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Research Note

Solving Nonlinear Ordinary Differential Equation as a Control Problem by Using Measure Theory

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In this paper, a new technique is introduced for finding the approximate solution of a nonlinear ordinary differential equation. In this method, first an ordinary differential equation problem is changed to an optimal control problem which itself is transformed into a measure theoric control problem, then this new problem is converted to an infinite dimensional linear programming. Finally, the solution of the infinite dimensional linear programming is approximated to the solution of a finite dimensional one and using the solution of this problem, an approximate solution for the original problem is obtained. Also, in this method, the total error of the approximate solution is found.

INTRODUCTION

There are many numerical methods for finding the solution of an ordinary differential equation. Here, a new technique is introduced for finding an approximate solution of a nonlinear ODE. A nonlinear ODE of the form:

$$y^{(n)} = g(t, y, y', ..., y^{(n-1)}), \qquad t \in J^\circ, \tag{1}$$

is considered with the initial conditions:

$$y(t_a) = y_0, y'(t_a) = y_1, ..., y^{(n-1)}(t_a) = y_{n-1}, \tag{2}$$

where $J = [t_a, t_b]$ with interior J°, $A = J \times Q$, Q is an n-cell in Euclidean space R^n and $g : A \to R$ is a continuous function. Using measure theory, an approximate solution $y(t)$, $t \in J$, of the initial value Problem 1 is found.

It should be noted that, due to the equivalency of an nth order ODE with a system of n first order ODEs, the above problem can be extended to find an approximate solution of a nonlinear system of first order ODEs as follows:

$$x' = f(t, x), \qquad (x, t) \in J^\circ \times Q, \tag{3}$$

$$x(t_a) = x_a, \tag{4}$$

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where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^t$ and $x_a = (x_1(t_a), x_2(t_a), ..., x_n(t_a))^t$ are vectors in R^n; also, $f = (f_1, f_2, ..., f_n)^t$ is a continuous function on A.

Rubio used measure theory to find a piecewise constant optimal control for classic optimal control problems [1]. A boundary optimal control problem for diffusion and wave equations, using measure theory, is considered by some authors (e.g., [2-7]); in [8], measure theory is used and an optimal control of an inhomogeneous wave problem with internal control is found.

CLASSICAL OPTIMAL CONTROL PROBLEM

Let:

$$x_1 = y, \; x_2 = y', ..., x_n = y^{n-1}, \tag{5}$$

then Problem 1 changes to the nonlinear system with initial Conditions 3 and 4, where $x_i(t_a) = y_{i-1}$, $i = 1, 2, ..., n$, and:

$$f_i(t, x) = \begin{cases} x_{i+1}(t) & i = 1, 2, ..., n-1 \\ g(t, x) & i = n. \end{cases}$$

Let k and k_1 be non-negative real constants such that $\|g(t, x)\| < k\|x\| + k_1$, for $(t, x) \in J \times R^n$; then:

$$\|f(t, x)\|^2 = \sum_{i=2}^{n} x_i^2 + g^2(t, x) \leq \|x\|^2 + (k\|x\| + k_1)^2$$
$$\leq (k_0\|x\| + k_1)^2 \quad \text{for} (t, x) \in J \times R^n \; ,$$

where $k_0 = k+1$. Therefore, using Theorem 3.2 of [9], there is a solution $x(t)$ of $x' = f(t,x)$ on $j = [t_a, t_b]$ for which $x(t_a) = x_0$.

Let $U = [a,b]$ be a closed subset of R such that $g(t,x) \in U$, $(t,x) \in A$ and $\Omega = J \times Q \times U$. Also, let control $u(\cdot)$ be a measurable function $u(t) : J \to U$. The vector function is defined as $g_0 = (g_1, g_2, ..., g_n)^t$, $g_0 : \Omega \to R^n$, where:

$$g_i(t,x,u) = \begin{cases} x_{i+1}(t) & i = 1,2,...,n-1 \\ u(t) & i = n. \end{cases}$$

A trajectory for $u(\cdot)$ is an absolutely continuous function $x(\cdot)$ such that:

$$x'(t) = g_0(t, x(t), u(t)).$$

Define $P(\cdot) = (x(\cdot), u(\cdot))$ as a trajectory-control pair, which is called admissible if $u(\cdot)$ is a control function and $x(\cdot)$ is a trajectory for u such that $x(t_a) = x_a$. The set of admissible pairs will be denoted by W. It is obvious that W is non-empty.

