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Weighted Composition Operators Induced by a Finite Number of Mappings on Uniform Spaces

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Let X be a compact Hausdorff space and $C(X)$ denote the space of all continuous complex-valued functions on X equipped with the Sup-norm. In this paper, compactness and nuclearity (in concrete situations) of operators on uniformly closed subspace A of $C(X)$ will be discussed, which are induced by a finite number of continuous mappings $\omega_i : X \longrightarrow X$ $(i = 1, \ldots, m)$ of the form $T = \sum_{i=1}^{m} T_i$, where $T_i = u_i \cdot (f \circ \omega_i)$ are weighted composition operators on A. Furthermore, some applications are provided.

INTRODUCTION

The endomorphisms of semisimple commutative Banach algebras (also any linear bounded operator on Banach spaces) can be represented as weighted composition operators; therefore, these operators and their finite sums are very interesting to study. Composition operators and also weighted composition operators on uniform algebras are being investigated from different points of view (such as compactness, spectrum, closedness of ranges, etc.) by many authors (see [1–6]); however, except for a few cases, they are essentially considered in concrete algebras of functions. The aim of this paper is to clarify the compactness and, in concrete situations, nuclearity conditions of finite sums of weighted composition operators, i.e., operators which are induced by a finite number of continuous mappings $\omega_i : X \longrightarrow X$ (i = 1,...,m) of the form $Tf = \sum_{i=1}^{m} u_i \cdot (f \circ \omega_i)$ (where $u_i \in C(X)$ are fixed functions) on uniformly closed subspaces of $C(X)$. In particular, Kamowitz [1] gives a compactness criterion for the weighted composition operator (i.e., of the form $f \longmapsto u \cdot (f \circ \omega)$) in the disc-algebra and described its spectrum when the operator is compact. In [6–8] Kamowitz results are extended to many other uniform spaces (including multidimensional analoges of disc-algebra) and in [4], an operator of the form T in the disc-algebra with two summands is considered. Here, sufficiently simple general compactness criteria, which for $m = 2$ and $m = 3$ (in the case of uniform algebras) the summands take sufficiently transparent form, and nuclearity criteria for operator T in the case of disc-algebra (under some assumptions) are given. Moreover, some applications of compactness of operator T, when the mappings $\omega_i (i = 1, \ldots, m)$ are a finite group will be presented.

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THE GENERAL CASE

Let X be a compact Hausdorff space, $C(X)$ the algebra of all continuous complex-valued functions on X equipped with the Sup-norm and A a closed subspace of $C(X)$.

Definition 1

The point $x_0 \in X$ is called a peak point (with respect to A) if there exists a sequence $\{f_n\}_{i=1}^{\infty}$ of elements of A, such that $\|f_n\| = f_n(x_0) = 1$ and $f_n(x) \longrightarrow 0$ uniformly outside any neighborhood of the point x_0.

The set of all peak points is denoted by Γ. Set $G = X \setminus \Gamma$. To each point $x \in X$ corresponds a functional $\delta_x : f \longmapsto f(x)$, which lies in the unit ball of conjugate space A^*. This induces on X the A^*- topology which is, in general, stronger than the original one. Further, it shall always be assumed that the original topology on G coincides with A^*- topology, G is everywhere dense in X and Γ is not empty. A typical example is given by the disc algebra, i.e., the algebra $A(D)$ of all analytic functions in the unit disc $D = \{z \in C : \|z\| < 1\}$ which is continuous on its closure $\overline{D}$. Here, $X = \overline{D}$, $\Gamma = \partial D = \overline{D} \setminus D$, $G = D$.

Consider the operator $T : A \longrightarrow C(X)$ of the form $f(x) \longmapsto \sum_{i=1}^{m} u_i(x)f(\omega_i(x))$, where $u_i \in C(X)$ are fixed functions and $\omega_i : X \longrightarrow X$ are such continuous mappings that $\omega_i(G) \subset G(i = 1,\dots,m)$. Except for easy degenerate cases, it will be assumed that $\omega_i \neq$ constant for all i.

