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Intrinsic Expressions for Arbitrary Stress Tensors Conjugate to General Strain Tensors

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In this paper, a unified explicit tensorial relation is sought between two stress tensors conjugate to arbitrary and general Hill strains. The approach used for deriving the tensorial relation is based on the eigenprojection method. The result is, indeed, a generalization of the relations that were derived by Farahani and Naghadabadi [1] in 2003 from a component to intrinsic form. The result is unified in the sense that it is valid for all cases of distinct and coalescent principal stretches. Also, in the case of three dimensional Euclidean inner product space, using the derived unified relation, some expressions for the conjugate stress tensors are presented.

INTRODUCTION

The rate of mechanical work per unit volume of the body in the reference configuration, $\dot{w}$, which is called stress power, is defined by [2]:

$$\dot{w} = J\boldsymbol{\sigma} : \mathbf{D} = \boldsymbol{\tau} : \mathbf{D}, \tag{1}$$

where J denotes the ratio of the volume of the material in the current configuration to the reference configuration, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{D}$ stands for the deformation rate tensor, i.e., the symmetric part of the velocity gradient in space, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress tensor and : represents the double scalar product.

Let $\{\lambda_i\}$ and $\{N_i\}$ be the eigenvalues and the subordinate orthonormal eigenvectors of the right stretch tensor, $\mathbf{U}$, respectively. Indeed, $\{\lambda_i\}$ are the principal stretches of the deformation. The general class of Lagrangean strain tensors was defined by Hill as [3,4]:

$$\mathbf{E}^{(f)} = \mathbf{E}^{(f)}(\mathbf{U}) = \sum_{i=1}^{n} f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{2}$$

where n is the dimension of the Euclidean space and the scale function $f(.)$ is an arbitrary, strictly-increasing scalar function satisfying conditions $f(1) = 0$ and $f'(1) = 1$. Also, the symbol $\otimes$ represents the dyadic or tensor product. If, in the special case, $f(.)$ is selected in the form of $f(\lambda) = (\lambda_i^r \quad 1)/r$, with r as an arbitrary integer, then, the Seth class of strains [5], with the notation $\mathbf{E}^{(r)}$, can be obtained. For example, the nominal strain, $\mathbf{E}^1 = (\mathbf{U} \quad \mathbf{I})$, Green's strain, $\mathbf{E}^2 = (\mathbf{U}^2 \quad \mathbf{I})/2$, and the logarithmic strain, $\mathbf{E}^{(0)} = \ln \mathbf{U}$, are strain measures in the Seth class that are given by the scale functions $f(\lambda) = (\lambda_i \quad 1)$, $f(\lambda) = (\lambda_i^2 \quad 1)/2$ and $f(\lambda) = \ln \lambda$, respectively.

The concept of energy conjugacy was introduced by Hill [3] and Macvean [6]. In this concept, a symmetric second order tensor, $\mathbf{T}$, is the conjugate stress to the Lagrangean strain measure, $\mathbf{E}$, if the double scalar product of $\mathbf{T}$ and $\mathbf{E}$ produce the stress power, i.e. $\dot{w}$.

$$\dot{w} = \mathbf{T} : \dot{\mathbf{E}}. \tag{3}$$

In the other words, the concept of energy conjugacy for stress and strain measures states that a stress tensor, $\mathbf{T}$, is conjugate to a strain measure, $\mathbf{E}$, if $\mathbf{T} : \dot{\mathbf{E}}$ provides the rate of change of the internal energy per unit reference volume of the body in an adiabatic process. For example, the Biot stress, $\mathbf{T}^{(1)}$, the second Piola-Kirchhoff stress, $\mathbf{T}^{(2)}$ and $\mathbf{T}^{(0)}$ are conjugate to $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ and $\mathbf{E}^{(0)}$, respectively.

The concept of energy conjugacy plays a great role in writing the internal power of a deforming body. Also, the virtual work required for finite element implementation, as a weak form of equilibrium equations, can be developed, in terms of a stress measure and the variation of its conjugate strain, as a basis for the analysis of a continuum.

In the study of nonlinear continuum mechanics, the aforementioned strain measures, their rates and

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their conjugate stresses, are basic quantities [4,7]. Finding expressions for these basic quantities has been a popular problem among researchers in the past decades. Hill derived component expressions for arbitrary stress tensor $\mathbf{T}^{(f)}$ conjugate to $\mathbf{E}^{(f)}$ in the principal basis, in terms of the components of Kirchhoff stress tensor $\boldsymbol{\tau}$ [4]. Guo and Dubey derived a more compact form of the component expressions for $\mathbf{T}^{(f)}$, in terms of [7].

Intrinsic (basis-free) expressions for the stress $\mathbf{T}^{(0)}$ conjugate to the Lagrangean logarithmic strain, $\ln \mathbf{U}$, were derived by Hoger [8]. Wang and Duan obtained intrinsic expressions for the stress conjugate to an arbitrary Hill strain measure, in terms of back rotated Cauchy stress, $\mathbf{R}^T\sigma\mathbf{R}$, where $\mathbf{R}$ is the rotation tensor of the deformation relative to the reference configuration [9]. Also, intrinsic expressions for the conjugate stress of an arbitrary Seth's measure of strain, excluding $r = 0$, were derived by Guo and Man [10].