Now, it is assumed that:

$$f_0(t, x(t), u(t)) = |u(t) - g(t,x)|, \tag{6}$$

considering the functional $I : W \to R$ defined by:

$$I(P) = \int_J f_0(t, x(t), u(t))dt. \tag{7}$$

The classical control problem consists in minimizing the functional I over the set W. If the optimal solution is zero, then $f_0 = 0$, so $u(t) = g(t,x)$ or $y^{(n)}(t) = u(t)$ and an exact solution $y(t)$, $t \in J$, for Problem 1 can be obtained; otherwise, if the optimal solution is approximately zero, an approximate solution for Problem 1 can be obtained.

Remark

In Equation 6, $f_0(t, x(t), u(t))$ can be defined as:

$$f_0(t, x(t), u(t)) = [u(t) - g(t,x)]^2.$$

Let B be an open ball in R^{n+1} containing A and $C'(B)$, the space of all continuously differentiable functions on B such that they and their first derivatives are bounded on B. Now for all $\phi \in C'(B)$, $\phi^g(t,x,u) = \phi_x(t,x)g_0(t,x,u) + \phi_t(t,x)$, $(t,x,,u) \in \Omega$ is defined; therefore, if $P(x(\cdot), u(\cdot))$ is an admissible pair,

$$\int_J \phi^g(t, x(t), u(t))dt = \phi(t_b, x_b) - \phi(t_a, x_a) \equiv \Delta\phi, \tag{8}$$

for all $\phi \in C'(B)$, where $x_b = (x_1(t_b), x_2(t_b), ..., x_n(t_b))^t$ is an unknown point in R^n. In particular, if $h(t, x(t), u(t))$ is a real-valued continuous function of time only, i.e., $h(t, x_1, u_1) = h(t, x_2, u_2)$ for all $t \in J$ and (x_1, u_1), $(x_2, u_2) \in Q \times U$; then:

$$\int_J h(t, x(t), u(t))dt = a_h, \tag{9}$$

where a_h is Lebesgue integral of h over J. Now consider $D(J^\circ)$, the space of all infinitely differentiable real functions with compact support in J°. Define:

$$\psi_j(t,x,u) = x_j\psi'(t) + g_j(t,x,u)\psi(t),$$

for $j = 1,2,...,n$ and all $\psi \in D(J^\circ)$. Then, it follows:

$$\int_J \psi_j(t, x(t), u(t))dt = \int_J x_j\psi'(t)dt + \int_J g_j(t, x(t), u(t))\psi(t)dt = x_j(t)\psi(t)|_J - \int [x_j' - g_j(t, x(t), u(t))]\psi(t)dt,$$

or:

$$\int_J \psi_j(t, x(t), u(t))dt = 0, \quad j = 1,2,...,n, \tag{10}$$

since ψ is a function with compact support in J° and $P = (x(\cdot), u(\cdot))$ is an admissible pair. Equalities 8 to 10 are properties of the admissible pairs in the classical formulation of the optimal control problem, thus, this problem can be transformed into a new non-classical one.

TRANSFORMATION

For an admissible pair P, the mapping:

$$\Lambda_P : F \to \int_J F(t,x,u)dt,$$

defines a positive linear funtional on the space $C(\Omega)$ of the continuous real-valued functions on Ω; therefore, an admissible pair P can be considered as a positive linear funtional Λ_P on the $C(\Omega)$ and Equations 8 to 10 can be rewritten as follows:

$$\begin{cases} \Lambda_P(\phi^g) = \Delta\phi & \phi \in C'(B) \\ \Lambda_P(f) = a_f & f \in C_1(\Omega) \\ \Lambda_P(\psi_j) = 0 & \psi \in D(J^\circ), \quad j = 1,2,...,n; \end{cases} \tag{11}$$

where $C_1(\Omega)$ denotes the subspace of $C(\Omega)$ of those functions which depend on time only.

