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Research Note

Markov Finite Approximation of Frobenius-Perron Operator for Higher-Dimensional Transformations with a Special Action Matrix

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In this paper, the set of finite rank approximations of Frobenius-Perron operators is extended for higher-dimensional transformations from projections to general finite rank operators through indicating an arbitrary action matrix. The convergence is proven in the case for which the action matrix is a special type of doubly-stochastic and tridiagonal.

INTRODUCTION

Let $I = [0, 1]$ and $\tau : I^n \to I^n$ be a piecewise expanding transformation. For $n = 1$, Lasota and Yorke [1] have proven the existence of an absolutely continuous invariant measure μ with respect to Lebesgue measure. If f is the density of μ with respect to Lebesgue measure m on I^n, then it is well-known that f is the fixed point of the Frobenius-Perron operator P_τ. Ulam conjectured that it might be possible to construct finite-dimensional operators which approximate P_τ and whose fixed points approximate the fixed point of P_τ [2]. In [3-6], this conjecture is proven for a class of one-dimensional piecewise expanding transformations. Recently, the above conjecture is proven for general finite rank operators through indicating an arbitrary doubly-stochastic tridiagonal action matrix [7]. In [8], Jablonski has shown that a class of piecewise C^2-transformations of the n-dimensional cube $[0, 1]^n$ has an absolutely continuous invariant measure. An n-dimensional version of Ulam conjecture was proven with finite rank projection operators for Jablonski transformations (see [9]). The aim of this paper is to prove the above version of Ulam conjecture for Jablonski transformations with general finite rank operators through indicating a particular doubly-stochastic tridiagonal action matrix.

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Let m_j denote Lebesgue measure on I^j. For $j = n$, let $m = m_n$. The space of all Lebesgue integrable functions on I^n is denoted by L^1 and the space of all essentially bounded (with respect to Lebesgue measure) functions on I^n by L^∞. The transformation $\tau : I^n \to I^n$ is written as:

$$\tau(x_1, \cdots, x_n) = (\varphi_1(x_1, \cdots, x_n), \cdots, \varphi_n(x_1, \cdots, x_n)),$$

where for $i = 1, \cdots, n$, $\varphi_i(x_1, \cdots, x_n)$ is a function from I^n into $[0, 1]$.

A measurable transformation $\tau : I^n \to I^n$ is nonsingular if $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. For nonsingular $\tau : I^n \to I^n$, Frobenius-Perron operator $P_\tau : L^1 \to L^1$ is defined by the formula:

$$\int_A P_\tau f dx = \int_{\tau^{-1}(A)} f dx,$$

where $A \subseteq I^n$ is measurable. It follows that for $x = (x_1, \cdots, x_n)$,

$$P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(\prod_{i=1}^n [0, x_i])} f(y) dy.$$

The operator P_τ has many interesting properties [10].

Suppose ℓ is a positive integer. An $\ell^n \times \ell^n$ matrix (called action matrix) will be associated to each finite rank approximation operator, which in the case of projection will be identity. The convergence of this new scheme is proven when the action matrix is a special type of doubly-stochastic and tridiagonal.

Let $\beta = \{D_1, \cdots, D_p\}$ be a partition of I^n such that $p < \infty$, i.e.,

$$\bigcup_{j=1}^{p} D_j = I^n \ , \ D_j \cap D_k = \emptyset \quad \text{for } j \neq k.$$

A partition β of I^n is called rectangular if for any $1 \leq j \leq p$, D_j is an n-dimensional rectangle.

Definition 1

A transformation $\tau : I^n \to I^n$ is called a Jablonski transformation if it is defined on a rectangular partition of I^n and is given by the formula:

$$\tau(x_1, \cdots, x_n) = (\varphi_{1j}(x_1), \cdots, \varphi_{nj}(x_n)),$$

where $(x_1, \cdots, x_n) \in D_j$, $1 \leq j \leq p$, $D_j = \prod_{i=1}^{n}[a_{ij}, b_{ij})$ and $\varphi_{ij} : [a_{ij}, b_{ij}] \to [0,1]$. If $b_{ij} = 1$ for some i, then $[a_{ij}, b_{ij})$ means $[a_{ij}, b_{ij}]$.