Xiao derived a unified explicit intrinsic relation for the stress conjugate to an arbitrary strain of Hill's class, in terms of the Biot(-Jaumann) stress tensor, $\mathbf{T}^{(1)}$ [11]. A new intrinsic expression for the stress tensor, $\mathbf{T}^{(3)}$, i.e. the conjugate stress to Seth's strain measure with $r = 3$, was derived by Dui and Ren through solving a tensor equation [12]. Dui et al. also have derived new and compact intrinsic expressions for stress tensor conjugate to an arbitrary strain of Seth's strain measures, excluding $r = 0$, in terms of $\mathbf{T}^{(1)}$ [13].

Farahani and Naghdabadi obtained the relation between components of two stress tensors, conjugate to arbitrary strains of Seth's class, in the principal coordinates of $\mathbf{U}$ [14]. They also derived the expressions relating the component of two stress tensors, conjugate to arbitrary Hill strain measures and obtained some intrinsic expressions between these stress tensors in special cases [1]. Recently, Dui has derived six intrinsic (basis-free) expressions for the conjugate of the logarithmic strain, i.e., $\mathbf{T}^{(0)}$ [15].

It should be pointed out that the representation of a tensor in component form might not be convenient for the purpose of theoretical and numerical studies. Thus, it is of merit to seek the basis-free or intrinsic representation of the desired tensors [9].

The main purpose of this work is to:

1. Generalize the component-form results of Farahani and Naghdabadi [1] (relating two arbitrary conjugate stresses to each other) to intrinsic expressions;
2. Obtain a unified intrinsic relation between two arbitrary conjugate stresses which is valid for a Euclidean inner product space with arbitrary dimension. The result is unified in the sense that it is valid for all different cases of distinct and coalescent principal stretches.

The outline of the paper is as follows. First, some basic relations in nonlinear continuum mechanics are reviewed. Then, using the eigenprojection method, a unified intrinsic expression is derived which relates two arbitrary conjugate stress tensors to each other. The derived expression is valid for all different cases of distinct and coalescent principal stretches. After that, in the special case of the three dimensional inner product space, the specific results are obtained from the derived unified expression presented previously. Finally, some examples are presented as applications of the results of this paper.

PRELIMINARIES

The deformation gradient tensor, $\mathbf{F}$, at a point of a deforming body, is written as a multiplication of the symmetric positive-definite right stretch tensor, $\mathbf{U}$, and the proper orthogonal tensor, $\mathbf{R}$, as follows:

$$\mathbf{F} = \mathbf{R}\mathbf{U}. \tag{4}$$

In the n-dimensional Euclidean inner product space, let $\lambda_1, \cdots, \lambda_m$ be all the distinct eigenvalues of $\mathbf{U}$ and $\mathbf{P}_1, \cdots, \mathbf{P}_m$, its subordinate eigenprojections. Thus, $\mathbf{U}$ can be written as:

$$\mathbf{U} = \sum_{i=1}^{m} \lambda_i \mathbf{P}_i, \tag{5}$$

where m is the number of the distinct eigenvalues of $\mathbf{U}$.

The characteristics of the eigenprojections result in:

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i, \tag{6}$$

$$\sum_{i=1}^{m} \mathbf{P}_i = \mathbf{I}, \tag{7}$$

where δ_{ij} is the Kronecker delta and $\mathbf{I}$ is the second order identity tensor over the Euclidean inner product space. It is noted that the summation convention is not used over dummy indices in this paper. Eigenprojections of any second order tensor, such as $\mathbf{U}$, are also expressible in terms of the tensor and its eigenvalues, as follows [16]:

$$\mathbf{P}_i = \begin{cases} \prod\limits_{\substack{j=1 \\ j\neq i}}^{m} \frac{\mathbf{U} \quad \lambda_j\mathbf{I}}{\lambda_i \quad \lambda_j} & m > 1 \\ \mathbf{I} & m = 1 \end{cases} \tag{8}$$

Using the eigenprojections, the general strain measures defined in Equation 2 are written in the following form:

$$\mathbf{E}^{(f)} = \sum_{i=1}^{m} f(\lambda_i)\mathbf{P}_i. \tag{9}$$

The material time derivative of $\mathbf{E}^{(f)}$ is equal to [11]:

$$\dot{\mathbf{E}}^{(f)} = \sum_{i,j=1}^{m} f_{ij}\mathbf{P}_i\dot{\mathbf{U}}\mathbf{P}_j = \mathbf{L}^f(\mathbf{U}) : \dot{\mathbf{U}}, \tag{10}$$

where the scalars, f_{ij}, are expressible, as follows:

$$f_{ij} = \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & i \neq j \\ f'(\lambda_i) & i = j \end{cases} \tag{11}$$

and $\mathbf{L}^f = \frac{\partial \mathbf{E}^{(f)}}{\partial \mathbf{U}}$ is a fourth order tensor valued function of $\mathbf{U}$. Also, $(.)'$ stands for the derivative, with respect to the argument. Considering Lin as the second order tensor space over the n-dimensional Euclidean inner product space and defining a bilinear map $* : \text{Lin} \times \text{Lin} \rightarrow L(\text{Lin}) : (\mathbf{A},\mathbf{B}) \mapsto \mathbf{A} * \mathbf{B}$ by [17]:

$$(\mathbf{A} * \mathbf{B}) : \mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^T; \qquad \forall \mathbf{C} \in \text{ Lin}, \tag{12}$$

then, $\mathbf{L}^f(\mathbf{U})$, presented in Equation 10, can be written, as follows:

$$\mathbf{L}^f(\mathbf{U}) = \sum_{i,j=1}^{m} f_{ij}\mathbf{P}_i * \mathbf{P}_j. \tag{13}$$