The set W can be considered as a set of positive linear functionals on $C(\Omega)$. To enlarge this set and perhaps overcome some of the difficulties associated with the classical formulation of the optimal control problem, a new framework is developed through considering all positive linear functionals on $C(\Omega)$ satisfying equalities akin to those in Statement 11. Through Riesz representation theorem, it is convenient to identify such functions with a positive Radon measure on Ω [10]; the set Q of all such measures can be taken into account which satisfies:

$$\begin{cases} \mu(\phi^g) = \Delta\phi & \phi \in C'(B) \\ \mu(f) = a_f & f \in C_1(\Omega) \\ \mu(\psi_j) = 0 & \psi \in D(J^\circ),\ j = 1,2,...,n \end{cases} \quad (12)$$

Therefore, the new optimization problem consists in minimizing the linear functional $I : Q \longrightarrow R$, defined by:

$$\mu \longrightarrow \mu(f_0) = \int_J f_0 d\mu,$$

over the set Q. It should be noted that this is an infinite dimensional linear programming problem. The existence of an optimal measure in set Q is considered for the functional I; a topology is defined on set Q induced by the weak*-topology on $M^+(\Omega)$, the set of all positive Radon measure on Ω.

Proposition

The measure-theoretical control problem, which is to find the minimum of the linear functional I over the set Q, attains its minimum μ^* in Q.

The proof of this theorem can be found in [1, Chapter 2]. In the following section, the approximation problem is considered.

APPROXIMATION

The above infinite-dimensional linear program is approximated by a finite-dimensional one, then the solution of this problem is approximated in a suitable way by an optimal admissible pair. These approximations have been effected in several stages.

First, the minimization of I over the subset of $M^+(\Omega)$ is considered, defined by a requirement that only a finite number of the constraints in Statement 12 be satisfied. This is achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate space, and then selecting a finite number of them. The underlying space is not finite -dimensional; however, the minimizing measure is a linear combination of a finite number of unitary atomic measure. Nevertheless, the support of these atomic measures are unknown; these supports can, however, be approximated by introducing a set dense in Ω.

Set Ω will be covered with a grid, by dividing the appropriate intervals for each component of t, x, u, into a number of equal subintervals. Let Ω be divided into N equal cells Ω_j, $j = 1,2,...,N$; points $z_j = (t_j, x_j, u_j) \in \Omega_j$ are chosen and $\sigma = \{z_j; j = 1,2,...,N\}$. The set σ is an approximate dense subset of the set Ω. Thus, the solution of the infinite dimensional linear programming problem, introduced above, is approximated by the following problem (see Chapter 4 of [1]):

Minimize:

$$\sum_{j=1}^{N} \alpha_j f_0(z_j) \quad (13)$$

over the set of coefficients $\alpha_j \geq 0,\ j = 1,2,...,N$

such that :

$$\begin{cases} \sum_{j=1}^{N} \alpha_j \phi_j^g(z_j) = \Delta\phi_i & i = 1,2,...,M_1 \\ \sum_{j=1}^{N} \alpha_j \chi_l(z_j) = 0 & l = 1,2,...,M_2 \\ \sum_{j=1}^{N} \alpha_j f_k(t_j) = a_k & k = 1,2,...,M_3 \end{cases} \quad (14)$$

where t_j is written for the corresponding components of z_j and $a_k = \int_j f_k(t)dt$. The functions $\{f_k; k = 1,2,...,M_3\}$ are defined by:

$$f_k(t) = \begin{cases} 1 & t \in J_k = (t_a + (k-1)d, t_a + kd) \\ 0 & \text{otherwise,} \end{cases}$$

where $d = \frac{t_b - t_a}{M_3}$. The functions $\{\phi_i;\ i = 1,2,...,M_1\}$ are monomials in the components of x only and $\{\chi_l;\ l = 1,2,...,M_2\} = \{\psi_j^s;\ j = 1,2,...,j_1, s = 1,2,...,n\}$, for some integer j_1; the functions ψ^s are of the form $\sin(2\pi st)$, $1-\cos(2\pi st)$.

The dimension of the linear programming problem is $M = M_1 + M_2 + M_3$ equalities by $N + n$ variables; because in Relation 14, i.e., $\Delta\phi_i = \phi_i(t_b, x_b) - \phi_i(t_a, x_a)$, the vector:

$$x_b = (x_1(t_b), x_2(t_b), ..., x_n(t_b)) = (\beta_1, \beta_2, ..., \beta_n),$$

is unknown and β_i, $i = 1,2,...,n$, must be selected as n free variables.