Cartesian product of the sets A_i is denoted by $\prod_{i=1}^{n} A_i$ and P_i, the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$, is given by:

$$P_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n).$$

Let $g : A \to \mathbb{R}$ be a function on the n-dimensional interval $A = \prod_{i=1}^{n}[a_i, b_i]$. For a fixed i, a function $V_i^A g$ with $n-1$ variables $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ is defined by the formula:

$$V_i^A g \equiv V_i g = \sup \left\{ \sum_{k=1}^{r} |g(x_1, \cdots, x_i^k, \cdots, x_n) - g(x_1, \cdots, x_i^{k-1}, \cdots, x_n)| : a_i = x_i^0 < x_i^1 < \cdots < x_i^r = b_i, \quad r \in \mathrm{N} \right\}.$$

For $f : A \to \mathbb{R}$, where $A = \prod_{i=1}^{n}[a_i, b_i]$, let:

$$\mathbf{V}_i^A f = \inf \left\{ \int_{P_i(A)} V_i g \ \ dm_{n-1} : g = f \text{ almost everywhere}, \ \ V_i g \text{ measurable} \right\},$$

and $\mathbf{V}^A f = \sup_{1 \leq i \leq n} \mathbf{V}_i^A f$. If $\mathbf{V}^A f < \infty$, then f is a bounded variation function on A and its total variation is $\mathbf{V}^A f$.

In the next section, the finite rank operators based on an action matrix will be presented and the corresponding convergence theorems are proven.

APPROXIMATION OF INVARIANT DENSITIES

Let $\tau : I^n \to I^n$ be a Jablonski transformation and for any positive integer ℓ, let I^n be divided into ℓ^n subsets of equal measure $I_1, I_2, \ldots, I_{\ell^n}$ with:

$$I_k = \left[\frac{r_1 - 1}{\ell}, \frac{r_1}{\ell}\right) \times \left[\frac{r_2 - 1}{\ell}, \frac{r_2}{\ell}\right) \times \cdots \times \left[\frac{r_n - 1}{\ell}, \frac{r_n}{\ell}\right),$$

for some $r_1, r_2, \ldots, r_n = 1, 2, \ldots, \ell$ and $m(I_k) = \frac{1}{\ell^n}$, $k = 1, 2, \ldots, \ell^n$. Suppose I_k's change according to the following quasi-code:

$$\begin{aligned}
&k := 0; \\
&\text{FOR } r_n := 1 \text{ TO } \ell \text{ DO} \\
&\quad \text{FOR } r_{n-1} := 1 \text{ TO } \ell \text{ DO} \\
&\qquad \vdots \\
&\qquad \text{FOR } r_1 := 1 \text{ TO } \ell \text{ DO} \\
&\qquad\quad k := k + 1; \\
&\qquad\quad I_k := [\tfrac{r_1 - 1}{\ell}, \tfrac{r_1}{\ell}) \times \cdots \times [\tfrac{r_n - 1}{\ell}, \tfrac{r_n}{\ell}) \ .
\end{aligned} \tag{1}$$

For simplicity, in what follows, sometimes ℓ^n is denoted by q. Let:

$$P_{st} = qm(I_s \cap \tau^{-1}(I_t)), \quad s, t = 1, \ldots, q,$$

and $P_\ell = (P_{ij})$ [9]. Suppose Δ_ℓ is the q-dimensional linear subspace of L^1, spanned by $\{\chi_k\}_{k=1}^{q}$, where χ_k denotes the characteristic function of I_k. Now, suppose that $A_\ell = (a_{ij})$ is a given $q \times q$ doubly-stochastic and tridiagonal (DST) matrix and let $\tilde{P}_\ell = P_\ell A_\ell$. Note that $\tilde{P}_\ell$ may be considered as an operator $\tilde{P}_\ell = \tilde{P}_\ell(\tau) : \Delta_\ell \to \Delta_\ell$, given by:

$$\tilde{P}_\ell(\tau)\chi_k = \sum_{t=1}^{q} \tilde{P}_{kt}\chi_t.$$

Since the product of two stochastic matrices is stochastic, it is concluded that if $\Delta_\ell^1 = \{\sum_{k=1}^{q} c_k\chi_k : c_k \geq 0$ and $\sum_{k=1}^{q} c_k = 1\}$, then $\tilde{P}_\ell$ maps Δ_ℓ^1 to a subset of Δ_ℓ^1; thus, there exists a fixed point $f_\ell \in \Delta_\ell$ of $\tilde{P}_\ell$ such that $\|f_\ell\| = 1$ for all ℓ [6].