It is noted that $L(\text{Lin})$ represents the second-order tensor space over Lin and $\mathbf{I} * \mathbf{I}$ is the second-order identity tensor over Lin. Also, the symmetry of each $\mathbf{P}_i$, together with the type of summation in Equation 13, results that $\mathbf{L}^f$ possess both major and minor symmetries. It is emphasized that the scale tensor, $f(.)$, can be replaced by any other proper scale tensor, such as $g(.)$, in the aforementioned relations. For example, by considering the definitions of Equations 11 and 13, one may obtain parameters, such as g_{ij} and $\mathbf{L}^g$, similar to f_{ij} and $\mathbf{L}^f$, respectively.

INTRINSIC EXPRESSION BETWEEN TWO STRESSES CONJUGATE TO ARBITRARY HILL'S STRAINS

Two different arbitrary Lagrangean strains $\mathbf{E}^{(f)}$ and $\mathbf{E}^{(g)}$, with scale functions $f(.)$ and $g(.)$ are considered, respectively. Let $\mathbf{T}^{(f)}$ and $\mathbf{T}^{(g)}$ be their conjugate stresses. Thus, in view of Equation 3, one can write:

$$\mathbf{T}^{(f)} : \dot{\mathbf{E}}^{(f)} = \mathbf{T}^{(g)} : \dot{\mathbf{E}}^{(g)}. \tag{14}$$

Now, substitution of Equation 10 into Equation 14 yields:

$$(\mathbf{T}^{(f)} : \mathbf{L}^f) : \dot{\mathbf{U}} = (\mathbf{T}^{(g)} : \mathbf{L}^g) : \dot{\mathbf{U}}. \tag{15}$$

Since Equation 15 is valid for every $\dot{\mathbf{U}}$, one obtains:

$$\mathbf{T}^{(f)} : \mathbf{L}^f = \mathbf{T}^{(g)} : \mathbf{L}^g. \tag{16}$$

Also, in view of the symmetries existing in the fourth-order tensors, $\mathbf{L}^f$ and $\mathbf{L}^g$, it is concluded that:

$$\mathbf{L}^f : \mathbf{T}^{(f)} = \mathbf{L}^g : \mathbf{T}^{(g)}. \tag{17}$$

Premultiplying Equation 17 by the inverse of the $\mathbf{L}^f$, one obtains:

$$\mathbf{T}^{(f)} = [(\mathbf{L}^f)^{-1} : \mathbf{L}^g] : \mathbf{T}^{(g)}, \tag{18}$$

where $(\mathbf{L}^f)^{-1}$ is the inverse of $\mathbf{L}^f$, in the sense that the double contraction, $(\mathbf{L}^f)^{-1} : \mathbf{L}^f$, results the identity tensor $\mathbf{I} * \mathbf{I}$ over Lin. It is obvious from Equation 7 that:

$$\sum_{i,j=1}^{m} \mathbf{P}_i * \mathbf{P}_j = \mathbf{I} * \mathbf{I}. \tag{19}$$

Considering Equations 6, 13 and 19, it is concluded that $(\mathbf{L}^f)^{-1}$ is in the form of:

$$(\mathbf{L}^f)^{-1} = \sum_{i,j=1}^{m} (f_{ij})^{-1}\mathbf{P}_i * \mathbf{P}_j. \tag{20}$$

So, Equations 6, 13 and 20 yield:

$$(\mathbf{L}^f)^{-1} : \mathbf{L}^g = \sum_{i,j=1}^{m} (f_{ij})^{-1} g_{ij}\mathbf{P}_i * \mathbf{P}_j. \tag{21}$$

With the help of Equation 8, the eigenprojections, $\mathbf{P}_i$, can be expressed in terms of a power series of $\mathbf{U}$, as follows:

$$\mathbf{P}_i = \frac{1}{d_i}\sum_{k=0}^{m-1} a_{i,m-1-k}\mathbf{U}^k, \tag{22}$$

where:

$$d_i = \prod_{\substack{j=1 \\ j\neq i}}^{m} (\lambda_i - \lambda_j), \tag{23}$$

and:

$$a_{i,j} = \begin{cases} 0 & j = 0 \\ (-1)^j \sum\limits_{1\leq i_1<\cdots<i_j\leq m}^{m-1} \lambda_{i_1}\cdots\lambda_{i_j}(1-\delta_{ii_1})\cdots(1-\delta_{ii_j}) & 0<j \end{cases}. \tag{24}$$

