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# On Sequences of Composition Operators

**S.D. Sharma* and U. Bhanu[1]**

In this paper, the closed graph theorem is used to demonstrate that every analytic self-map of the unit disc induces a composition operator on a vector-valued Hardy space. Conditions for the convergence of a sequence of composition operators in the weak, strong and uniform operator topologies, in terms of the convergence of the corresponding sequence of inducing maps, are also reported.

## INTRODUCTION

If $\phi$ is a analytic self-map of the open unit disc $D$, then Littlewood subordination theorem guarantees that without any additional assumptions about the behavior of $\phi$, composition transformation $C_\phi$, defined by $C_\phi f = fo\phi$ for a holomorphic $f$ in $D$, turns out to be a bounded operator on $H^P(D)$, and is called composition operator induced by $\phi$. Detailed study of these operators on scalar-valued Hardy spaces are given in [1-4]. In this paper, an attempt is made to study composition operators on a vector-valued Hardy space.

The paper is organised as follows. The next section is preliminary in nature. In this section, some known as well as unknown facts about vector-valued Hardy spaces are presented. Then, an appeal to the closed graph theorem is made to show that $C_\phi$ is bounded on $H^2_X(D)$. In the last section, necessary and sufficient conditions on the sequence $\{\phi_n\}$ of analytic self-maps of the unit disc $D$ are given so that the corresponding sequence $\{C_{\phi_n}\}$ of composition operators converges in the weak and strong operator topologies.

## PRELIMINARIES

Let $D$ be an open unit disc in the complex plane and $(X, \|\cdot\|_X)$ be a complex Banach space. For $0 < p < \infty$, the vector-valued Hardy space $H^p_X(D)$ consists of all functions $f : D \to X$ such that $x^* o f$ is holomorphic in $D$ for every $x^* \in X^*$, the dual of $X$ and:

$$\lim_{r\to 1} 1/2\pi \int_0^{2\pi} \|f(re^{i\theta})\|^p_X de < \infty.$$

$H^p_X(D)$, $1 \le p < \infty$, is a Banach space with:

$$|||f|||^p_p = \lim_{r\to 1} 1/2\pi \int_0^{2\pi} \|f(re^{i\theta})\|^p_X d\theta.$$

If $X = C$, $H^p_X(D)$ is simply denoted by $H^p(D)$ and $|||\cdot|||_p$ by $\|\cdot\|_p$.

It is remarkable to mention here that unlike $H^p(D)$, not every function $f \in H^p_X(D)$ has a radial limit a.e. An example is $X = C_o$, the Banach space of null sequences of complex numbers. Then $f : D \to C_o$, defined as: $f(z) = \{Z^n\}^\infty_{n=o}$, is in $H^p_{C_o}(X)$. However, $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \{e^{in\theta}\} \notin C_o$ for any $\theta$. Hence, in order to make $H^p_X(D)$ a proper Banach space, it becomes obligatory to choose $X$ in such a manner that every $f \in H^p_X(D)$ would have a radial limit a.e. As a matter of fact, here the interest lies in the case where $p = 2$ and $(X, <,>)$ is a separable Hilbert space. In this case, every function $f \in H^2_X(D)$ has a radial limit $f^*(e^{i\theta})$ a.e. [5, Theorem A, p 89], which for the sake of convenience is simply denoted by $f(e^{i\theta})$, and $H^2_X(D)$ becomes a Hilbert space under the inner product $<<,>>$ given by $<< f, g >> = 1/2\pi \int_o^{2\pi} < f(e^{i\theta}), g(e^{i\theta}) > d\theta$. For more details about vector-valued analytic functions and Hardy spaces, see [5-7] and for classical Hardy spaces, see [8].

A lemma is formulated which will be used to find kernel functions for $H^2_X(D)$.