The unit ball of the conjugate space $C(X)^*$ of $C(X)$ will be identified with all complex Borel measures on X with variation ≤ 1, any point $x \in X$ will correspond to a δ-measure. Since the compactness of an operator is equivalent to the compactness of its conjugate, it can easily be shown (see [9], Theorem VI,7.1) that the compactness of the operator T is equivalent to continuity of mapping $x \longmapsto T^*x = \sum_{i=1}^{m} u_i(x)\omega_i(x)$ acting on X with original topology into A^* with metric topology: if $z \longrightarrow \zeta$ in X with original topology, then $\sum_{i=1}^{m} u_i(z)\omega_i(z)$ must converge to $\sum_{i=1}^{m} u_i(\zeta)\omega_i(\zeta)$, with respect to A^*-topology. If $\zeta \in G$, then because of $\omega_i(\zeta) \in G$ and since the original topology coincides with A^*- topology on G, $\|\omega_i(z)-\omega_i(\zeta)\|_{A^*} \longrightarrow 0$ is obtained when $z \longrightarrow \zeta$ in the original topology, i.e., the above-mentioned mapping $x \longrightarrow T^*x$ for $x \in G$ is automatically continuous. For this reason, the continuity will be investigated only at peak points.

Definition 2

Let $\zeta \in \Gamma$ be a fixed point, the indices i, j are equivalent with respect to $\zeta \in \Gamma$, if $\omega_i(\zeta) = \omega_j(\zeta) \in \Gamma$. Equivalent classes will be denoted by K. K_0 will denote indices i such that $\omega_i(\zeta) \in G$. Indices i, j are called strongly equivalent, if $\|\omega_i(z) - \omega_j(z)\|_{A^*} \longrightarrow 0$ when $z \longrightarrow \zeta$. Equivalent classes of this kind will be denoted by L.

Lemma 1

If the operator T is compact, then for any class K, $\sum_{i\in K} u_i(\zeta) = 0$ is obtained.

Proof

Since $K \neq K_0$, there exists a point $\xi \in \Gamma$ such that $\omega_i(\zeta) = \xi$ for all $i \in K$. Since ξ is a peak point, a sequence of functions $f_n \in A$ can be found such that $\|f_n\| = f_n(\xi) = 1$ and outside any neighborhood of the point ξ the sequence tends to 0 uniformly as $n \longrightarrow \infty$. It may be assumed that the sequence Tf_n converges uniformly. Since G is invariant with respect to the mappings ω_i and is everywhere dense in X, then $\|Tf_n\| \longrightarrow 0$ and this completes the proof. ■

Conditions $\omega_i(G) \subset G$, $(i = 1,\dots,m)$ in Lemma 1 are essential as the following example shows.

Example 1

Let $X = \{z \in C : \mid z \mid\leq 1 \text{ or } \mid z-2 \mid\leq 1\}$ and A be the algebra of all continuous functions on X and analytic inside, set $\omega_1(z) = z$ for all $z \in X$,

$$\omega_2(z) = \begin{cases} z, & \text{for } \mid z \mid\leq 1, \\ 1, & \text{for } \mid z-2 \mid\leq 1. \end{cases}$$

$$\omega_3(z) = \begin{cases} 1, & \text{for } \mid z \mid\leq 1, \\ z, & \text{for } \mid z-2 \mid\leq 1. \end{cases}$$

Since $\omega_i(1) = 1$ for all i, then for $\zeta = 1$ all indices are equivalent. Since $-f\circ\omega_1 + f\circ\omega_2 + f\circ\omega_3 = f(1)$, the operator is compact, but $u_1 + u_2 + u_3 = 1$.

Theorem 1

The operator T is compact, iff for an arbitrary point $\zeta \in \Gamma$, $\sum_{i\notin K_0} u_i(\zeta)\omega_i(z) \longrightarrow 0$ with respect to A^*-norm as $z \longrightarrow \zeta$ (in original topology of X) and $\sum_{i\in K} u_i(\zeta) = 0$ for any class $K \neq K_0$.