Let $\Delta = [v_1, \ldots, v_p]$ be a p-dimensional subspace of L^1, spanned by $v_i \in L^1$, $i = 1, \ldots, p$. For given $u_i \in L^\infty$, $i = 1, \ldots, p$, the finite rank operator $Q_\ell : L^1 \to \Delta$ is defined by the tensor notation, $Q_\ell = \sum_{i=1}^{p} u_i \otimes v_i$, where for $f \in L^1$, $(u \otimes v)(f) = (\int_{I^n} fu)v$. The numbers

$$\int_{I^n} u_j v_i = a_{ij} \quad (i, j = 1, \ldots, p)$$

define a $p \times p$ matrix $A_\ell = (a_{ij})$, which is called the action matrix of the operator $\boldsymbol{Q_\ell}$ (for notation and terminology see [11]).

Here, the choice of the functions v_i and u_j is as follows:

$$v_i = \chi_i \qquad i = 1, \dots, q,$$

$$u_j = c_1^j \chi_1 + \cdots + c_q^j \chi_q \quad j = 1, \dots, q,$$

where:

$$c_j^j = qa_j, \quad c_i^{i+1} = c_{i+1}^i = qb_i,$$

$a_j \geq 0$ $(j = 1, \dots, q)$ are the elements of the main diagonal of the DST matrix A_ℓ and $b_i \geq 0$ $(i = 1, \dots, q-1)$ are the elements of the "diagonals" just above and below the main diagonal of the DST matrix A_ℓ.

The corresponding DST action matrix $A_\ell = tridiag(a_j, b_i)$ of the operator Q_ℓ satisfies:

$$\begin{cases} a_1 + b_1 = 1, \\ b_{i-1} + a_i + b_i = 1 \quad i = 2, \cdots, q-1, \\ b_{q-1} + a_q = 1. \end{cases}$$

It follows that:

$$u_1 = qa_1\chi_1 + qb_1\chi_2,$$

$$u_j = qb_{j-1}\chi_{j-1} + qa_j\chi_j + qb_j\chi_{j+1}, \quad j = 2, \dots, q-1,$$

$$u_q = qb_{q-1}\chi_{q-1} + qa_q\chi_q.$$

Definition 2

For $f \in L^1$ and for any positive integer ℓ, $Q_\ell : L^1 \to \Delta_\ell$ is defined by:

$$Q_\ell(f) = \sum_{k=1}^{q} (u_k \otimes v_k)(f),$$

where $v_k = \chi_k$, $u_k = \sum_{j=1}^{q} c_j^k \chi_j$ $(k = 1, \dots, q)$ and the corresponding action matrix of the operator Q_ℓ is the matrix A_ℓ defined as above.

Remark 1

It is not hard to prove that for any ℓ, the operator $Q_\ell : L^1 \to \Delta_\ell$ is a Markov operator, and $\tilde{P}_\ell f = Q_\ell P_\tau f$ for any $f \in \Delta_\ell$ [6,7].

In the rest of this paper, it shall be assumed that A_ℓ is a $q \times q$ DST matrix of the form:

$$A_\ell = \begin{bmatrix} A & & & & \\ & A & & \oslash & \\ & & A & & \\ & \oslash & & \ddots & \\ & & & & A \end{bmatrix}, \tag{2}$$

where A is an $\ell \times \ell$ DST matrix given by:

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \oslash & \\ & \oslash & & b_{\ell-2} & a_{\ell-1} & b_{\ell-1} \\ 0 & \cdots & & 0 & b_{\ell-1} & a_\ell \end{bmatrix} \tag{3}$$

In what follows it is assumed that the action matrix of the operator Q_ℓ in Definition 2 is of the Form 2. Clearly,

$$c_{j+p\ell}^{j+p\ell} = qa_j \quad (j = 1, \dots, \ell;\ p = 0, \dots, \ell^{n-1} - 1),$$

$$c_{i+p\ell}^{i+p\ell+1} = c_{i+p\ell+1}^{i+p\ell} = qb_i \quad (i = 1, \dots, \ell - 1;\ p = 0, \dots, \ell^{n-1} - 1).$$

Remark 2

It is clear that for $f \in L^1$ as $\ell \to \infty$, the sequence $\{Q_\ell f\}$ converges in L^1 to f and in particular if $f \in \Delta_\ell$, then the sequence $\{\tilde{P}_\ell f\}$ converges in L^1 to $P_\tau f$. For details see [6,7].

