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Research Note

Dynamic Error Analysis of Gantry Type Coordinate Measuring Machines

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Coordinate Measuring Machines (CMMs) are designed for precision inspection of complex industrial products. The mechanical accuracy of CMMs depends on both static and dynamic sources of error. In automated CMMs, one of the dynamic error sources is vibration of the probe, due to inertia forces resulting from parts acceleration and deceleration. Modeling of a gantry type CMM, based on the Timoshenko beam theory with moving mass effects, is developed and the dynamic errors of the probe resulting from the acceleration and deceleration of moving parts, are calculated. Findings from analytical solution and dynamic modeling software indicate high accuracy and good agreement between the results.

INTRODUCTION

Coordinate measuring machines are, nowadays, widely used for a large range of measuring tasks. These tasks are expected to be carried out with ever increasing accuracy, speed, flexibility and ability to operate under shop floor conditions. Research is necessary to meet these demands. CMMs are prone to many error sources. Based on functional components of a CMM, an overview has been given by Weekers [1] of the most important error sources affecting the accuracy of a CMM:

- *Geometric Errors*: Limited accuracy in manufacturing, assembling and adjustment of components, like guide ways and measurement systems;
- *Drive System*: For CNC operated CMMs, the axes are equipped with drives, transmission and a servo-control unit causing errors, such as mechanical load and structural vibration;
- *Measurement System*: The actual coordinates of measuring points are derived from the values indicated by the linear scales of the CMM. The main errors introduced by scales are inaccuracy of scale pitch, the misalignment and adjustment of the reading device and interpolation errors;
- *Errors Due to Mechanical Loads*: These are errors related to static or slowly varying forces on CMM components, in combination with the compliance of components, and are mainly caused by the weight of moving parts;
- *Thermally Induced Errors*: The difference between the temperature of the measuring scales of CMM and the work piece and temperature gradient in machine components are sources of error;
- *Dynamic Errors*: These are errors mainly caused by the deceleration of moving parts before stopping. These errors depend on the CMMs structural properties, like mass distribution, component stiffness and damping characteristics, as well as on the control system and disturbing forces.

This paper concentrates on the dynamic errors of CMMs caused by the deceleration of moving parts. Some studies on the error analysis of CMMs have been done. Weekers and Schellekenes [2] proposed a method for compensation of the dynamic errors of CMMs using inductive position sensors for online measurement of major dynamic errors. Barakat et al. [3] presented a kinematical and geometrical error compensation of CMMs, based on experiment. Nijs et al. [4] presented a very simple model of CMM for obtaining natural frequencies of a CMM. Several researchers have applied software compensation successfully on CMMs [5,6].

Most of these researchers have considered a very simple model for their analysis, while most of the studies are based on experiment. In the present study, a full CMM modeling is analyzed. All columns

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and guide ways are modeled as a Timoshenko beam with moving mass effects [7,8], and flexibility in all directions. All bearings are modeled as torsional springs and torsional deformation of the column and guide ways are considered, too. Dynamic equations of motion are derived using Hamilton's principle [9]. The derived equations are solved using the finite difference method. The results are compared with those obtained from the dynamic modeling software (ADAMS-FLEX). Determination of the natural frequencies of CMM at various positions of the probe and optimization of the system using a Genetic Algorithm are presented by Ramezani [10].

MODELING OF CMM STRUCTURE

Structural components of a gantry type CMM are shown in Figure 1. Because of the very small deformation of each component, the x-, y- and z-axes are assumed to remain in the same direction as in an undeformed state. The most important problem in this modeling is that the motions of the y-guide way, x-guide way and z-pinole are relative motions. They are the absolute deformation of each beam, but, they are not the absolute motion of each beam. So, the strain energy of each beam is only a function of its deformations, but, the kinetic energy of each component is influenced by the motion of other components. The system is assumed to be fixed at each position of the probe. The modeling of each component is presented in the following sections.

MODELING OF THE RIGHT COLUMN

For the right column, bending, torsional and longitudinal vibration is considered. The kinetic and strain energy of the column can be written as:

$$T_c = \frac{\rho_1}{2}\int_0^h [A_1\dot{A}^2(\xi,t) + I_{1z}\dot{\alpha}^2(\xi,t) + A_1\dot{B}^2(\xi,t) + I_{1y}\dot{\beta}^2(\xi,t) + I_{1z}\dot{\theta}_1^2(\xi,t) + A_1\dot{C}^2(\xi,t)]d\xi, \tag{1}$$

$$U_c = \frac{1}{2}\int_0^h \Big\{ E_1 I_{1x}{\alpha'}^2(\xi,t) + E_1 I_{1y}{\beta'}^2(\xi,t) + kG_1A_1\left(\left[\alpha(\xi,t) \quad \frac{\partial A(\xi,t)}{\partial \xi}\right]^2 + \left[\beta(\xi,t) \quad \frac{\partial B(\xi,t)}{\partial \xi}\right]^2\right) + G_1 I_{1z}{\theta'}_1^2(\xi,t) + E_1A_1{C'}^2(\xi,t)\Big\}d\xi, \tag{2}$$

where:

$A(\xi,t)$: bending of column in xz-plane,
$\alpha(\xi,t)$: rotation of column around y-axis,
$B(\xi,t)$: bending of column in yz-plane,
$\beta(\xi,t)$: rotation of column around x-axis,
$\theta_1(\xi,t)$: torsion of column around z-axis,
$C(\xi,t)$: longitudinal vibration in z-direction,
A_1, I_{1x}, I_{1y}, I_{1z}, E_1, G_1, ρ_1, h and k: cross sectional area, actual and equivalent moments of inertia, Young's modulus, shear modulus, density, length of column and shear factor, respectively.

Note that the symbols ($\dot{}$) and ($'$) denote derivation, with respect to time and coordinates, respectively.

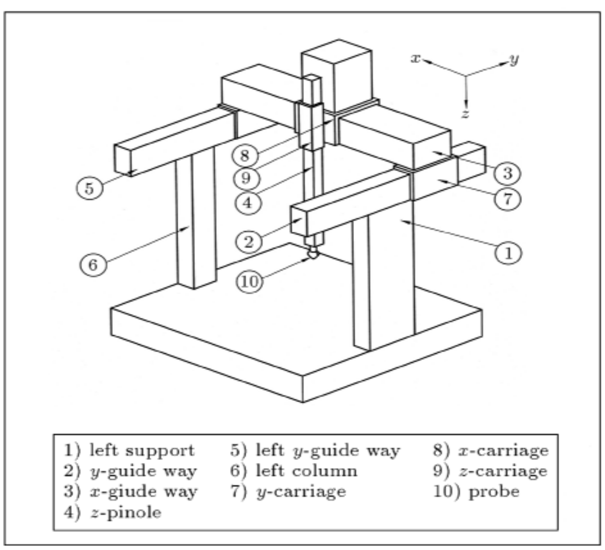

Figure 1. Schematic view of a gantry type CMM.

MODELING OF THE RIGHT Y-GUIDE WAY

For the right y-guide way, kinetic and strain energy have been considered to be as follows:

$$T_y = \frac{\rho_2}{2}\int_0^{L_y} \Big\{ A_2([\dot{E}(y,t) + y\dot{\alpha}(h,t) \quad \dot{C}(h,t)]^2 + \dot{A}^2(h,t) + [\dot{D}(y,t) + \dot{B}(h,t) + y\dot{\theta}_1(h,t)]^2) + I_{2z}\dot{\gamma}_1^2(y,t) + I_{2y}[\dot{\theta}_3(y,t) + \dot{\beta}(h,t)]^2 + I_{2x}\dot{\gamma}_2^2(y,t)\}dy, \tag{3}$$

$$U_y = \frac{1}{2}\int_0^{L_y}\Big\{E_2 I_{2z}{\gamma'}_1^2(y,t) + E_2 I_{2x}{\gamma'}_2^2(y,t)$$
$$+ kG_2A_2\left(\left[\gamma_2(y,t) \quad \frac{\partial D(y,t)}{\partial y}\right]^2\right.$$
$$\left. + \left[\gamma_1(y,t) \quad \frac{\partial E(y,t)}{\partial y}\right]^2\right)$$
$$+ G_2 I_{2y}{\theta'}_3^2(y,t)\Big\}dy, \tag{4}$$

where:

y_s: location of y-carriage on the y-guide way,
$D(y,t)$: bending in xy-plane,
$\gamma_1(y,t)$: rotation around z-axis,
$E(y,t)$: bending in yz-plane,
$\gamma_2(y,t)$: rotation around x-axis,
$\theta_3(y,t)$: torsion around y-axis,
A_2, I_{2x}, I_{2z}, I_{2y}, E_2, G_2, ρ_2 and L_y are cross sectional area, actual and equivalent moments of inertia, Young's modulus, shear modulus and density and length of y-guide way, respectively.