Substituting the eigenprojections from Equation 22 into Equation 21 and using Equations 23 and 24, $\mathbf{T}^{(f)}$ can be written as an isotropic tensor valued function of $\mathbf{U}$ and $\mathbf{T}^{(g)}$, and linear in $\mathbf{T}^{(g)}$, in the following form:

$$\mathbf{T}^{(f)} = \sum_{i,j=0}^{m-1} \alpha_{ij}\mathbf{U}^i\mathbf{T}^{(g)}\mathbf{U}^j, \tag{25}$$

where the scalars, α_{ij}, are given as:

$$\alpha_{ij} = \sum_{k,l=1}^{m} \frac{g_{kl}}{d_k d_l f_{kl}} a_{k,m-1-i}\, a_{l,m-1-j}. \tag{26}$$

Equation 25 is a unified intrinsic expression relating two arbitrary conjugate stresses, $\mathbf{T}^{(f)}$ and $\mathbf{T}^{(g)}$, which is valid for all cases of distinct and repeated principal stretches. Also, Equation 25 is valid for arbitrary dimensions of the Euclidean inner product space. This intrinsic expression, which relates two arbitrary conjugate stress tensors, has not been presented in the literature, as far as the authors know. Equation 25 is, indeed, the generalization of the results of Farahani and Naghdabadi [1], from a component to intrinsic form. As explained in the introduction, component expressions are not generally satisfactory. Thus, the need to find intrinsic expressions for the basic tensors is useful.

RESULTS FOR THREE DIMENSIONAL EUCLIDEAN INNER PRODUCT SPACE

In this section, the three dimensional Euclidean inner product space is considered, i.e., $n = 3$, and specific results are obtained from Equation 25 for all possible cases of distinct and coalescent principal stretches, i.e., for different values of m.

The Case of Non-Coalescent Principal Stretches, i.e., $m = 3$

Substitution of Equations 11, 23 and 24 for the case $m = 3$ into Equation 26 and, then, substitution of the results into Equation 25 yields:

$$\begin{aligned}\mathbf{T}^{(f)} &= \alpha_{00}\mathbf{T}^{(g)} + \alpha_{01}(\mathbf{T}^{(g)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(g)}) \\ &\quad + \alpha_{11}\mathbf{U}\mathbf{T}^{(g)}\mathbf{U} + \alpha_{02}(\mathbf{T}^{(g)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(g)}) \\ &\quad + \alpha_{12}(\mathbf{U}\mathbf{T}^{(g)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(g)}\mathbf{U}) + \alpha_{22}\mathbf{U}^2\mathbf{T}^{(g)}\mathbf{U}^2,\end{aligned} \tag{27}$$

where:

$$\begin{aligned}\alpha_{00} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_i^2\lambda_j^2(\lambda_i-\lambda_j)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ 2\mathrm{III}\lambda_k(\lambda_i-\lambda_j)^{-2}(\lambda_k-\lambda_i)^{-1} \\ &\times(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg],\end{aligned} \tag{28}$$

$$\begin{aligned}\alpha_{01} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ \mathrm{III}[2+\mathrm{III}\lambda_i^{-2}\lambda_j^{-2}\times(\lambda_i+\lambda_j)](\lambda_i-\lambda_j)^{-2} \\ &(\lambda_k-\lambda_i)^{-1}(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg],\end{aligned} \tag{29}$$

$$\begin{aligned}\alpha_{11} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i^2-\lambda_j^2)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ (\mathrm{I\,II}-\mathrm{III})(\lambda_i^2-\lambda_j^2)^{-1}\times(\lambda_i-\lambda_j)^{-1} \\ &(\lambda_k-\lambda_i)^{-1}(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg],\end{aligned} \tag{30}$$

$$\begin{aligned}\alpha_{02} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i-\lambda_j)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ \lambda_k(\lambda_i+\lambda_j)\times(\lambda_i-\lambda_j)^{-2} \\ &(\lambda_k-\lambda_i)^{-1}(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg],\end{aligned} \tag{31}$$

$$\begin{aligned}\alpha_{12} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ (\lambda_i+\lambda_j+2\,\mathrm{III}\,\lambda_i^{-1}\lambda_j^{-1})\times(\lambda_i-\lambda_j)^{-2} \\ &(\lambda_k-\lambda_i)^{-1}(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg],\end{aligned} \tag{32}$$

$$\begin{aligned}\alpha_{22} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i-\lambda_j)^2\frac{g'(\lambda_k)}{f'(\lambda_k)} \\ &+ 2(\lambda_i-\lambda_j)^{-2}\times(\lambda_k-\lambda_i)^{-1} \\ &(\lambda_j-\lambda_k)^{-1}\frac{g(\lambda_i)-g(\lambda_j)}{f(\lambda_i)-f(\lambda_j)}\Bigg].\end{aligned} \tag{33}$$

In Equations 28 to 33, I, II, III are the three principal invariants of $\mathbf{U}$ and Δ is expressible in terms of the principal invariants of $\mathbf{U}$, as follows:

$$\Delta = 18\,\mathrm{I\,II\,III} + \mathrm{I}^2\,\mathrm{II}^2 - 4\,\mathrm{I}^3\mathrm{III} - 4\,\mathrm{II}^3 - 27\,\mathrm{III}^2. \tag{34}$$