Lemma 1

If $f \in L^1$, then $\mathbf{V}^{I^n} Q_\ell f \leq 3\mathbf{V}^{I^n} f$.

Proof

For any $1 \leq k \leq q$, let $I_k = \prod_{i=1}^{n} [(r_i - 1)/\ell, r_i/\ell) = \prod_{i=1}^{n} J_{r_i}$ where, for $1 \leq i \leq n$, r_i assumes the values $1, 2, \dots, \ell$. I_k change according to the quasi-code Relation 1. For $k = 1, \cdots, q$, $m(I_k) = \prod_{i=1}^{n} m(J_{r_i}) = \frac{1}{q}$. Let:

$$Q_{\ell_1}(f) = \sum_{r_1=1}^{\ell} (u_{r_1} \otimes v_{r_1})(f),$$

where $v_{r_1} = \chi_{J_{r_1}}(x_1)$, $u_{r_1} = \sum_{j=1}^{\ell} c_j^{r_1} v_j$ $(r_1 = 1, \dots, \ell)$, $(u_{r_1} \otimes v_{r_1})(f) = (\int_I f u_{r_1} dx_1) v_{r_1}$ and the corresponding action matrix of the operator Q_{ℓ_1} is as Form 3. Hence, $c_j^j = \ell a_j$ and $c_\nu^{\nu+1} = c_{\nu+1}^\nu = \ell b_\nu$ $(j = 1, \dots, \ell;\ \nu = 1, \dots, \ell - 1)$. For $i = 2, \dots, n$, the following is defined:

$$Q_{\ell_i}(f) = \sum_{r_i=1}^{\ell} (u_{r_i} \otimes v_{r_i})(f),$$

where $v_{r_i} = \chi_{J_{r_i}}(x_i)$, $u_{r_i} = \sum_{j=1}^{\ell} c_j^{r_i} v_j$ $(r_i = 1, \dots, \ell)$, $(u_{r_i} \otimes v_{r_i})(f) = (\int_I f u_{r_i} dx_i) v_{r_i}$ and the corresponding action matrix of the operator Q_{ℓ_i} is an

$\ell \times \ell$ identity matrix. Hence, $c_j^j = \ell$ and $c_j^\nu = 0$ for $\nu \neq j$ $(j, \nu = 1, \dots, \ell)$. It is obtained that:

$$Q_\ell f(x) = Q_{\ell_1} Q_{\ell_2} \cdots Q_{\ell_n} f(x) = (\textstyle\prod_{i=1}^n Q_{\ell_i}) f(x).$$

It follows that (see [6,7]):

$$V_i^{I^n} Q_\ell f = V_i^{I^n} (\textstyle\prod_{j=1}^n Q_{\ell j}) f = V_i^{I^n} Q_{\ell_i} (\prod_{j=1, j\neq i}^n Q_{\ell_j}) f$$
$$\leq \begin{cases} V_i^{I^n} (\prod_{j=1,j\neq i}^n Q_{\ell_j}) f & \text{if } i \neq 1, \\ 3V_i^{I^n} (\prod_{j=1,j\neq i}^n Q_{\ell_j}) f & \text{if } i = 1. \end{cases}$$

Now it is shown that for any i,

$$\int_{I^{n-1}} V_i^{I^n} (\textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j}) f (\prod_{j=1,j\neq i}^n dx_j)$$
$$\leq \int_{I^{n-1}} V_i^{I^n} f (\textstyle\prod_{j=1,j\neq i}^n dx_j). \qquad (4)$$

For $i = 1$, the Relation 4 was proven in Lemma 6 of [9]. If $i \neq 1$, then for $t = 1, 2, \cdots, \ell^{n-1}$ let $I_t^{n-1} = \prod_{j=1,j\neq i}^n J_{r_j}$, where r_j assume the values $1, 2, \dots, \ell$. I_t^{n-1} change according to the quasi-code Relation 1 when k is replaced by t, I_k by I_t^{n-1} and the loop for r_i is eliminated. It is clear that for $t = 1, 2, \cdots, \ell^{n-1}$, $m(I_t^{n-1}) = \prod_{j=1,j\neq i}^n m(J_{r_j}) = \frac{1}{\ell^{n-1}}$. For simplification, notation $f(x_1, \cdots, x_i^k, \cdots, x_n) - f(x_1, \cdots, x_i^{k-1}, \cdots, x_n)$ is denoted by $f_i^{k,k-1}$, $\prod_{j=1,j\neq i}^n dx_j$ by $\prod_{j\neq i} dx_j$ and ℓ^{n-1} by M. It is obtained that:

$$\textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j} f(x_1, \cdots, x_i, \dots, x_n) =$$
$$\sum_{t=1}^M (u_t \otimes v_t)(f),$$

where:

$$v_t = \chi_{I_t^{n-1}}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \equiv \chi_t,$$

$$u_t = \sum_{\nu=1}^M c_\nu^t v_\nu \quad (t = 1, \cdots, M),$$

$$(u_t \otimes v_t)(f) =$$
$$\left(\int_{I^{n-1}} f u_t \, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n\right) v_t,$$

and the corresponding action matrix of the operator $\prod_{j=1,j\neq i}^n Q_{\ell_j}$ is similar to Form 2, but the number of iterations of the matrix A defined by Form 3 equals ℓ^{n-2}. Hence:

$$c_{j+p\ell}^{j+p\ell} = M a_j \quad (j = 1, \dots, \ell;\ p = 0, \dots, \ell^{n-2} - 1),$$

$$c_{\nu+p\ell}^{\nu+p\ell+1} = c_{\nu+p\ell+1}^{\nu+p\ell} = M b_\nu$$

$$(\nu = 1, \dots, \ell - 1;\ p = 0, \dots, \ell^{n-2} - 1).$$

For any $0 = x_i^0 < x_i^1 < \cdots < x_i^{r-1} < x_i^r = 1$,

$$\sum_{k=1}^r |\textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j} f(x_1, \cdots, x_i^k, \cdots, x_n)$$
$$- \textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j} f(x_1, \cdots, x_i^{k-1}, \cdots, x_n)|$$
$$= \sum_{k=1}^r |\sum_{t=1}^M \Big(\int_{I^{n-1}} f_i^{k,k-1}$$
$$(\sum_{\nu=1}^M c_\nu^t \chi_\nu) \textstyle\prod_{j\neq i} dx_j\Big) \chi_t|$$
$$\Rightarrow \int_{I^{n-1}} \sum_{k=1}^r |\textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j} f(x_1, \cdots, x_i^k, \cdots, x_n)$$
$$- \textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j} f(x_1, \cdots, x_i^{k-1}, \cdots, x_n)| (\prod_{j\neq i} dx_j)$$
$$\leq \int_{I^{n-1}} \sum_{k=1}^r \sum_{t=1}^M \Bigg(\int_{I^{n-1}} |f_i^{k,k-1}|$$
$$(\sum_{\nu=1}^M c_\nu^t \chi_\nu) \textstyle\prod_{j\neq i} dx_j \Bigg) \chi_t (\prod_{j\neq i} dx_j)$$
$$= \frac{1}{M} \sum_{k=1}^r \int_{I^{n-1}} \sum_{\nu=1}^M (\sum_{t=1}^M c_\nu^t) |f_i^{k,k-1}| \chi_\nu (\textstyle\prod_{j\neq i} dx_j)$$
$$= \frac{M}{M} \sum_{k=1}^r \int_{I^{n-1}} \sum_{\nu=1}^M |f_i^{k,k-1}| \chi_\nu (\textstyle\prod_{j\neq i} dx_j)$$
$$= \sum_{k=1}^r \sum_{\nu=1}^M \int_{I_\nu^{n-1}} |f_i^{k,k-1}| (\textstyle\prod_{j\neq i} dx_j)$$
$$= \sum_{k=1}^r \int_{I^{n-1}} |f_i^{k,k-1}| (\textstyle\prod_{j\neq i} dx_j)$$
$$= \int_{I^{n-1}} \sum_{k=1}^r |f(x_1, \cdots, x_i^k, \cdots, x_n)$$
$$- f(x_1, \cdots, x_i^{k-1}, \cdots, x_n)| (\textstyle\prod_{j=1,j\neq i}^n dx_j).$$

Hence:

$$\int_{I^{n-1}} V_i^{I^n} (\textstyle\prod_{j=1,j\neq i}^n Q_{\ell_j}) f (\prod_{j=1,j\neq i}^n dx_j)$$
$$\leq \int_{I^{n-1}} V_i^{I^n} f (\textstyle\prod_{j=1,j\neq i}^n dx_j).$$