Proof

The original topology on G coincides with A^*-topology; therefore, the compactness of operator T is equivalent to the following: for any point $\zeta \in \Gamma$, when $z \longrightarrow \zeta$ (converges with respect to original topology), the $\sum_{i=1}^{m} u_i(\zeta)[\omega_i(z) - \omega_i(\zeta)]$ converges to zero in A^*-topology. It is clear that in the sum one can restrict indices $i \notin K_0$. If the sum is divided into corresponding classes, then the sufficiency will be obvious. The necessity of $\sum_{i\in K} u_i(\zeta) = 0$ for $K \neq K_0$ follows from Lemma 1 and if the sum for $i \notin K_0$ is considered, the necessity of the first condition will hold. Therefore, the theorem is proven. ■

Corollary 1

if $\sum_{i\in L} u_i(\zeta) = 0$ for any subclass L and for any $\zeta \in \Gamma$, then the operator T is compact.

Indeed, the condition $\sum_{i\in K} u_i(\zeta) = 0$ is obvious $(K \neq K_0)$. Further, $\sum_{i\notin K_0} u_i(\zeta)\omega_i(z) = \sum_L(\sum_{i\in L} u_i(\zeta)\omega_i(z))$ and if $p \in L$, then $\sum_{i\in L} u_i(\zeta)\omega_i(z) = [\sum_{i\in L} u_i(\zeta)]\omega_p(z) + \sum_{i\in L} u_i\ (\zeta)[\omega_i(z) - \omega_p(z)] = \sum_{i\in L} u_i(\zeta)[\omega_i(z) - \omega_p(z)] \longrightarrow 0 as z \longrightarrow \zeta$.

The sufficiency of the Corollary 1 does not imply, in general, the necessity, as the next example shows.

Example 2

Let X and A be as in Example 1. Now set $\omega_1(z) = z$ and $\omega_2(z) = \frac{z+1}{2}$ for all $z \in X$,

$$\omega_3(z) = \begin{cases} z, & \text{if } \mid z \mid\leq 1, \\ \frac{z+1}{2}, & \text{if } \mid z-2 \mid\leq 1. \end{cases}$$

$$\omega_4(z) = \begin{cases} \frac{z+1}{2}, & \text{if } \mid z \mid\leq 1, \\ z, & \text{if } \mid z-2 \mid\leq 1. \end{cases}$$

It is obvious that $\omega_i(z) \in G$, when $z \in G$ and $\omega_i(1) = 1$ for all i, so for $\zeta = 1$ all indices are equivalent (consist of one class); however, it can be easily shown that no pairs of indices are strongly equivalent. Nevertheless, regardless of this, $-f\circ\omega_1 - f\circ\omega_2 + f\circ\omega_3 + f\circ\omega_4 = 0$, i.e., the operator corresponding to given ω_i and $u_i = \pm 1$ is zero.

THE CASE $M \leq 2$

For $m = 2$ and $m = 3$ (under additional conditions) the converse of Corollary 1 holds and the criteria of compactness can easily be verified.

Now these two cases are considered separately. The following theorem generalizes the results of [4].

Theorem 2

The operator $f \longmapsto u_1 \cdot f\circ\omega_1 + u_2 \cdot f\circ\omega_2$ is compact if, and only if, the following conditions hold for any point $\zeta \in \Gamma$: if indices are strongly equivalent, then $u_1(\zeta) + u_2(\zeta) = 0$; if they are not strongly equivalent, but $\omega_i(\zeta) \in \Gamma$, then $u_i(\zeta) = 0$.

Proof

The sufficiency is clear from Corollary 1. It is assumed that the operator is compact. Let $\zeta \in \Gamma$, $\omega_1(\zeta) \in \Gamma$ and $\omega_2(\zeta) \neq \omega_1(\zeta)$. According to Theorem 1, $u_1(\zeta)\omega_1(\zeta) + u_2(\zeta)\omega_2(\zeta) = 0$. Since $\omega_1(\zeta)$ is a peak point, $u_1(\zeta) = 0$. Now, if $\omega_1(\zeta) = \omega_2(\zeta) \in \Gamma$ is assumed (i.e., they are equivalent), then, from Theorem 1, $u_1(\zeta) + u_2(\zeta) = 0$ is obtained. If they are not strongly equivalent, then $\|\omega_1(z_n) - \omega_2(z_n)\| \geq \delta > 0$ for some sequence $z_n \longrightarrow \zeta$. Hence, using Theorem 1, $u_1(\zeta)\omega_1(z_n) + u_2(\zeta)\omega_2(z_n) \longrightarrow 0$ is obtained, so $u_1(\zeta)[\omega_1(z_n) - \omega_2(z_n)] \longrightarrow 0$ and from this, $u_1(\zeta) = u_2(\zeta) = 0$. ■

