tag{13}$$

where f is continuous and:

$$f(x, -x', x'', x''', x^{(4)}) = -f(x, x', x'', x''', x^{(4)}).$$

Now let a, $b > 0$ and $k \in (1, 2)$, such that:

$$k + \frac{1}{k} < \frac{a}{\sqrt{b}} < \frac{k-1}{2} + \frac{2}{k-1}, \tag{14}$$

then, if $m = |ax''' + bx' - f|_D$ is sufficiently small, where $D \in R^5$ contains the origins, then Equation 13 has a periodic solution.

Proof

Using Relation 14, one can write:

$$w_1^2 = \frac{a + \Delta}{2}, \qquad \Delta = \sqrt{a^2 - 4b}.$$

Next, Equation 13 can be written in the following equivalent form:

$$x''' = -w_1^2 x' + \varepsilon y',$$

$$y''' = -w_2^2 y' + \frac{1}{\varepsilon} g(x, y, x', y', x'', y''), \tag{15}$$

where it is assumed that $g = ax''' + bx' - f$ and ε is sufficiently small in comparison with w_2.

Now, considering Lemma 2 and assuming g is sufficiently small, Equation 15 has a solution satisfying initial conditions,

$$x'(0) = y'(0) = x'(T) = y'(T).$$

Next, the solution $(x(t), y(t))$ is extended periodically to $[0, 2T]$ by:

$$x = x(t), \qquad y = y(t), \qquad t \in [0, T],$$

$$x = x(2T - t), \qquad y = y(2T - t), \qquad t \in [T, 2T].$$

Example 1

The authors consider the 4th order equation:

$$x^{(4)} + 5x'' + 4x + xx' + 3/2(x'')^3 x''' = 0,$$

or the following equivalent system with the given initial conditions,

$$\begin{cases} x' = y \\ y' = z \\ z' = u \\ u' = -5z - 4x - xy - \frac{3}{2} z^3 u \end{cases}$$

$$x(0) = 0, \quad y(0) = 0.5, \quad z(0) = 0, \quad u(0) = -1.08128.$$

This system has a periodic solution with period $w = 6.40769$ (Figure 1), where:

$$|x(w)| < 2 \times 10^{-7}, \qquad y(w) = 0.5,$$

$$|z(w)| = 1.4 \times 10^{-6}, \qquad u(w) = -1.08128.$$

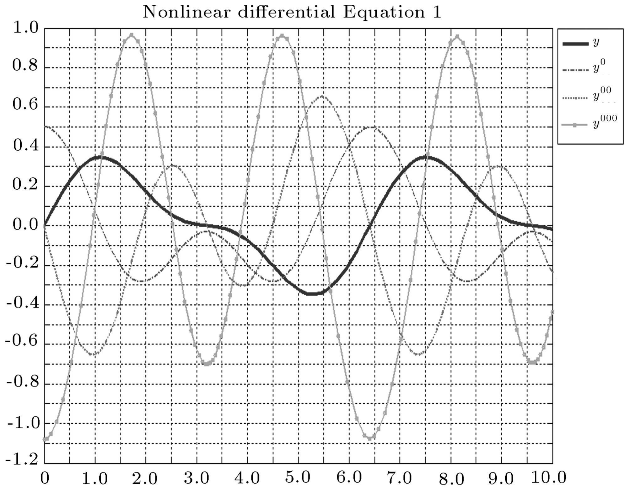

Figure 1. Periodic behavior of the solution of the system of Example 1.

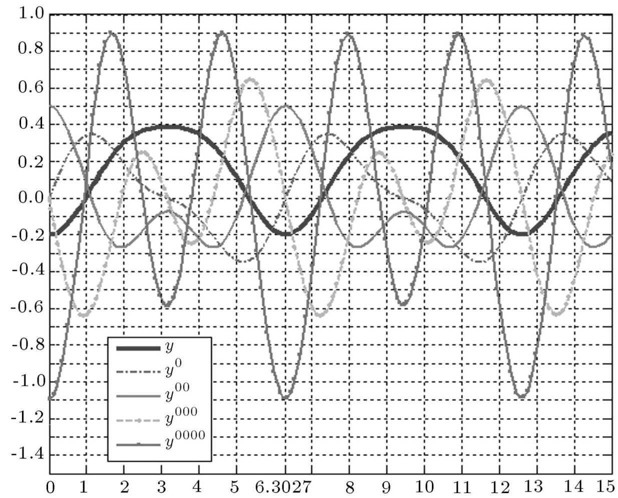

Figure 2. Periodic behavior of the solution of the system of Example 2.

Example 2

Next, look at the following 5th order equation:

$$x^{(5)} + 5x''' + 4x' + x'x^2 + 3/2(x''')^3x^4 = 0,$$

or the equivalent system:

$$x' = y, \qquad y' = z, \qquad z' = u,$$

$$u' = v, \qquad v' = -5u - 4y - yx^2 - 3/2u^3v,$$

with the following initial conditions:

$$x(0) = -0.2, \qquad y(0) = 0, \qquad z(0) = 0.50307,$$

$$u(0) = 0, \qquad v(0) = -1.1.$$

This system has a periodic solution with period $w = 6.30277$ (Figure 2), where:

$$x(w) = -0.2, \qquad |y(w) < 10^{-7}, \qquad z(w) = 0.05307,$$

$$u(w) < 10^{-7}, \qquad v(w) = -1.1.$$

CONCLUSION

Sufficient conditions are given for the existence of periodic solutions of a class of third order equations. Specific examples are also given to illustrate the behavior of the solutions as was indicated by the theorems.

ACKNOWLEDGMENT

This research has been supported by Research Department of Sharif University of Technology.

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