Recall that if $f, g \in L^1$ and $f = g$ a.e., then $Q_\ell f = Q_\ell g$ a.e. Also, the measurability of $V_i^{I^n} g$ implies the measurability of $V_i^{I^n} Q_\ell g$. Now,

$$\mathbf{V}_i^{I^n} Q_\ell f = \inf\left\{ \int_{I^{n-1}} V_i^{I^n} h(\prod_{j=1,j\neq i}^n dx_j) \right.$$

$$\left. : h = Q_\ell f \text{ a.e.}, \quad V_i^{I^n} h \quad \text{measurable} \right\}$$

$$\leq \inf\left\{ \int_{I^{n-1}} V_i^{I^n} Q_\ell g(\prod_{j=1,j\neq i}^n dx_j) \quad : g = f \right.$$

$$\left. \text{a.e.}, \quad V_i^{I^n} Q_\ell g \quad \text{measurable} \right\}$$

$$\leq \ 3\inf\left\{ \int_{I^{n-1}} V_i^{I^n} (\prod_{j=1,j\neq i}^n Q_{\ell_j}) g \right.$$

$$(\prod_{j=1,j\neq i}^n dx_j) : \quad g = f$$

$$\left. \text{a.e.}, \quad V_i^{I^n} g \quad \text{measurable} \right\}$$

$$\leq \ 3\inf\left\{ \int_{I^{n-1}} V_i^{I^n} g(\prod_{j=1,j\neq i}^n dx_j) \quad : g = f \right.$$

$$\left. \text{a.e.}, \quad V_i^{I^n} g \quad \text{measurable} \right\} = 3\mathbf{V}_i^{I^n} f.$$

Therefore,

$$\mathbf{V}^{I^n} Q_\ell f = \max_{1\leq i\leq n} \mathbf{V}_i^{I^n} Q_\ell f \leq 3 \max_{1\leq i\leq n} \mathbf{V}_i^{I^n} f = 3\mathbf{V}^{I^n} f. \blacksquare$$

The following result is established in [8].

Theorem 1

Let τ be a Jablonski transformation, where:

$$\tau(x) = (\varphi_{1j}(x_1), \cdots, \varphi_{nj}(x_n)), \ x \in D_j.$$

If $\lambda = \inf_{i,j}\{ \inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}|\} > 2$, then for any $f \in L^1$:

$$\mathbf{V}^{I^n} P_\tau f \leq K_\tau \|f\| + \alpha \mathbf{V}^{I^n} f,$$

where K_τ is a constant depending on τ and $\alpha = 2\lambda^{-1} < 1$.

Lemma 2

Let τ be a Jablonski transformation,

$$\tau(x) = (\varphi_{1j}(x_1), \cdots, \varphi_{nj}(x_n)), \ x \in D_j,$$

and $f_\ell \in \Delta_\ell$ be any fixed point of $\tilde{P}_\ell(\tau)$ with $\|f_\ell\| = 1$. If:

$$\lambda = \inf_{i,j}\{ \inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}|\} > 6,$$

then the sequence $\{\mathbf{V}^{I^n} f_\ell\}_{\ell=1}^\infty$ is bounded.

Proof

By Remark 1, $f_\ell = \tilde{P}_\ell f_\ell = Q_\ell P_\tau f_\ell$ for all ℓ. Hence, by Lemma 1 and Theorem 1, it is obtained that:

$$\mathbf{V}^{I^n} f_\ell = \mathbf{V}^{I^n} Q_\ell P_\tau f_\ell \leq 3\mathbf{V}^{I^n} P_\tau f_\ell$$

$$\leq 3(K_\tau \|f_\ell\| + \alpha \mathbf{V}^{I^n} f_\ell) = 3K_\tau + 3\alpha \mathbf{V}^{I^n} f_\ell,$$

where $K_\tau > 0$ and $0 < \alpha < \frac{1}{3}$. Since $\mathbf{V}^{I^n} f_\ell < \infty$,

$$\mathbf{V}^{I^n} f_\ell \leq 3K_\tau/(1-3\alpha). \blacksquare$$

The following self-adjoint property of Q_ℓ, which was not needed in [6], plays a vital role in the sequel.