Now the disc-algebra is considered, as the algebra of all continuous functions on the half- plane $Im\lambda \geq 0$ and analytic in $Im\lambda > 0$, such that they have a limit as $|\ \lambda\ | \longrightarrow \infty$. Let ω be a continuous mapping from $Im\lambda \geq 0$ into itself, which is holomorphic for $Im\lambda > 0$ and has a limit as $|\ \lambda\ | \longrightarrow \infty$. Then from Theorem 2, the next corollary is obtained: Let ω_1, ω_2 be such that they map $Im\lambda \geq 0$ into itself and have the above mentioned conditions as well as $Im\omega_i(\lambda) > 0$, for all λ and $\omega_i(\infty) = \infty$. Then, the operator $f \longmapsto u_1 \cdot f\circ\omega_1 + u_2 \cdot f\circ\omega_2$ (u_1 and u_2 are the elements of disc-algebra) is compact iff the following conditions hold: either $u_1(\infty) = u_2(\infty) = 0$, or $u_1(\infty) + u_2(\infty) = 0$ and $|\ \frac{\omega_1(\lambda) - \omega_2(\lambda)}{\omega_1(\lambda) - \overline{\omega_2(\lambda)}}\ | \longrightarrow 0$ when $|\ \lambda\ | \longrightarrow \infty$.

THE CASE $M \leq 3$ IN THE UNIFORM ALGEBRAS

In this section a uniform algebra A is considered and if m is not greater than 3, the following theorem is obtained.

Theorem 3

If A is a uniform algebra, then the operator $f \longmapsto u_1 \cdot f\circ\omega_1 + u_2 \cdot f\circ\omega_2 + u_3 \cdot f\circ\omega_3$ is compact iff, for any strongly equivalent class, L, $\sum_{i\in L} u_i(\zeta) = 0$.

Proof

The sufficiency is clear from Corollary 1.

For necessity, only the case where all indices are equivalent, but not strongly equivalent, is considered. According to Theorem 1, if $z \longrightarrow \zeta$, then:

$$u_1(\zeta)[\omega_1(z) - \omega_3(z)] + u_2(\zeta)[\omega_2(z) - \omega_3(z)] \longrightarrow 0.$$

A sequence $\{z_n\}$ is chosen, such that $\|\omega_1(z_n) - \omega_2(z_n)\| \geq \delta > 0$. Since A is a uniform algebra, for some sequence of uniformly bounded functionals $\{f_n\}$, $f_n(\omega_1(z_n)) - f_n(\omega_3(z_n)) = 1$ and $f_n(\omega_2(z_n)) - f_n(\omega_3(z_n)) = 0$ are obtained, hence $u_1(\zeta) = 0$. In the same way, $u_2(\zeta) = u_3(\zeta) = 0$. ■

NUCLEARITY

In this section, the foregoing general compactness criteria is applied in special cases such that it turns out to be a very easy criteria of compactness and nuclearity. As is known, $T \in L(E,F)$ ($L(E,F)$ is the space of all bounded linear operators from Banach space E to Banach space F) is called a nuclear operator if it can be represented in the form $T = \sum_{i=1}^{\infty} a_i \otimes f_i$, where $a_i \in E^*$ (E^* is the dual space of E) and $f_i \in F$, such that $\sum_{i=1}^{\infty} \|a_i\| \cdot \|f_i\| < \infty$, i.e., if it can be shown that the series $\sum_{i=1}^{\infty} T_i$ is absolutely convergent in $L(E,F)$ where $T_i = \langle\cdot, a_i\rangle f_i$ are one dimensional operators. Consequently, nuclear operators are compact ones.

If the operator T is regarded in the case when $A(X) = A(D)$ (the disc-algebra), then if all the functions $u_i (i = 1, ..., m)$ and any sum of them are not equal to zero on the boundary of D, the next result is obtained (by using Lemma 1).