MODELING OF THE Y-CARRIAGE AND BEARING

For the drive system effect, only the rotation of the y-carriage bearing around the y-axis is considered. Note that this part acts as a moving mass on the y-guide way. The kinetic and strain energy in this member can be written as:

$$T_{bc3} = \frac{1}{2}M_{bc3}\Big\{[\dot{A}(h,t)+v_y]^2 + \Big[y_s\dot{\theta}_1(h,t) + \dot{B}(h,t)$$
$$+ \left(\frac{\partial D(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial D(y,t)}{\partial y}\Big|_{y_s}\right)\Big]^2 + \Big[y_s\dot{\alpha}(h,t)$$
$$\dot{C}(h,t) + \left(\frac{\partial E(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial E(y,t)}{\partial y}\Big|_{y_s}\right)\Big]^2\Big\}$$
$$+ \frac{1}{2}J_{bc3y}\left[\frac{\partial\theta_3(y,t)}{\partial t}\Big|_{y_s} + \dot{\beta}(h,t)\right]^2 + \frac{1}{2}J_{bc3z}\Big[\dot{\theta}_2(t)$$
$$+ \left(\frac{\partial\gamma_1(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial\gamma_1(y,t)}{\partial y}\Big|_{y_s}\right)\Big]^2, \tag{5}$$

$$U_{bc3} = \frac{1}{2}K_{bc3}\theta_2(t)^2, \tag{6}$$

where:

$\theta_2(t)$: torsion around y-axis,
K_{bc3}, M_{bc3}, J_{bc3y} and J_{bc3z}: torsional stiffness of bearing and drive system, mass and mass moments of inertia of y-carriage and its bearing, respectively.

MODELING OF THE X-GUIDE WAY

For the x-guide way, bending and torsion is considered. The kinetic and strain energy can be written as:

$$T_x = \frac{1}{2}\rho_3\Big\{\int_0^{L_x} A_3\Big\{\Big[y_s\dot{\theta}_1(h,t) + \dot{B}(h,t)$$
$$+ \frac{\partial D(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial D(y,t)}{\partial y}\Big|_{y_s}\Big]^2 + \Big[\dot{V}(x,t)$$
$$x\left[\dot{\theta}_2(t) + \frac{\partial\gamma_1(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial\gamma_1(y,t)}{\partial y}\Big|_{y_s}\right]$$
$$+ \dot{A}(h,t) + v_y\Big]^2 + \Big[\frac{\partial E(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial E(y,t)}{\partial y}\Big|_{y_s}$$
$$+ \dot{W}(x,t) + y_s\dot{\alpha}(h,t) + x\Big[\frac{\partial\theta_3(y,t)}{\partial t}\Big|_{y_s}$$
$$+ \dot{\beta}(h,t)\Big] \quad \dot{C}(h,t)\Big]^2\Big\}dx + I_{3y}\int_0^{L_x}\dot{\psi}^2(x,t)dx$$
$$+ I_{3z}\int_0^{L_x}\dot{\varphi}^2(x,t)dx + I_{3x}\int_0^{L_x}\Big[\dot{\eta}(x,t) + \frac{\partial\gamma_2(y,t)}{\partial t}\Big|_{y_s}$$
$$+ v_y\frac{\partial\gamma_2(y,t)}{\partial y}\Big|_{y_s}\Big]^2 dx\Big\}, \tag{7}$$

$$U_x = \frac{1}{2}\int_0^{L_x}\Big\{E_3[I_{3y}{\psi'}^2(x,t) + I_{3z}{\varphi'}^2(x,t)]$$
$$+ G_3I_{3x}{\eta'}^2(x,t) + kG_3A_3\left(\left[\psi(x,t) \quad \frac{\partial W(x,t)}{\partial x}\right]^2\right.$$
$$\left. + \left[\varphi(x,t) \quad \frac{\partial V(x,t)}{\partial x}\right]^2\right)\Big\}dx, \tag{8}$$

where:

x_s: location of y-carriage on the y-guide way,
$W(x,t)$: bending in xz-plane,
$W_1(x,t)$: bending in xz-plane,
$\psi(x,t)$: rotation around y-axis,
$V(y,t)$: bending in xy-plane,
$\varphi(x,t)$: rotation around z-axis,
$\eta(x,t)$: torsion around x-axis,
A_3, I_{3x}, I_{3y}, I_{3z}, E_3, G_3, ρ_3 and L_x are area, moments of inertia, Young's modulus, shear modulus, density and length of x-guide way, respectively.

MODELING OF THE BEARINGS AT *Z*-PINOLE

The kinetic and strain energy resulting from the motion of the z-pinole and torsion of bearings in the z-pinole assembly is, as follows:

$$\begin{aligned}
T_{bcz} = \frac{1}{2}M_{bcz}\Bigg\{\Bigg[& y_s\dot{\theta}_1(h,t) + \dot{B}(h,t) + v_x \\
&+ \frac{\partial D(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial D(y,t)}{\partial y}\Big|_{y_s}\Bigg]^2 \\
&+ \Bigg[y_s\dot{\alpha}(h,t) \quad \dot{C}(h,t) + \frac{\partial E(y,t)}{\partial t}\Big|_{y_s} \\
&+ v_y\frac{\partial E(y,t)}{\partial y}\Big|_{y_s} + \frac{\partial W(x,t)}{\partial t}\Big|_{x_s} + v_x\frac{\partial W(x,t)}{\partial x}\Big|_{x_s} \\
&+ x_s\Bigg[\dot{\beta}(h,t) + \frac{\partial \theta_3(y,t)}{\partial t}\Big|_{y_s}\Bigg]^2 + \Bigg[\dot{A}(h,t) + v_y \\
&\quad x_s\Bigg[\dot{\theta}_2(t) + \frac{\partial \gamma_1(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial \gamma_1(y,t)}{\partial y}\Big|_{y_s}\Bigg] \\
&+ v_x\frac{\partial V(x,t)}{\partial x}\Big|_{x_s} + \frac{\partial V(x,t)}{\partial t}\Big|_{x_s}\Bigg]^2\Bigg\} \\
&+ \frac{1}{2}\Bigg\{J_{bc4x}\Bigg[\dot{\theta}_4(t) + \frac{\partial \eta(x,t)}{\partial t}\Big|_{x_s} + \frac{\partial \gamma_2(y,t)}{\partial t}\Big|_{y_s} \\
&+ v_y\frac{\partial \gamma_2(y,t)}{\partial y}\Big|_{y_s}\Bigg]^2 + J_{bc6y}\Bigg[\dot{\theta}_6^2(t) + \frac{\partial \psi(x,t)}{\partial t}\Big|_{x_s} \\
&+ v_x\frac{\partial \psi(x,t)}{\partial x}\Big|_{x_s}\Bigg]^2 + J_{bc5x}\Bigg[\dot{\theta}_4(t) + \dot{\theta}_5(t) \\
&+ \frac{\partial \eta(x,t)}{\partial t}\Big|_{x_s} + \frac{\partial \gamma_2(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial \gamma_2(y,t)}{\partial y}\Big|_{y_s}\Bigg]^2 \\
&+ J_{4y}\Bigg[\frac{\partial \psi(x,t)}{\partial t}\Big|_{x_s} + v_x\frac{\partial \psi(x,t)}{\partial x}\Big|_{x_s}\Bigg]^2 \\
&+ J_{4z}\Bigg[\frac{\partial \varphi(x,t)}{\partial t}\Big|_{x_s} + v_x\frac{\partial \varphi(x,t)}{\partial x}\Big|_{x_s}\Bigg]^2\Bigg\},
\end{aligned} \tag{9}$$

$$U_{bcz} = \frac{1}{2}[K_{bc4}\theta_4(t)^2 + K_{bc5}\theta_5(t)^2 + K_{bc6}\theta_6(t)^2], \tag{10}$$

where:

$\theta_4(t)$: torsion of x-carriage bearing around x-axis,
$\theta_5(t)$: torsion of z-axis bearing around x-axis,
$\theta_6(t)$: torsion of z-axis bearing around y-axis.