Lemma 3

For any $f \in L^1$, $\ell = 1, 2, \cdots$, and measurable subset A of I^n

$$\int_{I^n} \chi_A Q_\ell f dx = \int_{I^n} f Q_\ell \chi_A dx.$$

Proof

$$\int_{I^n} \chi_A(x) Q_\ell f(x) dx$$

$$= \int_{I^n} \chi_A(x) \sum_{k=1}^q (\int_{I^n} f(y) \sum_{j=1}^q c_j^k \chi_j(y) dy) \chi_k(x) dx$$

$$= \sum_{k=1}^q (\int_{I^n} f(y) \sum_{j=1}^q c_j^k \chi_j(y) dy) \int_{I^n} \chi_A(x) \chi_k(x) dx$$

$$= \sum_{k=1}^q (\int_{I^n} f(y) \sum_{j=1}^q c_j^k \chi_j(y) dy) \int_{I_k} \chi_A(x) dx$$

$$= \sum_{k=1}^q \sum_{j=1}^q \int_{I_j} \int_{I_k} f(y) c_j^k \chi_A(x) dx dy$$

$$= \sum_{k=1}^q \sum_{j=1}^q \int_{I_j} \int_{I_k} f(y) c_k^j \chi_A(x) dx dy$$

$$= \int_{I^n} f(x) \sum_{j=1}^q (\int_{I^n} \chi_A(y) \sum_{k=1}^q c_k^j \chi_k(y) dy) \chi_j(x) dx$$

$$= \int_{I^n} f(x) Q_\ell \chi_A(x) dx. \blacksquare$$

Theorem 2

Let τ be a nonsingular Jablonski transformation with partition $\{D_1, \cdots, D_p\}$ and

$$\lambda = \inf_{i,j}\{ \inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}|\} > 6.$$

Suppose P_τ has a unique fixed point. Then for any positive integer ℓ, $\tilde{P}_\ell(\tau)$ has a fixed point f_ℓ in Δ_ℓ with $\|f_\ell\| = 1$ and the sequence $\{f_\ell\}$ converges weakly to the fixed point of P_τ.

Proof

From Lemma 3 of [8] and Lemma 2, it is known that the set $\{f_\ell\}_{\ell=1}^{\infty}$ is weakly and relatively compact in L^1. Let $\{f_{\ell_j}\}$ be a weakly convergent subsequence of $\{f_\ell\}_{\ell=1}^{\infty}$ and let $f = \lim_{j\to\infty} f_{\ell_j}$ weakly. Then, for any $g \in L^\infty$,

$$\left|\int_{I^n} g(f - P_\tau f)dx\right| \le \left|\int_{I^n} g(f - f_{\ell_j})dx\right| + \left|\int_{I^n} g(f_{\ell_j} - Q_{\ell_j}P_\tau f_{\ell_j})dx\right| + \left|\int_{I^n} g(Q_{\ell_j}P_\tau f_{\ell_j} - P_\tau f)dx\right|. \quad (5)$$

The first term approaches zero since f_{ℓ_j} converges weakly to f as $j \to \infty$. From Remark 1, $Q_{\ell_j}P_\tau f_{\ell_j} = \tilde{P}_{\ell_j} f_{\ell_j} = f_{\ell_j}$, hence, the second term is identically zero.

Now the last term is considered. Because of the weak continuity of P_τ [10, p 43], $P_\tau f_{\ell_j}$ converges weakly to $P_\tau f$ as $j \to \infty$. It will be proven that $Q_{\ell_j}P_\tau f_{\ell_j}$ converges weakly to $P_\tau f$ as $j \to \infty$. It is enough to show that for any measurable subset A of I^n, the following is obtained:

$$\lim_{j\to\infty}\int_{I^n} \chi_A Q_{\ell_j} h_{\ell_j} dx = \int_{I^n} \chi_A h dx,$$

where $h_{\ell_j} = P_\tau f_{\ell_j}$ and $h = P_\tau f$.