**Lemma 1**

Let $f \in H^2_X(D)$. Then,

$$\|f(z)\|_X \le \frac{|||f|||_2}{\{1 - |z|^2\}^{1/2}}.$$

*. *Corresponding Author, Department of Mathematics, University of Jammu, Jammu 180004, India.*
1. *Department of Mathematics, University of Jammu, Jammu 180004, India.*

***Proof***

let $f(z) = \sum_{n=o}^{\infty} a_n z^n \in H_X^2(D)$. Then, $\sum_{n=o}^{\infty} ||a_n||_X^2 < \infty$ and:

$$||f(z)||_X \leq \sum_{n=o}^{\infty} ||a_n||_X |z|^n$$
$$\leq \left\{\sum_{n=o}^{\infty} ||a_n||_X^2\right\}^{1/2} \left\{\sum_{n=o}^{\infty} |z|^{2n}\right\}^{1/2}$$
$$= \frac{|||f|||_2}{\{1-|z|^2\}^{1/2}}. \blacksquare$$

Let $N = \{0,1,2,...\}$ and $\{e_n : n \in N\}$ be an orthonormal basis for $X$. For $m, n \in N$, $e_{mn} : D \to X$ is defined as:

$$e_{mn}(z) = z^m e_n \quad \text{for every} \quad z \in D.$$

Then, clearly, $\{e_{mn} | m, n \in N\}$ is an orthonormal subset of $H_X^2(D)$. Furthermore, if $f \in H_X^2(D)$, then:

$$<<f, e_{mn}>> = 0$$
$$\Rightarrow 1/2\pi \int_0^{2\pi} < f(e^{i\theta}), e_{mn}(e^{i\theta}) > d\theta = 0$$
$$\Rightarrow 1/2\pi \int_0^{2\pi} e^{-im\theta} < f(e^{i\theta}), e_n > d\theta = 0.$$

Since $g_n$ defined by $g_n(z) =< f(z), e_n >$ is analytic in $D$, it is concluded that:

$$< f(z), e_n >= 0 \quad \text{for every} \quad z \in D \quad \text{and} \quad n \in N.$$

This further implies that $f \equiv 0$. Hence, $\{e_{mn} : m, n \in N\}$ is a basis for $H_X^2(D)$.

For each $z \in D$ and $j \in N$, $E_z^j : H_X^2(D) \to C$ is defined as follows:

$$E_z^j(f) =< f(z), e_j > \quad \text{for every} \quad f \in H_X^2(D).$$

Then, $E_z^j \in (H_X^2(D))^*$ and so by Riesz representation theorem, there exists $K_z^j \in H_X^2(D)$ such that:

$$E_z^j f =<< f, K_z^j >> \quad \text{for every} \quad f \in H_X^2(D).$$

$K_z^j$ is designated as a generalized reproducing kernel or simply a kernel function whenever there is no confusion. The next task is to find these kernel functions.

**Theorem 1**

For $z \in D$ and $j \in N$, the generalized reproducing kernel $K_z^j$ is given by:

$$K_z^j(w) = \frac{e_j}{1 - \bar{z}w}.$$

Further, $|||K_z^j|||_2^2 = \frac{1}{1-|z|^2}$.

***Proof***

By Parseval identity,

$$K_z^j(w) = \sum_{m,n \in N} << K_z^j, e_{mn} >> e_{mn}(w)$$
$$= \sum_{m,n \in N} \overline{E_z^j(e_{mn})}\, e_{mn}(w)$$
$$= \sum_{m,n \in N} < \overline{z^m e_n, e_j} > e_{mn}(w)$$
$$= \sum_{m \in N} (\bar{z}w)^m e_j = \frac{e_j}{1 - \bar{z}w}.$$

and $|||K_z^j|||_2^2 = \frac{1}{1-|z|^2}$. $\blacksquare$

**Proposition 1**

The subspace $[K_z^j : (z,j) \in D \times N]$, the span of all generalized reproducing kernel functions, is dense in $H_X^2(D)$.

***Proof***

Let $f \in H_X^2(D)$ be such that:

$$<< f, K_z^j >>= 0 \quad \text{for every} \quad (z,j) \in D \times N,$$
$$\Rightarrow < f(z), e_j >= 0 \quad \text{for every} \quad (z,j) \in D \times N$$
$$\Rightarrow f \equiv 0$$
$$\Rightarrow [K_z^j : (z,j) \in D \times N] \text{ is dense in } H_X^2(D). \blacksquare$$

## COMPOSITION OPERATORS ON $H_X^2(D)$

Using the bounded linear functionals $E_z^j$ and the closed graph theorem, it shall be shown that every analytic self-mapping $\phi$ of the unit disc induces a bounded operator on $H_X^2(D)$.