The Case of Double Coalescent Principal Stretches, i.e. $m = 2$; $\lambda_1 \neq \lambda_2$

The expansion of Equation 25, with $m = 2$, leads to:

$$\mathbf{T}^{(f)} = \beta_{00}\mathbf{T}^{(g)} + \beta_{01}(\mathbf{T}^{(g)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(g)}) + \beta_{11}\mathbf{U}\mathbf{T}^{(g)}\mathbf{U}, \tag{35}$$

where:

$$\beta_{00} = \frac{1}{(\lambda_1 - \lambda_2)^2}\left(\lambda_1^2\frac{g'(\lambda_2)}{f'(\lambda_2)} + \lambda_2^2\frac{g'(\lambda_1)}{f'(\lambda_1)} - 2\lambda_1\lambda_2\frac{g(\lambda_1) - g(\lambda_2)}{f(\lambda_1) - f(\lambda_2)}\right), \tag{36}$$

$$\beta_{01} = \frac{1}{(\lambda_1 - \lambda_2)^2}\left(\lambda_1\frac{g'(\lambda_2)}{f'(\lambda_2)} + \lambda_2\frac{g'(\lambda_1)}{f'(\lambda_1)} - (\lambda_1 + \lambda_2)\frac{g(\lambda_1) - g(\lambda_2)}{f(\lambda_1) - f(\lambda_2)}\right), \tag{37}$$

$$\beta_{11} = \frac{1}{(\lambda_1 - \lambda_2)^2}\left(\frac{g'(\lambda_2)}{f'(\lambda_2)} + \frac{g'(\lambda_1)}{f'(\lambda_1)} - 2\frac{g(\lambda_1) - g(\lambda_2)}{f(\lambda_1) - f(\lambda_2)}\right). \tag{38}$$

The Case of Triple Coalescent Principal Stretches, i.e., $m = 1$; $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda_0$

Equation 25, with m=1, can be simplified to:

$$\mathbf{T}^{(f)} = \gamma_{00}\mathbf{T}^{(g)}, \tag{39}$$

where:

$$\gamma_{00} = \frac{g'(\lambda_0)}{f'(\lambda_0)}. \tag{40}$$

EXAMPLES

In this section, some illustrative examples are presented as applications of the derived expressions. The results of the previous sections are general, so, for obtaining the relation between two specific conjugate stresses, it suffices to replace the scale functions, $f(\lambda)$ and $g(\lambda)$, with the appropriate ones in the corresponding equations, determining the scalar coefficients.

The Biot Stress $\mathbf{T}^{(1)}$ in Terms of the Second Piola-Kirchhof Stress $\mathbf{T}^{(2)}$

The Biot stress, $\mathbf{T}^{(1)}$, the stress conjugate to the nominal strain $\mathbf{E}^{(1)}$, is expressible in terms of the second Piola-Kirchhoff stress, $\mathbf{T}^{(2)}$, and the right stretch tensor, as follows [10]:

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}). \tag{41}$$

Here, this expression is obtained using the derived equations in this paper. This example has been selected as validation of the general results derived in this paper. With the aid of Equation 27, in the case of distinct principal stretches, it is written that:

$$\mathbf{T}^{(1)} = \overline{\alpha}_{00}\mathbf{T}^{(2)} + \overline{\alpha}_{01}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}) + \overline{\alpha}_{11}\mathbf{U}\mathbf{T}^{(2)}\mathbf{U} + \overline{\alpha}_{02}(\mathbf{T}^{(2)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(2)}) + \overline{\alpha}_{12}(\mathbf{U}\mathbf{T}^{(2)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(2)}\mathbf{U}) + \overline{\alpha}_{22}\mathbf{U}^2\mathbf{T}^{(2)}\mathbf{U}^2, \tag{42}$$

where the scalar coefficients, $\overline{\alpha}_{ij}$, are calculated using Equations 28-33, by replacing $f(\lambda)$ and $g(\lambda)$ with $(\lambda - 1)$ and $(\lambda^2 - 1)/2$, respectively. So, one obtains:

$$\overline{\alpha}_{00} = \overline{\alpha}_{11} = \overline{\alpha}_{02} = \overline{\alpha}_{12} = \overline{\alpha}_{22} = 0, \tag{43}$$

and:

$$\overline{\alpha}_{01} = \frac{1}{2}. \tag{44}$$

Substitutions of these coefficients into Equation 42 results in Equation 41. In a similar manner, for the case of double coalescent principal stretches, using Equations 35 to 38 by replacing $(\lambda - 1)$ and $(\lambda^2 - 1)/2$ for $f(\lambda)$ and $g(\lambda)$, respectively, one arrive at Equation 41. In the case of triple coalescent principal stretches, i.e. $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda_0$, Equations 39 and 40, with $f'(\lambda) = (\lambda - 1)' = 1$ and $g'(\lambda) = (\lambda^2 - 1)'/2 = \lambda$, give us:

$$\mathbf{T}^{(1)} = \lambda_0\mathbf{T}^{(2)}. \tag{45}$$

Noting that, in the case of $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda_0 =$, the right stretch tensor, $\mathbf{U}$, can be written as $\lambda_0\mathbf{I}$ and by substituting this into Equation 41, one arrives at Equation 45.