The parameters K_{bc4}, K_{bc5}, K_{bc6}, M_{bcz}, J_{bc4x}, J_{bc5x}, J_{bc6y}, J_{4y} and J_{4z} are torsional stiffness of bearings and drive system, mass and mass moments of inertia of z-pinole carriage about x-axis, z-bearing and housing about x-axis, z-bearing about y-axis, z-bearing and z-axis assembly about y- and z-axis, respectively.

MODELING OF THE *Z*-PINOLE

Here, the bending of z-pinole in the xz- and yz-planes is considered. Kinetic and strain energy can be written, as follows:

$$\begin{aligned}
T_z = \frac{1}{2}\rho_4\Bigg\{\int_0^{L_z} A_4\Bigg\{\Bigg[& \dot{S}(z,t) + y_s\dot{\theta}_1(h,t) + \dot{B}(h,t) \\
&+ v_x + (z+z_0)\dot{\theta}_6(t) + \frac{\partial D(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial D(y,t)}{\partial y}\Big|_{y_s}\Bigg]^2 \\
&+ \Bigg[\dot{R}(z,t) + \dot{A}(h,t) + v_y \quad x_s\Bigg[\dot{\theta}_2(t) + \frac{\partial \gamma_1(y,t)}{\partial t}\Big|_{y_s} \\
&+ v_y\frac{\partial \gamma_1(y,t)}{\partial y}\Big|_{y_s}\Bigg] + (z+z_0)\Bigg[\dot{\theta}_4(t) + \dot{\theta}_5(t) \\
&+ \frac{\partial \eta(x,t)}{\partial t}\Big|_{x_s} + \frac{\partial \gamma_2(y,t)}{\partial t}\Big|_{y_s} + v_y\frac{\partial \gamma_2(y,t)}{\partial y}\Big|_{y_s}\Bigg] \\
&+ \frac{\partial V(x,t)}{\partial t}\Big|_{x_s} + v_x\frac{\partial V(x,t)}{\partial x}\Big|_{x_s}\Bigg]^2 + \Bigg[y_s\dot{\alpha}(h,t)
\end{aligned}$$

$$+ \frac{\partial E(y,t)}{\partial t}\Big|_{y_s} + v_y \frac{\partial E(y,t)}{\partial y}\Big|_{y_s} + x_s \left[\frac{\partial \theta_3(y,t)}{\partial t}\Big|_{y_s} \right.$$

$$\left. + \dot{\beta}(h,t) \right] \quad \dot{C}(h,t) + \frac{\partial W(x,t)}{\partial t}\Big|_{x_s} + v_x \frac{\partial W(x,t)}{\partial x}\Big|_{x_s}$$

$$+ l \left[\dot{\theta}_4(t) + \dot{\theta}_5(t) + \frac{\partial \eta(x,t)}{\partial t}\Big|_{x_s} + \frac{\partial \gamma_2(y,t)}{\partial t}\Big|_{y_s} \right.$$

$$\left. \left. \left. + v_y \frac{\partial \gamma_2(y,t)}{\partial y}\Big|_{y_s} \right] \right]^2 \right\} dz + I_{4x} \int_0^{L_z} \dot{\eta}_1^2(z,t) dz$$

$$\left. + I_{4y} \int_0^{L_z} \dot{\eta}_2^2(z,t) dz \right\}, \qquad (11)$$

$$U_x = \frac{1}{2} \int_0^{L_z} \left\{ E_4 I_{4x} {\eta'}_1^2(z,t) + E_4 I_{4x} {\eta'}_2^2(z,t) \right.$$

$$+ kG_4 A_4 \left[\eta_1(z,t) \quad \frac{\partial R(z,t)}{\partial z} \right]^2$$

$$\left. + \left[\eta_2(z,t) \quad \frac{\partial S(z,t)}{\partial z} \right]^2 \right\} dz, \qquad (12)$$

where:

$R(z,t)$: bending in yz-plane,
$\eta_1(z,t)$: rotation around x-axis,
$S(z,t)$: bending in xz-plane,
$\eta_2(z,t)$: rotation around y-axis,
A_4, I_{4x}, I_{4y}, I_{4z}, E_4, G_4, ρ_4 and L_z: cross sectional area, moments of inertia, Young's modulus, shear modulus, density and length of z-axis, respectively.

MODELING OF THE LEFT COLUMN AND Y-GUIDE WAY

For the left-side support column and y-guide way, the same motions $C(\xi,t)$, $E(y,t)$ and $\gamma_2(y,t)$ have been considered in a similar fashion to the right side column and y-guide way. The kinetic and strain energy can be written, as follows:

$$T_L = \frac{1}{2} \left\{ \rho_5 A_5 \int_0^h \dot{C}^2(\xi,t) d\xi \right.$$

$$\left. + \rho_6 A_6 \int_0^{L_y} \left[\dot{E}(y,t) \quad \dot{C}(h,t) \right]^2 dy \right\}, \qquad (13)$$

$$U_L = \frac{1}{2} \left\{ E_5 A_5 \int_0^h {C'}^2(\xi,t) d\xi + \int_0^{L_y} \left\{ E_6 I_{6x} {\gamma'}_2^2(y,t) \right. \right.$$

$$\left. \left. + kG_6 A_6 \left[\left(\gamma_2(y,t) \quad \frac{\partial D(y,t)}{\partial y} \right]^2 \right\} dy \right\}, \qquad (14)$$

where A_5, A_6, I_{6x}, E_5, E_6, G_5, G_6, ρ_5 and ρ_6 are cross sectional area, moments of inertia, Young's modulus, shear modulus, density of left column and left y-guide way, respectively.

Note that because of the effect of the drive system, some motions, such as rotation of the x-carriage around the z-axis, longitudinal vibration of the z-axis and horizontal and vertical motion of the y-carriage, have been neglected. Furthermore, the gravitational strain energy is neglected.

EQUATIONS OF MOTION

Using the Hamilton's principle equations of motion can be found, as follows:

$$\int_{t_1}^{t_2} (\delta T \quad \delta U + \delta W) dt = 0, \qquad (15)$$

where δ denotes the variation operator and:

$$T = T_c + T_x + T_y + T_z + T_{bcz} + T_L, \qquad (16)$$

$$U = U_c + U_x + U_y + U_z + U_{bcz} + U_L. \qquad (17)$$

Now, defining:

$$M_1 = \rho_1 A_1 h, \quad M_2 = \rho_2 A_2 L_y, \quad M_3 = \rho_3 A_3 L_x,$$

$$M_4 = \rho_4 A_4 L_z, \quad M = M_3 + M_4 + M_{bcz} + M_{bc3}, \qquad (18)$$

and neglecting higher order terms, virtual work done by the drive system and gravitational effects can be written as:

$$\delta W = M \left[a_y \delta y_s \quad \delta C(h,t) g + x_s \delta \beta(h,t) g \right.$$

$$\left. + x_s \theta_3(y,t) \overline{\delta}(y_s) g + E(y,t) \overline{\delta}(y_s) g \right]$$

$$+ [M_{bcz} + M_4][a_x \delta x_s + W(x,t) \overline{\delta}(x_s) g]. \qquad (19)$$

Terms resulting from the moving mass effect are in the following form:

$$\overline{N}(\mu,t) = \frac{\partial^2 N(\mu,t)}{\partial t^2} + 2 v_\mu \frac{\partial^2 N(\mu,t)}{\partial \mu \partial t}$$

$$+ v_\mu^2 \frac{\partial^2 N(\mu,t)}{\partial \mu^2} + a_\mu \frac{\partial N(\mu,t)}{\partial \mu}, \qquad (20)$$

where the variable N can be E, D, γ_1, γ_2, V, W, φ and ψ and the parameter μ can be y or x, respectively. Besides, other terms, resulting from the moving mass effect, will appear in the equations of the y- and x-guide ways.