Now, $y(\cdot)$ is constructed (an approximate solution of the original Problem 1) from the solution $\{\alpha_j; j = 1,2,...,N\}$ of the above linear programming problem. Nevertheless, only the control function needs to be constructed; since $y^{(n)}(t) = u(t)$, therefore, by initial Condition 2, $y(t)$, $t \in J$ can be obtained. It is noted that $y(t) \in C^{n-1}(J)$, the space of all continuously differentiable functions on J of the order $n-1$.

It is well-known that the linear form Statement 13 attains its minimum at an extreme point of the feasible set defined by Relation 14 and that such a point has, at most, $M + n$ number of non-zero elements. Let J be divided into R equal subintervals and $N = R \times s$, pair (i, j) can be correlated to m as:

$$m = (i-1)s + j,$$

$$i = 1, 2, ..., R, \quad j = 1, 2, ..., s,$$

(for $m = 1, 2, ..., N$). Also, let $k_{ij} = \alpha_m$. Then, a piecewise-constant control is defined as:

$$u(t) = u_m, \quad t \in B_{ij}, \tag{15}$$

where u_m is a component of a triple $z_m = (t_m, x_m, u_m)$ and:

$$B_{ij} = [t_{i-1} + \sum_{m<j} k_{im}, t_{i-1} + \sum_{m\leq j} k_{im}). \tag{16}$$

Example 1

Consider the initial value problem:

$$\begin{cases} y'(t) = \frac{1}{e}e^t + y(t) & t \in (0,1) \\ y(0) = 0. \end{cases} \tag{17}$$

The exact solution of this problem can be written as:

$$y(t) = \frac{t}{e}e^t, \quad t \in [0,1]. \tag{18}$$

Now, an approximate solution of the above ordinary differential equation is computed. Let $J = [0,1]$, $A = [0,1]$, $U = [0,2]$, $M_1 = 1$, $M_2 = 8$, $M_3 = 10$, $M = 19$, $\phi_1 = x(t) = y(t)$, $\Delta x = 0.125$, $\Delta u = 0.25$ and $R = 10$; therefore, $N = 640$. Also, let $Z_j = (t_j, x_j, u_j)$, $j = 1, 2, ..., 640$; t_j, x_j, and u_j are selected as:

$$t_{64i+1} = t_{64i+2} = ... = t_{64i+64} = 0.11i$$

$$i = 0, 1, ..., 9\ ,$$

$$x_{64i+j+1} = x_{64i+8j+2} = ... = x_{64i+8j+8} = 0.14j$$

$$i = 0, 1, ..., 9, \ j = 0, 1, ..., 7\ ,$$

$$u_i = u_{i+8} = u_{i+16} = ... = u_{i+632} = 0.28i$$

$$i = 1, 2, ..., 8.$$

Consequently, using Statements 13 and 14, the following linear programming problem is obtained:

Minimize

$$\sum_{j=1}^{640} [u_j - \frac{1}{e}e^{t_j} - x_j]^2 \alpha_j$$

subject to:

$$\begin{cases} -\sum_{j=1}^{640} [\frac{1}{e}e^{t_j} + x_j]\alpha_j + \beta = 0 \\ \sum_{j=1}^{640} [2\pi s x_j \cos(2\pi s t_j) + \\ (\frac{1}{e}e^{t_j} + x_j)\sin(2\pi s t_j)]\alpha_j = 0, \quad s = 1,2,3,4 \\ \sum_{j=1}^{640} [2\pi(s-4)x_j \sin(2\pi(s-4)t_j) + \\ (\frac{1}{e}e^{t_j} + x_j)(1-\cos(2\pi(s-4)t_j))]\alpha_j = 0 \quad s = 5,6,7,8 \\ \alpha_{64j+1} + \alpha_{64j+2} + ... + \\ \alpha_{64j+64} = 0.1, \quad j = 0,1,2,...,9 \\ \beta \text{ is a free variable, } \alpha_j \geq 0, \quad j = 0,1,2,...,640. \end{cases}$$

Using the solution of the above linear programming problem, the optimal value of the objective function is found as:

$I^* = 0.0033$,

also, the final value of approximated Solution 15 is as follows:

$y(1) = \beta = 1.008$.