The Second Piola-Kirchhof Stress, $\mathbf{T}^{(2)}$, in Terms of the Biot Stress, $\mathbf{T}^{(1)}$

The expression of $\mathbf{T}^{(2)}$, in terms of $\mathbf{T}^{(1)}$, in the case of distinct principal stretches was derived in Farahani and Naghdabadi [1] by solving a system of six linear equations for obtaining six scalar coefficients. But, with the aid of the method presented in this paper, the desired expression can be obtained using Equations 27 to 33, only by considering $f(\lambda) = (\lambda^2 \quad 1)/2$ and $g(\lambda) = (\lambda \quad 1)$ in the corresponding equations. Hence, one gets:

$$\begin{aligned}\mathbf{T}^{(2)} &= \overline{\overline{\alpha}}_{00}\mathbf{T}^{(1)} + \overline{\overline{\alpha}}_{01}(\mathbf{T}^{(1)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(1)}) \\ &+ \overline{\overline{\alpha}}_{11}\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + \overline{\overline{\alpha}}_{02}(\mathbf{T}^{(1)}\mathbf{U}^{(2)} + \mathbf{U}^2\mathbf{T}^{(1)}) + \\ &+ \overline{\overline{\alpha}}_{12}(\mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^{(2)} + \mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}) \\ &+ \overline{\overline{\alpha}}_{22}\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2, \end{aligned} \tag{46}$$

where:

$$\begin{aligned}\overline{\overline{\alpha}}_{00} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_k^{\ 1}\lambda_i^2\lambda_j^2(\lambda_i \quad \lambda_j)^2 \\ &+ 4\ \mathrm{III}\ \lambda_k(\lambda_i \quad \lambda_j)^{\ 2}(\lambda_k \quad \lambda_i)^{\ 1} \\ &(\lambda_j \quad \lambda_k)^{\ 1}(\lambda_i+\lambda_j)^{\ 1}\Bigg], \end{aligned} \tag{47}$$

$$\begin{aligned}\overline{\overline{\alpha}}_{01} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i+\lambda_j)(\lambda_i \quad \lambda_j)^2\lambda_k^{\ 1} \\ &+ 2\ \mathrm{III}[2+\mathrm{III}\ \lambda_i^{\ 2}\lambda_j^{\ 2}(\lambda_i+\lambda_j)] \times (\lambda_i \quad \lambda_j)^{\ 2} \\ &(\lambda_k \quad \lambda_i)^{\ 1}(\lambda_j \quad \lambda_k)^{\ 1}(\lambda_i+\lambda_j)^{\ 1}\Bigg], \end{aligned} \tag{48}$$

$$\begin{aligned}\overline{\overline{\alpha}}_{11} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i^2 \quad \lambda_j^2)^2\lambda_k^{\ 1} \\ &+ 2(\mathrm{I}\ \mathrm{II} \quad \mathrm{III})(\lambda_i^2 \quad \lambda_j^2)^{\ 1}(\lambda_i \quad \lambda_j)^{\ 1} \\ &\times (\lambda_k \quad \lambda_i)^{\ 1}(\lambda_j \quad \lambda_k)^{\ 1}(\lambda_i+\lambda_j)^{\ 1}\Bigg], \end{aligned} \tag{49}$$

$$\begin{aligned}\overline{\overline{\alpha}}_{02} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i \quad \lambda_j)^2\lambda_k^{\ 1} \\ &+ 2\lambda_k(\lambda_i \quad \lambda_j)^{\ 2}(\lambda_k \quad \lambda_i)^{\ 1}(\lambda_j \quad \lambda_k)^{\ 1}\Bigg], \end{aligned} \tag{50}$$

$$\begin{aligned}\overline{\overline{\alpha}}_{12} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i+\lambda_j)(\lambda_i \quad \lambda_j)^2\lambda_k^{\ 1} \\ &+ 2(\lambda_i+\lambda_j+2\ \mathrm{III}\ \lambda_i^{\ 1}\lambda_j^{\ 1})(\lambda_i \quad \lambda_j)^{\ 2} \\ &\times (\lambda_k \quad \lambda_i)^{\ 1}(\lambda_j \quad \lambda_k)^{\ 1}(\lambda_i+\lambda_j)^{\ 1}\Bigg], \end{aligned} \tag{51}$$

$$\begin{aligned}\overline{\overline{\alpha}}_{22} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \Bigg[&\frac{1}{\Delta}(\lambda_i \quad \lambda_j)^2\lambda_k^{\ 1} + 4(\lambda_i \quad \lambda_j)^{\ 2} \\ &(\lambda_k \quad \lambda_i)^{\ 1}(\lambda_j \quad \lambda_k)^{\ 1}(\lambda_i+\lambda_j)^{\ 1}\Bigg], \end{aligned} \tag{52}$$

with Δ as given in Equation 34.