Now, the equations of each member can be written, as follows.

EQUATIONS OF THE RIGHT COLUMN

$$\rho_1 \ddot{A}(\xi,t) + kG_1\left[\frac{\partial\alpha(\xi,t)}{\partial\xi} \quad \frac{\partial^2 A(\xi,t)}{\partial\xi^2}\right] = 0, \tag{21}$$

$$\rho_1 I_{1x}\ddot{\alpha}(\xi,t) \quad E_1 I_{1x}\alpha''(\xi,t) + kG_1A_1\left[\alpha(\xi,t) \quad \frac{\partial A(\xi,t)}{\partial\xi}\right] = 0, \tag{22}$$

$$\rho_1 \ddot{B}(\xi,t) + kG_1\left[\frac{\partial\beta(\xi,t)}{\partial\xi} \quad \frac{\partial^2 B(\xi,t)}{\partial\xi^2}\right] = 0, \tag{23}$$

$$\rho_1 I_{1y}\ddot{\beta}(\xi,t) \quad E_1 I_{1y}\beta''(\xi,t) + kG_1A_1\left[\beta(\xi,t) \quad \frac{\partial B(\xi,t)}{\partial\xi}\right] = 0, \tag{24}$$

$$[\rho_1A_1+\rho_5A_5]\ddot{C}(\xi,t)+[E_1A_1+E_5A_5]C''(\xi,t)=0, \tag{25}$$

$$\rho_1 I_{1z}\ddot{\theta}_1(\xi,t) + G_1I_{1z}\theta_1''(\xi,t) = 0. \tag{26}$$

EQUATIONS OF RIGHT Y-GUIDE WAY

$$M_2\Big[\ddot{D}(y,t) + \ddot{B}(h,t) + y\ddot{\theta}_1(h,t)] + L_y k(G_2A_2 + G_6A_6)\left[\frac{\partial\gamma_1(y,t)}{\partial y} \quad \frac{\partial^2 D(y,t)}{\partial y^2}\right] + \left\{M\Big[\overline{D}(y,t) + 2v_y\dot{\theta}_1(h,t) + y_s\ddot{\theta}_1(h,t)+\ddot{B}(h,t)\Big] + \Big[M_{bcz}+M_4\Big]a_x + M_4\frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\ddot{\theta}_6(t)\right\}\overline{\delta}(y_s) = 0, \tag{27}$$

$$\rho_2 I_{2z}\ddot{\gamma}_1(y,t) \quad (E_2I_{2z}+E_6I_{6z})\gamma_1''(y,t) + k(G_2A_2+G_6A_6)\left[\gamma_1(y,t) \quad \frac{\partial D(y,t)}{\partial y}\right] + \left\{\Big[\ddot{\theta}_2(t)+\overline{\gamma}_1(y,t)\Big]\Big[J_{bc3z}+x_s^2(M_{bcz}+M_4) + \frac{M_3L_x^2}{3}\Big] \quad \Big[a_y+\ddot{A}(h,t)\Big]\Big[\frac{M_3L_x}{2}+x_s(M_{bcz}+M_4)\Big] + \Big[M_{bcz}+M_4\Big]\Big[x_sv_x(v_y+\dot{A}(h,t)) + \frac{\partial V^2(x,t)}{\partial t^2} + v_x\frac{\partial^2 V(x,t)}{\partial x\partial t} + a_x\frac{\partial V(x,t)}{\partial x}\Big] + M_4\Big[\ddot{\theta}_4(t) + \ddot{\theta}_5(t) + \ddot{\gamma}_2(y_s,t) + \ddot{\eta}(x_s,t) + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_2(x,t)}{\partial x}\Big]\frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\right\}\overline{\delta}(y_s)=0, \tag{28}$$

$$\rho_2A_2[\ddot{E}(y,t) \quad \ddot{C}(h,t)+y\ddot{\alpha}(h,t)] + \rho_6A_6[\ddot{E}(y,t) \quad \ddot{C}(h,t)] + k(G_2A_2+G_6A_6)\left[\frac{\partial\gamma_2(y,t)}{\partial y} \quad \frac{\partial^2 E(y,t)}{\partial y^2}\right] + \left\{M[\overline{E}(y,t)+y_s\ddot{\alpha}(h,t) + 2v_y\dot{\alpha}(h,t)] + [M_3\frac{L_x}{2}+M_4x_s][\ddot{\theta}_3(y_s,t) + \dot{\beta}(h,t)] + [M_{bcz}+M_4]\left[\frac{\partial W^2(x,t)}{\partial t^2} + v_x\frac{\partial^2 W(x,t)}{\partial x\partial t} + a_x\frac{\partial W(x,t)}{\partial x}\right]\Bigg|_{x=x_s} + M_4l\Big[\ddot{\theta}_4(t) + \ddot{\theta}_5(t) + \ddot{\gamma}_2(y_s,t) + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_2(x,t)}{\partial x} + \ddot{\eta}(x_s,t)\Big]\right\}\overline{\delta}(y_s) = ME(y,t)\overline{\delta}(y_s), \tag{29}$$

$$\rho_2A_2[\ddot{E}(y,t) \quad \ddot{C}(h,t)+y\ddot{\alpha}(h,t)] + \rho_6A_6[\ddot{E}(y,t) \quad \ddot{C}(h,t)] + k(G_2A_2+G_6A_6)\left[\frac{\partial\gamma_2(y,t)}{\partial y}\right.$$

$$\left.\frac{\partial^2 E(y,t)}{\partial y^2}\right] + \left\{ M[\overline{E}(y,t) + y_s\ddot{\alpha}(h,t)\right.$$

$$+ 2v_y\dot{\alpha}(h,t)] + \left[M_3\frac{L_x}{2} + M_4 x_s\right][\ddot{\theta}_3(y_s,t) + \dot{\beta}(h,t)]$$

$$+ [M_{bcz} + M_4]\left[\frac{\partial^2 W(x,t)}{\partial t^2} + v_x\frac{\partial^2 W(x,t)}{\partial x\partial t}\right.$$

$$\left.\left. + a_x\frac{\partial W(x,t)}{\partial x}\right]\right|_{x=x_s} + M_4 l[\ddot{\theta}_4(t) + \ddot{\theta}_5(t) + \ddot{\gamma}_2(y_s,t)$$

$$\left. + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_2(x,t)}{\partial x} + \ddot{\eta}(x_s,t)]\right\}\overline{\delta}(y_s)$$

$$= ME(y,t)\overline{\delta}(y_s), \tag{30}$$

$$(\rho_2 I_{2x} + \rho_6 I_{6x})\ddot{\gamma}_2(y,t) \quad (E_2 I_{2x} + E_6 I_{6x})\gamma_2''(y,t)$$

$$+ k(G_2A_2 + G_6A_6)\left[\gamma_2(y,t) \quad \frac{\partial E(y,t)}{\partial y}\right]\left\{\overline{\gamma}_2(y,t)\right.$$

$$+ [\rho_3 I_{3x} + J_{bc4x} + J_{bc5x} + 2M_4] + \left[\ddot{\theta}_4(t)\right.$$

$$\left.\left. + \ddot{\eta}(x_s,t)\right|_{x=x_s}\right]\left[J_{bc4x} + J_{bc5x}\right.$$