**Theorem 2**

Let $\phi : D \to D$ be analytic. Then $C_\phi$ is bounded.

***Proof***

By Theorem C of [5], $fo\phi \in H_X^2(D)$ for every $f \in H_X^2(D)$. Hence, $C_\phi$ is a mapping from $H_X^2(D)$ into $H_X^2(D)$. To prove the boundedness of $C_\phi$, let $\{(f_n, C_\phi f_n)\}$ be a sequence in the graph of $C_\phi$ which converges to $(f, g)$. Then:

$$f_n \to f \quad \text{and} \quad C_\phi f_n \to g,$$

which implies that:

$$E_{\phi(z)}^j f_n \to E_{\phi(z)}^j f \quad \text{and} \quad E_z^j C_\phi f_n \to E_z^j g,$$

i.e., $< f_n(\phi(z)), e_j > \to < f(\phi(z)), e_j >$ and $< f_n(\phi(z)), e_j > \to < g(z), e_j >$ for every $z \in D$ and for every $j \in N$. Since $\{e_j : j \in N\}$ is a basis for $X$, it is concluded that $C_\phi f = g$. This shows that the graph of $C_\phi$ is closed. Hence, by the closed graph theorem, $C_\phi$ is bounded. $\blacksquare$

## SEQUENCES OF COMPOSITION OPERATORS

In this section, conditions are provided on a given sequence of analytic self-maps of the open unit disc, so that the corresponding sequence of composition operators converges in the weak, strong and uniform operator topologies.

The First theorem of this section gives a necessary and sufficient condition for the weak convergence of a sequence of composition operators.

**Theorem 3**

Let $\{\phi_n\}$ be a sequence of analytic self-maps of the unit disc $D$ and $\phi : D \rightarrow D$ be analytic. Then, the sequence $\{C_{\phi_n}\}$ converges in the weak operator topology to $C_\phi$ on $H^2_X(D)$ if and only if the sequence $\{\phi_n\}$ converges to $\phi$ uniformly on $D$.

Prior to proving this theorem, it is noted here that if $C_\phi$ is a composition operator on $H^2_X(D)$, then for $(z,j) \in D$, $C^*_\phi K^j_z = K^j_{\phi(z)}$. In fact:

$$<< f, C^*_\phi K^j_z >>$$
$$=<< C_\phi f, K^j_z >>$$
$$=< C_\phi f(z), e_j >$$
$$=<< f, K^j_{\phi(z)} >>, \text{ for every } f \in H^2_X(D),$$

which implies that $C^*_\phi K^j_z = K^j_{\phi(z)}$.

***Proof***

It is first assumed that $\{C_{\phi_n}\}$ converges to $C_\phi$ weakly. Then $C_{\phi_n} f \rightarrow C_\phi f$ weakly for every $f \in H^2_X(D)$ and so $\lim_n | << C_{\phi_n} f, g >> - << C_\phi f, g >> | = 0$ for every $f, g \in H^2_X(D)$. In particular, taking $f = e_{1j}$ and $g = K^j_z$, it is obtained that:

$$\lim_n | << e_{1j}, C^*_{\phi_n} K^j_z >> - << e_{1j}, C^*_\phi K^j_z >> |$$
$$= 0 \text{ for every } z \in D \text{ and } j \in N.$$

Using the above remark:

$$\lim_n | << e_{1j}, K^j_{\phi_n(z)} >> - << e_{1j}, K^j_{\phi(z)} >> | = 0,$$

which further implies that:

$$\lim_n |\phi_n(z) - \phi(z)| = 0 \text{ for every } z \in D.$$

Hence, $\{\phi_n\}$ converges to $\phi$ uniformly on $D$.