Using the result of this finite dimensional linear programming and Solution 15, an approximated piecewise constant control function is obtained; since $y'(t) = x(t) = u(t)$, an approximated solution $y(\cdot) \in C(J)$ can be attained for Statements 17 and 18 as:

$$y(t) = \begin{cases} 0.56t & t \in [0.0000, 0.0671) \\ 0.40t - 0.056364 & t \in [0.0671, 0.1000) \\ 0.28t + 0.055636 & t \in [0.1000, 0.1958) \\ 0.56t + 0.000812 & t \in [0.1958, 0.3792) \\ 1.40t - 0.317716 & t \in [0.3792, 0.4000) \\ 0.56t + 0.018284 & t \in [0.4000, 0.4699) \\ 1.40t - 0.376432 & t \in [0.4699, 0.5000) \\ 0.56t + 0.043568 & t \in [0.5000, 0.5571) \\ 1.40t - 0.424396 & t \in [0.5571, 0.6000) \\ 0.84t - 0.088396 & t \in [0.6000, 0.6666) \\ 1.68t - 0.648340 & t \in [0.6666, 0.7000) \\ 0.84t - 0.060340 & t \in [0.7000, 0.7445) \\ 1.68t - 0.685720 & t \in [0.7445, 0.8000) \\ 0.84t - 0.013720 & t \in [0.8000, 0.8228) \\ 1.68t - 0.704872 & t \in [0.8228, 0.9000) \\ 1.96t - 0.956872 & t \in [0.9000, 1.0000] \end{cases}$$

The approximated solution $y(\cdot)$ and the exact solution $Y(\cdot)$ can be seen in Figure 1.

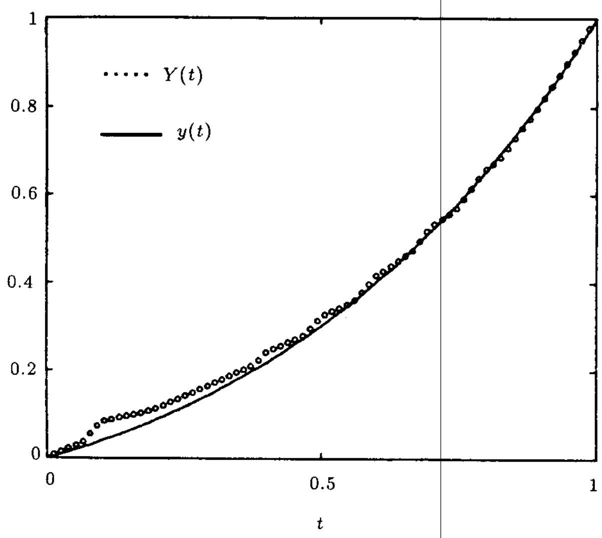

Figure 1. Final status for Example 1.

Example 2

Consider the nonlinear differential equation:

$$y''(t) + ty'^2(t) - y(t) = -2(t - \frac{1}{3}) - \frac{1}{3}(t - \frac{1}{3})^3(3t^2 - t + 1), \quad t \in (0,1), \tag{19}$$

with the initial conditions:

$$y(0) = \frac{1}{81}, \quad y'(0) = \frac{-1}{9}. \tag{20}$$

The exact solution of this problem can be written as:

$$y(t) = \frac{-1}{3}(t - \frac{1}{3})^3, \quad t \in [0,1].$$

As in Example 1, an approximate solution of this problem can be computed. Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $J = [0,1]$, $A = [-0.1, 0.1] \times [-0.5, 0]$, $U = [\frac{-4}{3}, \frac{2}{3}]$, $M_1 = 2$, $M_2 = 8$, $M_3 = 10$, $M = 20$, $\phi_1(t) = x_1(t)$, $\phi_2(t) = x_2(t)$, $\Delta x_1 = 0.05$, $\Delta x_2 = 0.125$, $\Delta u = 0.5$ and $R = 10$; therefore, $N = 640$. Also, $Z_i = (t_i, x_{1i}, x_{2i}, u_i), i = 1, 2, ..., 640$, where t_i, x_{1i}, x_{2i} are selected for $i = 1, 2, ..., 640$, as mid-points of intervals Δt, Δx_1, Δx_{2i}, respectively (like Example 1); moreover, it is assumed that:

$$u_{1+i} = u_{5+i} = u_{9+i} = ... = u_{637+i} = \frac{-2}{3} + \frac{5}{8}i, \quad i = 0, 1, 2, 3.$$