$$\left. + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right] + \ddot{\theta}_5(t)\left[J_{bc5x}\right.$$

$$\left. + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right]$$

$$+ M_4\frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\left\{\ddot{A}(h,t) + a_y \quad v_x\left[\dot{\theta}_2(t)\right.\right.$$

$$\left. + \frac{\partial\gamma_1(y,t)}{\partial t} + v_y\frac{\partial\gamma_1(y,t)}{\partial y}\right] \quad x_s[\ddot{\theta}_2(t) + \overline{\gamma}_1(y,t)]$$

$$\left. + \left[\ddot{V}(x,t) + a_x\frac{\partial V(x,t)}{\partial x} + v_x\frac{\partial^2 V(x,t)}{\partial x\partial t}\right]\overline{\delta}(x_s)\right\}$$

$$+ M_4 l\left\{2v_y\dot{\alpha}(h,t) + y_s\ddot{\alpha}(h,t) + \overline{E}(y,t)\right.$$

$$+ v_x[\dot{\theta}_3(y,t) + \dot{\beta}(h,t)] + x_s\left[\ddot{\theta}_3(y,t) + \ddot{\beta}(h,t)]\right.$$

$$\ddot{C}(h,t) + \left[\ddot{W}(x,t) + a_x\frac{\partial W(x,t)}{\partial x}\right.$$

$$\left.\left.\left. + v_x\frac{\partial^2 W(x,t)}{\partial x\partial t}\right]\overline{\delta}(x_s)\right\}\right\}\overline{\delta}(y_s) = 0, \tag{31}$$

$$M_2[\ddot{\theta}_3(y,t) + \ddot{\beta}(h,t)] + L_y G_2 I_{2y}\theta_3''(y,t) + \left\{[\ddot{\theta}_3(y,t)\right.$$

$$+ \ddot{\beta}(h,t)]\left[J_{bc3y} + M_3\frac{L_x^2}{3} + (M_{bcz} + M_4)x_s^2\right]$$

$$+ \left[\ddot{E}(y,t) + a_y\frac{\partial E(y,t)}{\partial y} + v_y\frac{\partial^2 E(y,t)}{\partial y\partial t} + v_y\dot{\alpha}(h,t)\right.$$

$$\left. + y_s\alpha(h,t)\right]\left[M_3\frac{L_x}{2} + (M_{bcz} + M_4)x_s\right]$$

$$+ x_s(M_{bcz} + M_4)\left[\left.\ddot{W}(x,t)\right|_{x=x_s} + \left. a_x\frac{\partial W(x,t)}{\partial x}\right|_{x=x_s}\right.$$

$$\left.\left. + v_x\frac{\partial^2 W(x,t)}{\partial x\partial t}\right|_{x=x_s}\right] \quad \ddot{C}(h,t) + x_s M_4 l\left[\ddot{\theta}_4(t)\right.$$

$$+ \ddot{\theta}_5(t) + \left.\ddot{\eta}(x_s,t)\right|_{x=x_s} + \ddot{\gamma}_2(y,t) + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t}$$

$$\left.\left. + a_y\frac{\partial\gamma_2(x,t)}{\partial x}\right]\right\}\overline{\delta}(y_s) = 0. \tag{32}$$

In the above equations, the term $\overline{\delta}$ is the Kronecker delta function and is defined in the following form:

$$\overline{\delta}(y_s) = \begin{cases} 1 & \text{if} \quad y = y_s \\ 0 & \text{if} \quad y \neq y_s \end{cases} \tag{33}$$

The coefficients of $\overline{\delta}(y_s)$ are terms resulting from moving mass effects.

EQUATION CORRESPONDING TO ROTATION θ_2

$$\left[J_{bc3z} + M_3\frac{L_x^2}{3} + M_{bcz}x_s^2 + M_4x_s^2\right]\left[\ddot{\theta}_2(t)\right.$$

$$+ \left(\ddot{\gamma}_1(y,t) + v_y\frac{\partial^2\gamma_1(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_1(x,t)}{\partial x}\right)\overline{\delta}(y_s)$$

$$\begin{aligned}
&K_{bc3}\theta_2(t) + \Bigg[\ddot\theta_4(t) + \ddot\theta_5(t) + \ddot\eta(x_s,t) + \Bigg(\ddot\gamma_2(y,t) \\
&+ v_y \frac{\partial^2\gamma_2(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_2(x,t)}{\partial x}\Bigg)\Bigg|_{y_s}\Bigg] \\
&+ M_4 x_s \frac{(L_z+z_0)^2 \quad z_0^2}{2L_z} + \Bigg[M_3\frac{L_z}{2} + M_{bcz}x_s \\
&+ M_4 x_s\Bigg]\ddot A(h,t) + x_s(M_{bcz}+M_4)\Bigg[\ddot V(x,t) \\
&+ a_x \frac{\partial V(x,t)}{\partial x} + v_x\frac{\partial^2 V(x,t)}{\partial x\partial t}]\bar\delta(x_s) + a_y\Bigg[M_3\frac{L_x}{2} \\
&+ M_{bcz}x_s + M_4\frac{x_s^2}{L_z}\Bigg] = 0.
\end{aligned} \tag{34}$$

EQUATIONS OF *X*-GUIDE WAY

$$\begin{aligned}
&M_3[\ddot V(x,t) + \ddot A(h,t) \quad x\ddot\theta_2(t) + a_y] \\
&\quad + kG_3A_3L_x\Bigg[\frac{\partial\varphi(x,t)}{\partial x} \quad \frac{\partial^2 V(x,t)}{\partial x^2}\Bigg] \\
&\quad + \Bigg\{(M_4 + M_{bcz})\Bigg[\ddot A(h,t) + a_y \quad v_x\Bigg[\dot\theta_2(t) \\
&\quad + \Bigg(\frac{\partial\gamma_1(y,t)}{\partial t} + v_y\frac{\partial\gamma_1(y,t)}{\partial y}\Bigg)\Bigg|_{y_s}\Bigg] \\
&\qquad x_s\Bigg[\Bigg(\ddot\gamma_1(y,t) + v_y\frac{\partial^2\gamma_1(y,t)}{\partial y\partial t} + a_y\frac{\partial\gamma_1(x,t)}{\partial x}\Bigg)\Bigg|_{y_s} \\
&\quad + \ddot\theta_2(t)\Bigg] + M_4\frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\Bigg[\ddot\theta_4(t) + \ddot\theta_5(t) \\
&\quad + \ddot\eta(x,t) + v_x\frac{\partial^2\eta(x,t)}{\partial x\partial t} + \Bigg(\ddot\gamma_2(y,t) + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t} \\
&\quad + a_y\frac{\partial\gamma_2(x,t)}{\partial x}\Bigg)\bar\delta(y_s)]\Bigg\}\bar\delta(x_s) = 0,
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\rho_3 I_{3z}\ddot\varphi(x,t) \quad E_3 I_{3z}\varphi''(x,t) + kG_3A_3\Bigg[\varphi(x,t) \\
&\quad \frac{\partial V(x,t)}{\partial x}\Bigg] + J_{4z}\overline{\varphi}(x,t)\bar\delta(x_s) = 0,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&M_3\Bigg[\ddot W(x,t) \quad \ddot C(h,t) + x\ddot\theta_3(y_s,t) + \ddot E(y_s,t) \\
&\quad + y_s\ddot\alpha(h,t) + a_y\frac{\partial E(y,t)}{\partial y}\Bigg|_{y_s} + v_y\frac{\partial^2 E(y,t)}{\partial y\partial t}\Bigg|_{y_s}\Bigg] \\
&\quad + L_x kG_3A_3\Bigg[\frac{\partial\psi(x,t)}{\partial x} \quad \frac{\partial^2 W(x,t)}{\partial x^2}\Bigg] \\
&\quad + \Bigg\{\Bigg[v_y\dot\theta_1(h,t) + y_s\ddot\theta_1(h,t) + \ddot B(h,t) + a_x \\
&\quad + \ddot D(y_s,t) + a_y\frac{\partial D(y,t)}{\partial y}\Bigg|_{y_s} + v_y\frac{\partial^2 D(y,t)}{\partial y\partial t}\Bigg|_{y_s} \\
&\quad + v_y\dot\alpha(h,t) + y_s\ddot\alpha(h,t) \quad \ddot C(h,t) + \ddot E(y_s,t) \\
&\quad + a_y\frac{\partial E(y,t)}{\partial y}\Bigg|_{y_s} + v_y\frac{\partial^2 E(y,t)}{\partial y\partial t}\Bigg|_{y_s} + \overline{W}(x,t) \\
&\quad + v_x(\dot\theta_3(y_s,t) + \dot\beta(h,t)) + \ddot\beta(h,t) \\
&\quad + x_s(\ddot\theta_3(y_s,t))(M_{bcz}+M_4)] + M_4 l\Bigg[\ddot\theta_4(t) \\
&\quad + \ddot\theta_5(t) + \ddot\eta(x_s,t) + \ddot\gamma_2(y_s,t) + a_y\frac{\partial\gamma_2(y,t)}{\partial y}\Bigg|_{y_s} \\
&\quad + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t}\Bigg|_{y_s}\Bigg\}\bar\delta(x_s) \\
&\quad = (M_{bcz}+M_4)W(x,t)\bar\delta(x_s)g,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\rho_3 I_{3y}\ddot\psi(x,t) \quad E_3 I_{3y}\psi''(x,t) + kG_3A_3\Bigg[\psi(x,t) \\
&\quad \frac{\partial W(x,t)}{\partial x}\Bigg] + (J_{bc6y} + J_{4y})\overline{\psi}(x,t)\bar\delta(x_s) = 0,
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\rho_3\ddot\eta(x,t) + G_3\eta''(x,t) + \Bigg\{\Bigg[\ddot\theta_4(t) + \ddot\eta(x_s,t) + \ddot\gamma_2(y_s,t) \\
&\quad + a_y\frac{\partial\gamma_2(y,t)}{\partial y}\Bigg|_{y_s} + v_y\frac{\partial^2\gamma_2(y,t)}{\partial y\partial t}\Bigg|_{y_s}\Bigg](J_{bc4x} + J_{bc5x}
\end{aligned}$$