From Corollary IV.8.11 in [12, p 294],

$$\int_E h_{\ell_j}(x)dx \to 0 \quad \text{as } m(E) \to 0$$

uniformly in j. Because $\|h_{\ell_j}\| = 1$ and $h_{\ell_j} \ge 0$, from Theorem 7.5.3 in [13, p 296], h_{ℓ_j}'s are uniformly integrable, i.e.,

$$\int_{\{|h_{\ell_j}| \ge K\}} |h_{\ell_j}|dx \to 0 \text{ as } K \to \infty,$$

uniformly in j. Therefore, for any $\epsilon > 0$, there exists $K > 0$ such that for all j:

$$2\int_{\{|h_{\ell_j}| \ge K\}} |h_{\ell_j}|dx < \epsilon.$$

Hence:

$$\begin{aligned}
&\left|\int_{I^n} h_{\ell_j}(Q_{\ell_j}\chi_A - \chi_A)dx\right| \\
&\le \int_{I^n} |h_{\ell_j}||Q_{\ell_j}\chi_A - \chi_A|dx \\
&= \int_{\{|h_{\ell_j}| \ge K\}} |h_{\ell_j}||Q_{\ell_j}\chi_A - \chi_A|dx \\
&\quad + \int_{\{|h_{\ell_j}| < K\}} |h_{\ell_j}||Q_{\ell_j}\chi_A - \chi_A|dx \\
&\le 2\int_{\{|h_{\ell_j}| \ge K\}} |h_{\ell_j}|dx + K\int_{\{|h_{\ell_j}| < K\}} |Q_{\ell_j}\chi_A - \chi_A|dx \\
&\le 2\int_{\{|h_{\ell_j}| \ge K\}} |h_{\ell_j}|dx \\
&\quad + K\int_{I^n} |Q_{\ell_j}\chi_A - \chi_A|dx.
\end{aligned}$$

The first term is less than ϵ and by Remark 2 the second term approaches zero as $j \to \infty$. Thus:

$$\lim_{j\to\infty}\int_{I^n} h_{\ell_j}(Q_{\ell_j}\chi_A - \chi_A)dx = 0.$$

By Lemma 3,

$$\begin{aligned}
\lim_{j\to\infty}\int_{I^n} \chi_A Q_{\ell_j} h_{\ell_j} dx &= \lim_{j\to\infty}\int_{I^n} h_{\ell_j} Q_{\ell_j}\chi_A dx \\
&= \lim_{j\to\infty}\int_{I^n} h_{\ell_j}(Q_{\ell j}\chi_A - \chi_A)dx \\
&\quad + \lim_{j\to\infty}\int_{I^n} h_{\ell_j}\chi_A dx \\
&= \int_{I^n} h\chi_A dx.
\end{aligned}$$

This means that the last term in Relation 5 approaches zero.

Therefore, it is established that for any $g \in L^\infty$,

$$\int_{I^n} g(x)(f(x) - P_\tau f(x))dx = 0.$$

It follows that $P_\tau f(x) = f(x)$ almost everywhere. Therefore, any weakly convergent subsequence of $\{f_\ell\}$ converges weakly to a unique fixed point of P_τ. Hence, $f_\ell \to f$ weakly as $\ell \to \infty$. ∎

Corollary 1

If the fixed point of P_τ is not unique in Theorem 2, then any weak limit point of $\{f_\ell\}_{\ell=1}^{\infty}$ is a fixed point of P_τ.

Theorem 3

Let τ be a nonsingular Jablonski transformation with $\lambda = \inf_{i,j}\{\inf_{[a_{ij},b_{ij}]} |\varphi'_{ij}|\} > 1$. Suppose P_τ has a unique **fixed** point. For an integer k such that $\lambda^k > 6$, let $\phi = \tau^k$ and f_ℓ be a fixed point of $\tilde{P}_\ell(\phi)$. Define:

$$g_\ell = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f_\ell.$$

Then, $\{g_\ell\}$ converges weakly to the fixed point of P_τ.

Proof

Since P_τ is a weakly continuous operator [10, p 43], Theorem 2 implies that $g_\ell \to g = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f$ weakly as $\ell \to \infty$. Therefore,

$$P_\tau g = \frac{1}{k}\sum_{j=1}^{k} P_{\tau^j} f = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f = g,$$

where f is the fixed point of $P_\phi = P_{\tau^k}$, i.e., $P_{\tau^k} f = f$. ■

Corollary 2

If the fixed point of P_τ is not unique in Theorem 3, then any weak limit point f of $\{f_\ell\}_{\ell=1}^{\infty}$ is a fixed point of P_ϕ and $g = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f$ is a fixed point of P_τ. If $f_{\ell_i} \to f$ weakly as $i \to \infty$, then $g_{\ell_i} = \frac{1}{k}\sum_{j=0}^{k-1} P_{\tau^j} f_{\ell_i} \to g$ weakly as $i \to \infty$.

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