Using Statements 13 and 14, the following linear programming problem is obtained:

Minimize

$$\sum_{j=1}^{640} |[U_j + t_j x_{2j}^2 - x_{1j} + 2(t_j - \frac{1}{3}) + \frac{1}{3}(t_j - \frac{1}{3})^3(3t^2 - t + 1)]|\alpha_j$$

subject to :

$$\begin{cases} -\sum_{j=1}^{640} x_{2j}\alpha_j + \beta_1 = \frac{1}{81}, \\ \sum_{j=1}^{640} u_j\alpha_j + \beta_2 = \frac{1}{9} \\ \sum_{j=1}^{640} [2\pi s x_{1j}\cos(2\pi s t_j) + x_{2j}\sin(2\pi s t_j)]\alpha_j = 0 \quad s = 1,2 \\ \sum_{j=1}^{640} [2\pi s x_{2j}\cos(2\pi s t_j) + u_j\sin(2\pi s t_j)]\alpha_j = 0 \quad s = 1,2 \\ \sum_{j=1}^{640} [2\pi s x_{1j}\sin(2\pi s t_j) + x_{2j}(1 - \cos(2\pi s t_j)]\alpha_j = 0 \quad s = 1,2 \\ \sum_{j=1}^{640} [2\pi s x_{2j}\sin(2\pi s t_j) + u_j(1 - \cos(2\pi s t_j)]\alpha_j = 0 \quad s = 1,2 \\ \alpha_{10i+1} + \alpha_{10i+2} + ... + \alpha_{10i+64} = 0.1 \quad i = 0,1,2,...,9 \\ \beta_1 \text{ and } \beta_2 \text{ are free variables.} \quad \alpha_j \geq 0 \quad j = 0,1,2,...,640. \end{cases}$$

In this example, the cost function takes a value of $I^* = 0.0104$. Also, using the solution of the above linear programming problem, $y(1) = \beta_1 = x_1(1) = -0.1017$. The outcomes of this finite dimensional linear programming as well as Solution 15 result in the following approximated piecewise constant control function:

$$u(t) = \begin{cases} \frac{13}{24} & t \in [0.0000, .2000) \\ \frac{-2}{24} & t \in [0.2000, 0.2822) \\ \frac{13}{24} & t \in [0.2822, 0.3000) \\ \frac{-2}{24} & t \in [0.3000, 0.5000) \\ \frac{-17}{24} & t \in [0.5000, 0.8000) \\ \frac{-32}{24} & t \in [0.8000, 1.0000] \end{cases}$$

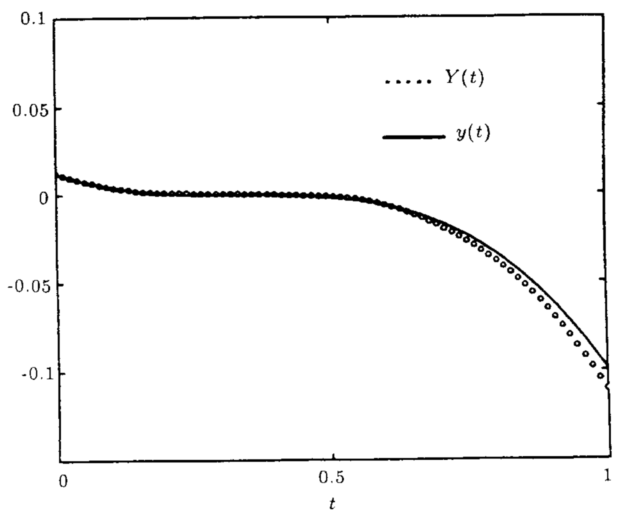

Figure 2. Final status for Example 2.