In the case of double coalescent principal stretches, i.e. $\lambda_1 \neq \lambda_2 = \lambda_3$, using Equations 35 to 38, with $f(\lambda) = (\lambda^2 \quad 1)/2$ and $g(\lambda) = (\lambda \quad 1)$, one obtains:

$$\mathbf{T}^{(2)} = \overline{\overline{\beta}}_{00}\mathbf{T}^{(1)} + \overline{\overline{\beta}}_{01}(\mathbf{T}^{(1)}\mathbf{U}+\mathbf{U}\mathbf{T}^{(1)}) + \overline{\overline{\beta}}_{11}\mathbf{U}\mathbf{T}^{(1)}\mathbf{U}, \tag{53}$$

where:

$$\begin{aligned}\overline{\overline{\beta}}_{00} = \frac{1}{(\lambda_1 \quad \lambda_2)^2}\Bigg(&\lambda_1^2\lambda_2^{\ 1} + \lambda_2^2\lambda_1^{\ 1} \\ &4\lambda_1\lambda_2(\lambda_1+\lambda_2)^{\ 1}\Bigg), \end{aligned} \tag{54}$$

$$\overline{\overline{\beta}}_{01} = \frac{1}{(\lambda_1 \quad \lambda_2)^2}\Bigg((\lambda_1^2+\lambda_2^2)\lambda_1^{\ 1}\lambda_2^{\ 1} \quad 2\Bigg), \tag{55}$$

$$\overline{\overline{\beta}}_{11} = \frac{1}{(\lambda_1 \quad \lambda_2)^2}\Bigg((\lambda_1+\lambda_2)\lambda_1^{\ 1}\lambda_2^{\ 1} \quad 4(\lambda_1+\lambda_2)^{\ 1}\Bigg). \tag{56}$$

In the case of triple coalescent principal stretches, i.e., $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda_0$, Equations 39 and 40, with $g'(\lambda) = (\lambda \quad 1)' = 1$ and $f'(\lambda) = (\lambda^2 \quad 1)'/2 = \lambda$, give us:

$$\mathbf{T}^{(2)} = \frac{1}{\lambda_0}\mathbf{T}^{(1)}. \tag{57}$$

The Biot Stress, $\mathbf{T}^{(1)}$, in terms of $\mathbf{T}^{(0)}$

Here, invariant expressions for the Biot stress, $\mathbf{T}^{(1)}$, in terms of the stress, $\mathbf{T}^{(0)}$, conjugate to the logarithmic strain, are presented. Similar to the previous examples, it suffices to use the results of the previous section, with suitable scale functions. In the case of distinct principal stretches, using Equations 27 to 33, with $g(\lambda) = \ln\lambda$ and $f(\lambda) = (\lambda \;\; 1)$, one obtains:

$$\mathbf{T}^{(1)} = \tilde{\alpha}_{00}\mathbf{T}^{(0)} + \tilde{\alpha}_{01}(\mathbf{T}^{(0)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(0)}) + \tilde{\alpha}_{11}\mathbf{U}\mathbf{T}^{(0)}\mathbf{U} + \tilde{\alpha}_{02}(\mathbf{T}^{(0)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(0)}) + + \tilde{\alpha}_{12}(\mathbf{U}\mathbf{T}^{(0)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(0)}\mathbf{U}) + \tilde{\alpha}_{22}\mathbf{U}^2\mathbf{T}^{(0)}\mathbf{U}^2, \tag{58}$$

where:

$$\tilde{\alpha}_{00} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}\lambda_k^{\;\;1}\lambda_i^2\lambda_j^2(\lambda_i \;\; \lambda_j)^2 + 2\,\mathrm{III}\,\lambda_k(\lambda_i \;\; \lambda_j)^{\;\;2}(\lambda_k \;\; \lambda_i)^{\;\;1} (\lambda_j \;\; \lambda_k)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1}\ln\frac{\lambda_i}{\lambda_j}\right], \tag{59}$$

$$\tilde{\alpha}_{01} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i+\lambda_j)(\lambda_i \;\; \lambda_j)^2\lambda_k^{\;\;1} + \mathrm{III}\,[2+\mathrm{III}\,\lambda_i^{\;\;2}\lambda_j^{\;\;2}(\lambda_i+\lambda_j)](\lambda_i \;\; \lambda_j)^{\;\;2} (\lambda_k \;\; \lambda_i)^{\;\;1}(\lambda_j \;\; \lambda_k)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1}\ln\frac{\lambda_i}{\lambda_j}\right], \tag{60}$$

$$\tilde{\alpha}_{11} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}(\lambda_i^2 \;\; \lambda_j^2)^2\lambda_k^{\;\;1} + (\mathrm{I\,II} \;\; \mathrm{III})(\lambda_i^2 \;\; \lambda_j^2)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1} (\lambda_k \;\; \lambda_i)^{\;\;1}(\lambda_j \;\; \lambda_k)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1}\ln\frac{\lambda_i}{\lambda_j}\right], \tag{61}$$

$$\tilde{\alpha}_{02} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}\lambda_i\lambda_j(\lambda_i \;\; \lambda_j)^2\lambda_k^{\;\;1} + \lambda_k(\lambda_i+\lambda_j) (\lambda_i \;\; \lambda_j)^{\;\;2}(\lambda_k \;\; \lambda_i)^{\;\;1}(\lambda_j \;\; \lambda_k)^{\;\;1} (\lambda_i \;\; \lambda_j)^{\;\;1}\ln\frac{\lambda_i}{\lambda_j}\right], \tag{62}$$