$$+ l^2 M_4) + (J_{bc5x} + l^2 M_4)\ddot{\theta}_5(t)\Big\}\overline{\delta}(x_s) = 0. \quad (39)$$

Again the coefficients of $\overline{\delta}(x_s)$ are terms resulting from moving mass effects.

EQUATIONS CORRESPONDING TO THE ROTATIONS θ_4, θ_5 AND θ_6

$$\left[J_{bc4x} + J_{bc5x} + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right]\ddot{\theta}_4(t)$$

$$+K_{bc4}\theta_4 + \left[J_{bc5x} + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right]\ddot{\theta}_5(t)$$

$$M_4 x_s \frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\left[\ddot{\gamma}_1(y_s,t) + a_y \frac{\partial \gamma_1(y,t)}{\partial y}\bigg|_{y_s}\right.$$

$$\left. + v_y \frac{\partial^2 \gamma_1(y,t)}{\partial y \partial t}\bigg|_{y_s} + \ddot{\theta}_2(t)\right] + \left[\ddot{\eta}(x_s) + a_y \frac{\partial \gamma_2(y,t)}{\partial y}\bigg|_{y_s}\right.$$

$$\left.\left. + v_y \frac{\partial^2 \gamma_2(y,t)}{\partial y \partial t}\bigg|_{y_s} + \ddot{\gamma}_2(y_s,t)\right]\right]\left[J_{bc4x} + J_{bc5x}\right.$$

$$\left. + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right] + \left[\ddot{A}(h,t) + a_y\right.$$

$$+ \ddot{V}(x_s,t) + a_x \frac{\partial V(x,t)}{\partial x}\bigg|_{x_s}$$

$$\left. + v_x \frac{\partial \dot{V}(x,t)}{\partial x}\bigg|_{x_s}\right] M_4 x_s \frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}$$

$$+ M_4 l^2 \left[\ddot{W}(x_s,t) + a_y \frac{\partial W(x,t)}{\partial x}\bigg|_{x_s} + v_x \frac{\partial \dot{W}(x,t)}{\partial x}\bigg|_{x_s}\right]$$

$$+ M_4 l \left[\quad \ddot{C}(h,t) + a_z + y_s \ddot{\alpha}(h,t)\ddot{E}(y_s,t)\right.$$

$$\left. + a_y \frac{\partial E(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 E(y,t)}{\partial y \partial t}\bigg|_{y_s}\right]$$

$$+ M_4 l x_s \ddot{\theta}_3(y_s,t) = 0, \quad (40)$$

$$\left[J_{bc5x} + M_4\left(\frac{(L_z+z_0)^3 \quad z_0^3}{3L_z} + l^2\right)\right]\left[\ddot{\theta}_4(t) + \ddot{\theta}_5(t)\right]$$

$$+ K_{bc5}\theta_5(t) \quad x_s M_4 \frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\left[\ddot{\theta}_2(t)\right.$$

$$\left. + \ddot{\gamma}_1(y_s,t) + a_y \frac{\partial \gamma_1(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 \gamma_1(y,t)}{\partial y \partial t}\bigg|_{y_s}\right]$$

$$+ \left[J_{bc5x} + M_4\left(l^2 + \frac{(L_z+z_0)^3 \quad z_0^3}{3L_z}\right)\right]\left[\ddot{\eta}(x_s,t)\right.$$

$$\left. + \ddot{\gamma}_2(y_s,t) + a_y \frac{\partial \gamma_2(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 \gamma_2(y,t)}{\partial y \partial t}\bigg|_{y_s}\right]$$

$$+ M_4 \frac{(L_z+z_0)^3 \quad z_0^3}{3L_z}\left[\ddot{A}(h,t) + a_y + \ddot{V}(x_s,t)\right.$$

$$\left. + a_x \frac{\partial V(x,t)}{\partial x}\bigg|_{x_s} + v_x \frac{\partial^2 V(x,t)}{\partial x \partial t}\bigg|_{x_s}\right] + M_4 l^2 \left[\ddot{W}(x_s,t)\right.$$

$$+ a_y \frac{\partial W(x,t)}{\partial x}\bigg|_{x_s} + v_y \frac{\partial^2 W(x,t)}{\partial x \partial t}\bigg|_{x_s} \quad \ddot{C}(h,t)$$

$$+ x_s \ddot{\theta}_3(y_s,t) + y_s \ddot{\alpha}(h,t) + \ddot{E}(y_s,t)$$

$$\left. + a_y \frac{\partial E(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 E(y,t)}{\partial y \partial t}\bigg|_{y_s} + a_z\right] = 0, \quad (41)$$

$$\left[J_{bc6y} + M_4 \frac{(L_z+z_0)^3 \quad z_0^3}{3L_z}\right]\ddot{\theta}_6(t) + K_{bc6}\theta_6(t)$$

$$+ M_4 \frac{(L_z+z_0)^2 \quad z_0^2}{2L_z}\left[a_x + y_s \ddot{\theta}_1(h,t) + \ddot{B}(h,t)\right.$$

$$\left. + \ddot{D}(y_s,t) + a_y \frac{\partial D(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 D(y,t)}{\partial y \partial t}\bigg|_{y_s}\right] = 0. \quad (42)$$