Since $y''(t) = x_2'(t) = u(t)$, an approximated solution $y(\cdot) \in C^1(J)$ can be computed for Statements 17 and 18 as:

$$y(t) = \begin{cases} \frac{13}{48}t^2 - 0.1111t + 0.0123 & t \in [0.0000, .2000) \\ \frac{-2}{48}t^2 + 0.0138t - 0.0001 & t \in [0.2000, 0.2822) \\ \frac{13}{48}t^2 - 0.1624t + 0.0247 & t \in [0.2822, 0.3000) \\ \frac{-2}{48}t^2 + 0.0250t - 0.0033 & t \in [0.3000, 0.5000) \\ \frac{-17}{48}t^2 + 0.3375t - 0.0815 & t \in [0.5000, 0.8000) \\ \frac{-32}{48}t^2 + 0.8375t - 0.2815 & t \in [0.8000, 1.0000] \end{cases}$$

The approximated solution $y(\cdot)$ and the exact solution $Y(\cdot)$ can be seen in Figure 2.

CALCULATION OF ABSOLUTE ERROR

A bound is first found for error in the above mentioned method. Let $Y(t)$ and $y(t)$, respectively, denote the exact and approximated solutions of Problem 1; it is clear that $Y^{(i)}(t_a) = y^{(i)}(t_a) = y_i, i = 0, 1, ..., n$; thus, error can be considered as follows:

$$\|Y - y\|_1 = \int_J |Y(t) - y(t)| dt$$

$$= \int_J | \int_{t_a}^{t} [Y'(t_1) - y'(t_1)] dt_1 | dt$$

$$\leq \int_J \int_J |Y'(t_1) - y'(t_1)| dt_1 dt$$

$$\leq \int_J \int_J \int_J |Y''(t_2) - y''(t_2)| dt_2 dt_1 dt \leq ...$$

$$\leq \int_J \int_J ... \int_J |Y^{(n)}(t_n) - y^{(n)}(t_n)| dt_n ... dt_1 dt.$$

However, using Equations 6 and 7,

$$I = \int_J | g(t, x) - u(t) | dt$$

$$= \int_J | Y^{(n)}(t) - y^{(n)}(t) | dt,$$

where the functional I is an objective function for the classical optimal control problem for which the solution is approximated by the solution of the linear Programmings 13 and 14; the minimum value of the objective function in Statements 13 and 14 is equal to I^*; therefore,

$$\|Y - y\|_1 \leq \int_J ... \int_J I^* dt_{n-1} ... dt_1 dt$$

or,

$$\|Y - y\|_1 \leq (\Delta t)^n I^* ,$$

where $\Delta t = t_b - t_a$.

Due to the above discussion, the error of Example 1 is equal to 0.0033; similarly, the error of Example 2 is equal to 0.0104.

It is also noted that the error in transformation of an optimal control problem to a linear programming problem is considered in [11].

REFERENCES

1. Rubio, J.E., *Control and Optimization: The Linear Treatment of Nonlinear Problems*, Manchester University Press, Manchester, UK (1986).
2. Wilson, D.A. and Rubio, J.E. "Existence of optimal controls for the diffusion equation", *Journal of Optimization Theory and Applications*, **22**, pp 91-102 (1977).
3. Kamyad, A.V., Rubio, J.E. and Wilson, D.A. "The optimal control of the multidimensional diffusion equation", *Journal of Optimization Theory and Applications*, **70**(1), pp 191-209 (1991).
4. Kamyad, A.V., Rubio, J.E. and Wilson, D.A. "An optimal control problem for the multidimensional diffusion equation with a generalized control variable", *Journal of Optimization Theory and Applications*, **75**(1), pp 101-132 (1992).
5. Farahani, M.H., Rubio, J.E. and Wilson, D.A. "The optimal control of the linear wave equation", *International Journal of Control*, **63**(5), pp 833-848 (1996).

6. Farahi, M.H., Rubio, J.E. and Wilson, D.A. "The global control of a nonlinear wave equation", *International Journal of Control*, **65**(1), pp 1-15 (1996).

7. Kamyad, A.V. "Strong controllability of the diffusion equation in n-dimensions", *Bulletin of the Iranian Mathematical Society*, **18**(1), pp 39-49 (1992).

8. Alavi, S.A., Kamyad, A.V. and Farahi, M.H. "The optimal control of an inhomogeneous wave problem with internal control and their numerical solution", **23**(2), pp 9-36 (1997).

9. Reid, G., *Ordinary Differential Equation*, John Wiley & Sons (1971).

10. Choquest, G., *Lectures on Analysis*, J. Morsdon, T. Lance and S. Galbort, Eds., New York, Benjamin (1969).

11. Farahi, M.H. "The boundary control of the wave equation", Ph.D. Thesies, Leeds University, UK.