$$\tilde{\alpha}_{12} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}(\lambda_i+\lambda_j)(\lambda_i \;\; \lambda_j)^2\lambda_k^{\;\;1} + (\lambda_i+\lambda_j+2\,\mathrm{III}\,\lambda_i^{\;\;1}\lambda_j^{\;\;1})(\lambda_i \;\; \lambda_j)^{\;\;2} (\lambda_k \;\; \lambda_i)^{\;\;1}(\lambda_j \;\; \lambda_k)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1}\ln\frac{\lambda_i}{\lambda_j}\right], \tag{63}$$

$$\tilde{\alpha}_{22} = \sum_{\substack{(i,j,k)=(1,2,3),\\(2,3,1),(3,1,2)}} \left[\frac{1}{\Delta}(\lambda_i \;\; \lambda_j)^2\lambda_k^{\;\;1} + 2(\lambda_i \;\; \lambda_j)^{\;\;2} (\lambda_k \;\; \lambda_i)^{\;\;1}(\lambda_j \;\; \lambda_k)^{\;\;1}(\lambda_i \;\; \lambda_j)^{\;\;1}\frac{\lambda_i}{\lambda_j}\right]. \tag{64}$$

In the case of double coalescent principal stretches, i.e. $\lambda_1 \neq \lambda_2 = \lambda_3$, Equations 35 to 38, with $g(\lambda) = \ln\lambda$ and $f(\lambda) = (\lambda \;\; 1)$, give:

$$\mathbf{T}^{(1)} = \tilde{\beta}_{00}\mathbf{T}^{(0)} + \tilde{\beta}_{01}(\mathbf{T}^{(0)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(0)}) + \tilde{\beta}_{11}\mathbf{U}\mathbf{T}^{(0)}\mathbf{U}, \tag{65}$$

where:

$$\tilde{\beta}_{00} = \frac{1}{(\lambda_1 \;\; \lambda_2)^2}\left(\lambda_1^2\lambda_2^{\;\;1} + \lambda_2^2\lambda_1^{\;\;1} \;\; 2\lambda_1\lambda_2(\lambda_1 \;\; \lambda_2)^{\;\;1}\ln\frac{\lambda_1}{\lambda_2}\right), \tag{66}$$

$$\tilde{\beta}_{01} = \frac{1}{(\lambda_1 \;\; \lambda_2)^2}\left((\lambda_1^2+\lambda_2^2)\lambda_1^{\;\;1}\lambda_2^{\;\;1} \;\; (\lambda_1+\lambda_2)(\lambda_1 \;\; \lambda_2)^{\;\;1}\ln\frac{\lambda_1}{\lambda_2}\right), \tag{67}$$

$$\tilde{\beta}_{11} = \frac{1}{(\lambda_1 \;\; \lambda_2)^2}\left((\lambda_1+\lambda_2)\lambda_1^{\;\;1}\lambda_2^{\;\;1} \;\; 2(\lambda_1 \;\; \lambda_2)^{\;\;1}\ln\frac{\lambda_1}{\lambda_2}\right). \tag{68}$$

In the case of triple coalescent principal stretches, i.e. $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda_0$, considering Equations 39 and 40, with $g'(\lambda) = (\ln\lambda)' = 1/\lambda$ and $f'(\lambda) = (\lambda \;\; 1)' = 1$, one gets:

$$\mathbf{T}^{(1)} = \frac{1}{\lambda_0}\mathbf{T}^{(0)}. \tag{69}$$

The expressions representing the Biot stress, $\mathbf{T}^{(1)}$, in terms of the stress $\mathbf{T}^{(0)}$, have not been presented in the literature, as far as the authors know.

CONCLUSIONS

In this paper, a general and unified intrinsic relation between two arbitrary stresses, $\mathbf{T}^{(f)}$ and $\mathbf{T}^{(g)}$, conjugate to arbitrary strain tensors, $\mathbf{E}^{(f)}$ and $\mathbf{E}^{(g)}$, in Hill's general class of strains, are obtained. The approach is based on the eigenprojection expansion of symmetric tensors. The result that is obtained in Equation 25 is unified, in the sense that is generally valid for all different cases of coalescent and distinct principal stretches. Also, this equation is valid for a Euclidean inner product space with arbitrary dimensions. Equation 25 represents $\mathbf{T}^{(f)}$ as an isotropic tensor valued function of $\mathbf{T}^{(g)}$ and $\mathbf{U}$ that is linear in $\mathbf{T}^{(g)}$. Moreover, for three-dimensional Euclidean inner product space, i.e., $n = 3$, specific results are derived from Equation 25 for all cases of repeated and distinct principal stretches, i.e., for $m = 1, 2, 3$. The merit of the derived general results of this paper is that they can be used to obtain specific expressions for conjugate stresses, only by substituting the appropriate scale functions in the corresponding equations. As applications of the derived general results, three examples are presented, which relate some specific conjugate stress tensors to each other through intrinsic expressions for different cases of coalescence principal stretches.

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