EQUATIONS IN Z-PINOLE

$$\rho_4 A_4 \Big\{\ddot{R}(z,t) + \ddot{A}(h,t) \quad x_s \ddot{\theta}_2(t) + a_y + \left[\ddot{\eta}(x_s)\right.$$

$$+ \ddot{\theta}_4(t) + \ddot{\theta}_5(t) + a_y \frac{\partial \gamma_2(y,t)}{\partial y}\bigg|_{y_s} + v_y \frac{\partial^2 \gamma_2(y,t)}{\partial y \partial t}\bigg|_{y_s}$$

$$\left. + \ddot{\gamma}_2(y_s,t)\right](z+z_0) + a_x \frac{\partial V(x,t)}{\partial x}\bigg|_{x_s} + v_x \frac{\partial^2 V}{\partial x \partial t}\bigg|_{x_s}$$

$$+\ddot{V}(x_s,t)\Bigg\} + kG_4A_4\left[\frac{\partial\eta_1(z,t)}{\partial z} \quad \frac{\partial^2 R(z,t)}{\partial z^2}\right] = 0, \tag{43}$$

$$\rho_4 A_4\Bigg[\ddot{S}(z,t) + (z+z_0)\ddot{\theta}_6(t) + a_x + \ddot{B}(h,t)$$

$$+ y_s\ddot{\theta}_1(h,t) + a_y\left.\frac{\partial D(y,t)}{\partial y}\right|_{y_s} + v_y\left.\frac{\partial^2 D(y,t)}{\partial y\partial t}\right|_{y_s}$$

$$+\ddot{D}(y_s,t)\Bigg] + kG_4A_4\left[\frac{\partial\eta_2(z,t)}{\partial z} \quad \frac{\partial^2 S(z,t)}{\partial z^2}\right] = 0 \tag{44}$$

$$\rho_4 I_{4y}\ddot{\eta}_2(z,t) \quad E_4 I_{4y}\eta_2''(z,t)$$

$$+ kG_4A_4\left[\eta_2(z,t) \quad \frac{\partial S(z,t)}{\partial z}\right] = 0. \tag{45}$$

BOUNDARY CONDITIONS

Columns are assumed to be clamped at $\xi = 0$. At $\xi = h$, the boundary conditions are determined from Hamilton's principle. The y-guide way is clamped at $y = 0$ and free at $y = L_y$. The x-guide way is clamped-free in the xy-plane and simply supported in the xz-plane. The z-pinole is clamped free, too. Here, for brevity, representation of the equations of the boundary conditions are neglected, but, they can be easily obtained from Hamilton's principle. Furthermore, most of them are the well known "simple" boundary conditions. As an example, boundary conditions corresponding to the right y-guide way are, as follows:

$$\begin{cases} D(0,t) = \gamma_1(0,t) = \left.\frac{\partial\gamma_1(y,t)}{\partial y}\right|_{y=L_y} = 0 \\ \left.\frac{\partial D(y,t)}{\partial y}\right|_{y=L_y} \quad \gamma_1(L_y,t) = 0 \end{cases},$$

$$\begin{cases} E(0,t) = \gamma_2(0,t) = \left.\frac{\partial\gamma_2(y,t)}{\partial y}\right|_{y=L_y} = 0 \\ \left.\frac{\partial E(y,t)}{\partial y}\right|_{y=L_y} \quad \gamma_2(L_y,t) = 0 \end{cases},$$

$$\theta_3(0,t) = \left.\frac{\partial\theta_3(y,t)}{\partial y}\right|_{y=L_y} = 0. \tag{46}$$

METHOD OF SOLUTION

An explicit time integration method [11] is used for solving the equations of motion. This method of solution is conditionally stable and should be used with a small time step size. A central difference equation is used for position differentiation and forward difference is used for time differentiation. For every function, it can be written that:

$$\left.\frac{\partial^2 N}{\partial\mu^2}\right|_{m\Delta\mu}^{n\Delta t} = \frac{N\Big|_{(m+1)\Delta\mu}^{n\Delta t} \quad 2N\Big|_{m\Delta\mu}^{n\Delta t} + N\Big|_{(m\ \ 1)\Delta\mu}^{n\Delta t}}{(\Delta\mu)^2}, \tag{47}$$

$$\left.\frac{\partial N}{\partial t}\right|_{m\Delta\mu}^{n\Delta t} = \frac{N\Big|_{m\Delta\mu}^{(n+1)\Delta t} \quad N\Big|_{m\Delta\mu}^{n\Delta t}}{(\Delta t)^2}, \tag{48}$$

where $\Delta\mu$ is the step size in position, Δt is the time step, n is counter of time and m is the node number. The input of the system is the profile of acceleration of moving parts versus time. Defining the time derivative of each variable as a new variable, the second order derivatives, with respect to the time, will be presented into the first order derivative form and the descritized equations will be presented in two times $n\Delta t$ and $(n+1)\Delta t$. The final equation is in the following form:

$$[A]^{(n)}\{V\}^{(n+1)} = [B]^{(n)}\{V\}^{(n)} + \{C\}^{(n)}, \tag{49}$$

where the matrices $[A]^{(n)}$ and $[B]^{(n)}$ are known coefficient matrices at time $n\Delta t$, the vector $\{V\}^{(n+1)}$ is a vector containing all variables in all nodes, except at the first node, at time $(n+1)\Delta t$. The vector $\{C\}^{(n)}$, is a known vector resulting from the acceleration of the moving parts. Note that because of the motion of x- and y-guide ways, each matrix should be constructed at each time step. Solving for velocities, one may obtain the corresponding displacements.

In order to illustrate an example of modeling, a CMM with the following specifications is modeled.

$\rho_i = 7850$ kg/m^3,	$E_i = 200$ Gpa,	$G_i = 70$ Gpa,
$h = 0.8$ m,	$L_y = 0.6$ m,	$L_x = 1.2$ m,
$L_z = 0.75$ m,	$l = 0.16$ m,	
$z_0 = 0.125$ m,	$K_{\text{bearings}} = 5e7$ N.m/rad,	
$A_1 = 7.6e\ \ 3$ m^2,	$A_2 = 4.6e\ \ 3$ m^2,	
$A_3 = 7.6e\ \ 3$ m^2,	$A_4 = 1.0e\ \ 3$ m^2,	
$A_5 = 4.6e\ \ 3$ m^2,	$A_6 = 4.6e\ \ 3$m^2,	
$I_{1x} = 7.86e\ \ 5$ m^4,	$I_{1y} = 1.3e\ \ 5$ m^4,	
$I_{1z} = 3.58e\ \ 5$ m^4,	$I_{2x} = 1.35e\ \ 5$ m^4,	
$I_{2y} = 1.38e\ \ 6$ m^4,	$I_{2z} = 2.04e\ \ 5$ m^4,	
$I_{3x} = 5.94e\ \ 5$ m^4,	$I_{3y} = 2.82e\ \ 5$ m^4,	
$I_{3z} = 6.35e\ \ 5$ m^4,	$I_{4x} = 4.21e\ \ 7$ m^4,	
$I_{4y} = 4.21e\ \ 7$ m^4,	$I_{5x} = 1.35e\ \ 5$ m^4,	
$I_{5z} = 2.04e\ \ 5$ m^4,	$I_{6x} = 1.35e\ \ 5$ m^4,	
$I_{6y} = 2.04e\ \ 5$ m^4,	$J_{bc3z} = 0.18$ kgm^4,	
$J_{bc3z} = 0.3$ kgm^4,	$J_{bc4x} = 2.7$ kgm^4,	
$J_{bc5x} = 0.05$ kgm^4,	$J_{bc6y} = 0.05$ kgm^4,	
$J_{4y} = 3.4$ kgm^4,	$J_{4z} = 1.3$ kgm^4,	
$M_{bc3} = 10$ kg,	$M_{bcz} = 35$ kg.	

In Figure 2, the profile of accelerations a_x and a_y is shown for two motions. In motion 1, the accelerations and decelerations are applied rapidly (in the step function form) and, in the second motion, the accelerations are applied in the semi-sinusoidal form. The maximum value of the accelerations is assumed to be 1.5 m/s^2 and maximum velocities are 0.6 m/s. Note that, in both cases, the systems start from rest and the course of traveling is 120 cm and 60 cm in the x-and y-directions, respectively.