Theorem 4

If all functions $u_i (i = 1, ..., m)$ and any sum of them are not equal to zero on the boundary of D, then the operator $T : f \longrightarrow \sum_{i=1}^{m} u_i.\ f\circ\omega_i : A(D) \longrightarrow C(D)$ is nuclear (or compact) if, and only if, there exists a constant $c : 0 < c < 1$, such that for any i, either $\|\omega_i\| \leq c$ or $\omega_i =$ constant.

W.C.O. INDUCED BY A FINITE GROUP OF MAPPINGS

In this section, the operator $T : A \longrightarrow C(X)$ is considered, such that $(Tf)(x) = \sum_{i=1}^{N} u_i(x) f(\omega_i(x))$, ($u_i \in C(X)$), where ω_i's are invertible and make a finite group of transformations on X, i.e., $\omega_i : X \longrightarrow X$

are surjective in this case. The compactness of this operator will be investigated under the conditions that $\omega_i(G) \subset G$ (for all i), where $G = X \setminus \Gamma$, Γ is the peak point set of A and the original topology on G coincides with A^*-topology. The importance of this kind of operators is that they are applicable in solving and examing the existence of solutions of equations of the form $\sum_{i=1}^{N} u_i(x)f(\omega_i(x)) = g(x)$, i.e., differential-functional equations, which contain both the argument and its shift.

Let $Y = \langle\omega\rangle$ be a cyclic group induced by the mapping ω, such that $\mid Y \mid = N$:, in other words, if $\omega_1 = \omega, \omega_2 = \omega \circ \omega_1, \ldots, \omega_N = \omega \circ \omega_{N-1}$, then $Y = \{\omega_1, ..., \omega_N\}$, where $\omega_N =$ id (identity).

For peak points Γ of $A(X)$ the notion of equivalence degree, with respect to Y, is introduced.

Definition 3

For any point $\zeta \in \Gamma$, the least positive integer N_ζ satisfying the equality $\omega_{N_\zeta}(\zeta) = \zeta$ is called equivalent degree for ζ. It is clear that such a number exists (because for any $\zeta \in \Gamma$ the equality $\omega_N(\zeta) = \zeta$ holds and also $1 \leq N_\zeta \leq N$). If K_j denote all peak points such that their equivalent degrees are j, then the peak set Γ has a disjoint union of classes K_j, i.e., $\Gamma = \bigcup_{j=1}^{N} K_j$, where $K_i \bigcap K_j \neq \emptyset$ for any $1 \leq i,j \leq N$, which $i \neq j$.

It is clear that if $\mid i - j \mid$, where $1 \leq i,j \leq N$, is divisible by N_ζ, then for $\zeta \in \Gamma$, $\omega_i(\zeta) = \omega_j(\zeta)$.

Since at the point ζ, the group $\{\omega_i(\zeta)\}_{i=1}^{N_\zeta}$ is a subgroup of $Y = \langle\omega(\zeta)\rangle$, N is divisible by N_ζ and, therefore, the following lemma holds.

Lemma 2

At any point $\zeta \in \Gamma$, the indices are separated into N_ζ (denoted by $K_1, K_2, \ldots, K_{N_\zeta}$) equivalent classes, such that each power is equal to $N : N_\zeta$.

Corollary 2

If N is a prime number, then all indices are equivalent.

Using the above mentioned facts for the operator T induced by the group $Y = \langle\omega\rangle$, the following compactness criterion is obtained.

Theorem 5

The operator T is compact iff for any point $\zeta \in K_{N_\zeta}$, $\sum_{i=1}^{N} u_i(\zeta)\omega_i(z) \longrightarrow 0$ with respect to A^*-norm, as $z \longrightarrow \zeta$ (with respect to original topology on X) and $\sum_{i\in A} u_i(\zeta) = 0$, where $A = K_1, \ldots, K_{N_\zeta}$.

In particular, if N is a prime number, then, from Theorem 5 and Corollary 2, the following simple criterion of compactness is obtained.

Theorem 6

If N is a prime number, then the finite sum of weighted composition operators T induced by the group $Y = \langle\omega\rangle = \{\omega_1, \ldots, \omega_N\}$ is compact iff for any peak point $\zeta \in \Gamma$, $u_i(\zeta) = 0$, for any $i = 1, \ldots, N$.

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