Conversely, suppose $\phi_n \rightarrow \phi$ uniformly on $D$. Let $f \in H^2_X(D)$. Then $< f(.), x >: D \rightarrow \mathbb{C}$ is analytic and so continuous. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$| < f(z), x > - < f(w), x > | < \varepsilon$$
$$\text{whenever } |z - w| < \delta, \text{ for every } x \in X. \qquad (1)$$

Also, since $\phi_n \rightarrow \phi$ uniformly on $D$, there exists a postitive integer $n_o$ such that for $n \geq n_o$,

$$|\phi_n(z) - \phi(z)| < \delta \text{ for every } z \in D. \qquad (2)$$

From Equations 1 and 2, it is obtained for $n \geq n_o$ that:

$$| < f(\phi_n(z)), x > - < f(\phi(z)), x > | < \varepsilon$$
$$\text{for every } z \in D \text{ and for every } x \in X.$$

In particular, taking $x = e_j$, for $n \geq n_o$:

$$| < (C_{\phi_n} f)(z), e_j > - < (C_\phi f)(z), e_j > | < \varepsilon$$
$$\text{for every } (z,j) \in D \times N$$
$$\text{or} | << C_{\phi_n} f, K^j_z >> - << C_\phi f, K^j_z >> | < \varepsilon$$
$$\text{for every } (z,j) \in D \times N.$$

This, by Proposition 1, implies that $\{C_{\phi_n}\}$ converges to $C_\phi$ weakly.

Next, a necessary and sufficient condition is given for strong convergence of a sequence of composition operators.

**Theorem 4**

The sequence $\{C_{\phi_n}\}$ converges to $C_\phi$ strongly if, and only if, the corresponding sequence $\{\phi_n\}$ of analytic self-maps of the unit disc converges to $\phi$ in $(H^2(D), ||.||_2)$.

***Proof***

If $\{C_{\phi_n}\}$ converges to $C_\phi$ strongly, then:

$$\lim_n |||C_{\phi_n} f - C_\phi f|||_2 = 0 \text{ for every } f \in H^2_X(D).$$

In particular, taking $f = e_{1j}$, it is obtained that:

$$\lim_n 1/2\pi \int_0^{2\pi} |\phi_n(e^{i\theta}) - \phi(e^{i\theta})|^2 d\theta = 0,$$

i.e., $\lim_n ||\phi_n - \phi||^2_2 = 0$.

This implies that $\{\phi_n\}$ converges to $\phi$ strongly in $(H^2(D), ||.||_2)$.

The proof of the sufficient part folllows from a scalar-valued version of this result simply by replacing $|.|$ by $||.||_X$ and using the fact that polynomials with coefficients in $X$ are dense in $H^2_X(D)$ (see [1, Theorem 4.2]). ■

**Corollary 1**

If $\{\phi^*_n\}$ converges to $\phi^*$ a.e. on the unit circle $\partial D$, then the sequence $\{C_{\phi_n}\}$ converges strongly to $C_\phi$.

For proof, see [1]. ■

This section is concluded with a necessary condition for the uniform convergence of a sequence of composition operators.

**Theorem 5**

If $\{C_{\phi_n}\}$ converges to $C_\phi$ uniformly on $H^2_X(D)$, then $\{\phi^j_n\}$ converges strongly to $\phi^j$ uniformly in $j$ in $H^2(D)$.

***Proof***

Suppose $\{C_{\phi_n}\}$ converges to $C_\phi$ uniformly on $H^2_X(D)$. Then, for any $\varepsilon > 0$, there exists a positive integer $n_o$ such that:

$$||C_{\phi_n} - C_\phi|| < \varepsilon \text{ for every } n \geq n_o.$$

This implies that for $n \geq n_o$,

$$|||C_{\phi_n} f - C_\phi f|||_2 < \varepsilon$$

$$\text{for every } f \in H^2_X(D) \text{ with } |||f|||_2 \leq 1.$$

In particular, taking $f = e_{jK}$, it is obtained for $n \geq n_o$ that:

$$|||C_{\phi_n} e_{jK} - C_\phi e_{jK}|||_2 < \varepsilon,$$

$$\text{i.e., } \{1/2\pi \int_0^{2\pi} ||e_{jK}(\phi_n(e^{i\theta})) - e_{jK}(\phi(e^{i\theta}))||^2_X d\theta\}^{1/2} < \varepsilon.$$

This implies that for $n \geq n_o$,

$$||\phi^j_n - \phi^j||_2 < \varepsilon \text{ for any } j \in N.$$

Hence, $\{\phi^j_n\}$ converges to $\phi^j$ uniformly in $j$. ■

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