The values of error for step time acceleration and deceleration in the x-, y- and z-directions are shown in Figures 3 to 5. The absolute value of error for this type of excitation is shown in Figure 6. It can be seen that the main part of the error is produced in the y-

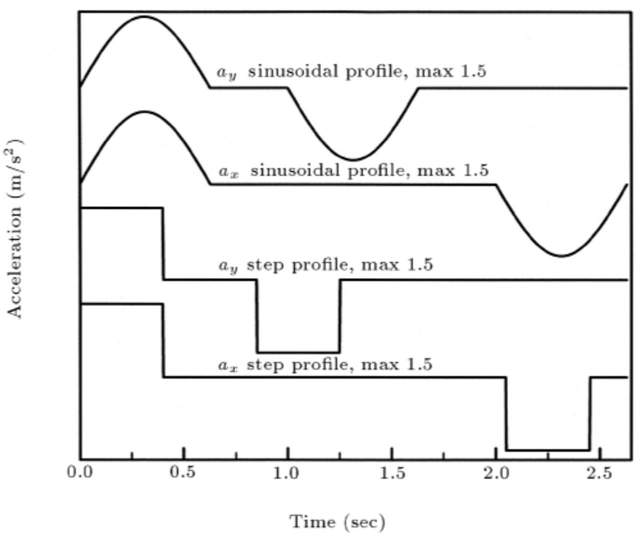

Figure 2. Acceleration and deceleration profiles for two motions (step and sinusoidal profiles).

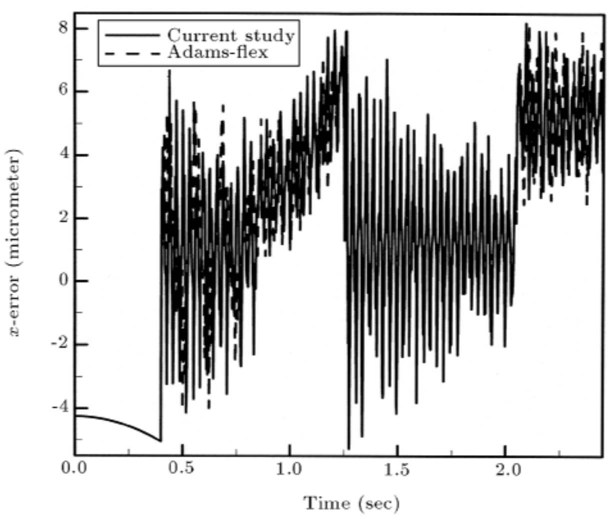

Figure 3. Dynamic error of probe in x-direction for step type acceleration and deceleration.

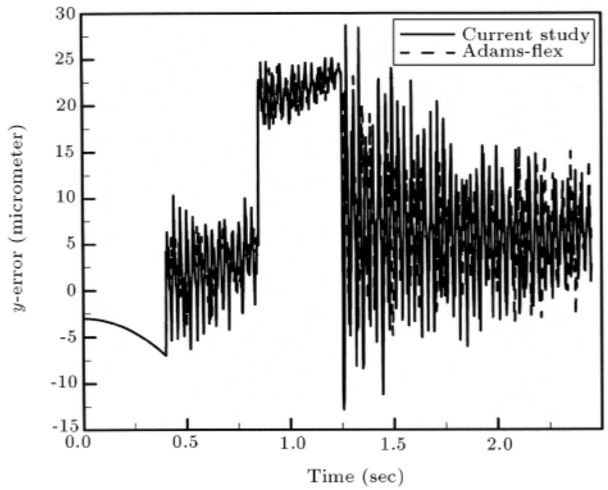

Figure 4. Dynamic error of probe in y-direction for step type acceleration and deceleration.

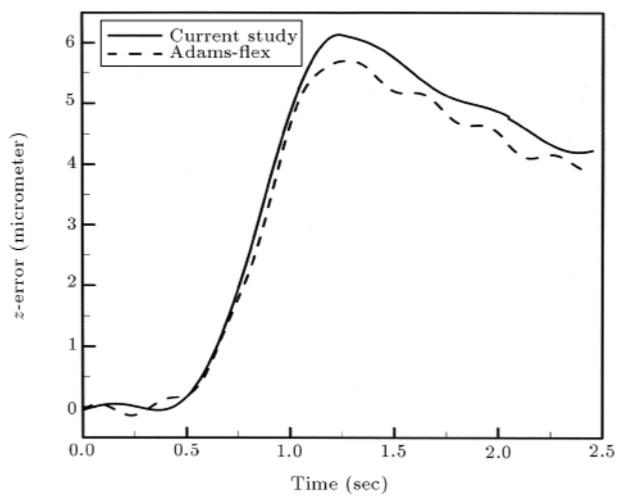

Figure 5. Dynamic error of probe in z-direction for step type acceleration and deceleration.

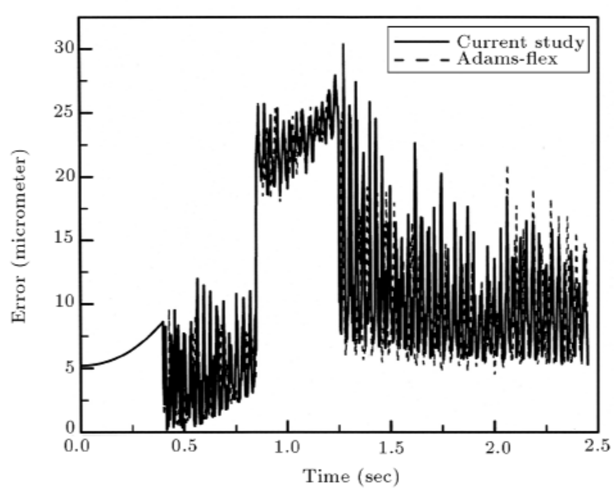

Figure 6. Magnitude of dynamic error of probe for step type acceleration and deceleration.

direction, since the total stiffness of the structure is small in this direction in comparison with the other two directions. Similarly, the values of error for the semi-sinusoidal acceleration and deceleration in the different directions are shown in Figures 7 to 9. The absolute value of error for this type of excitation is shown in Figure 10. Again, the main part of the error can be seen in the y-direction. The results indicate good agreement with those obtained from the dynamic modeling software ADAMS-FLEX. Furthermore, it is clear that, in order to decrease the oscillations of the probe, a smooth profile (such as a semi-sinusoidal one) of acceleration should be applied on the CMM.

CONCLUSION

The modeling of a gantry type CMM, based on the Timoshenko beam theory, subjected to moving mass was developed and the dynamic errors of the probe have been calculated. The equations of motion and boundary conditions have been obtained using Hamilton's principle.

Results indicate that, in order to decrease the oscillatory motion of the probe, a smooth function of the acceleration or deceleration function should be applied on the CMM. The maximum value of the dynamic error in the step type excitation is larger than the value obtained from semi-sinusoidal excitation. So, a smooth function of excitation enforces a smaller value of absolute error to the machine.

In both step and semi-sinusoidal types of excitation applied to the machine, the main part of the error is produced in the y-direction. The value of error in this direction is approximately three times the error induced in the x- and z-directions. This fact indicates that the total stiffness of the system in the y-direction is small, in comparison with the other two directions.

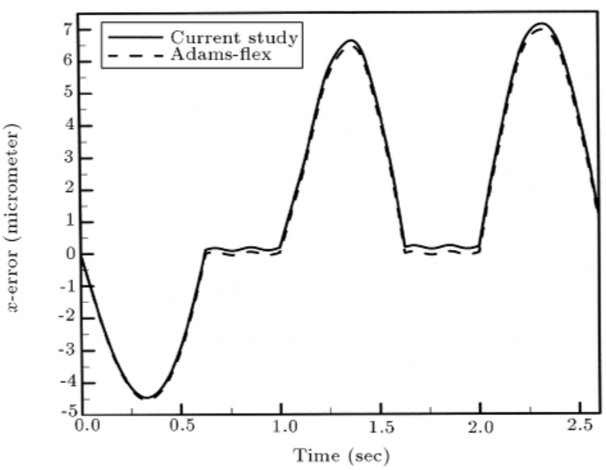

Figure 7. Dynamic error of probe in x-direction for semi-sinusoidal acceleration and deceleration.

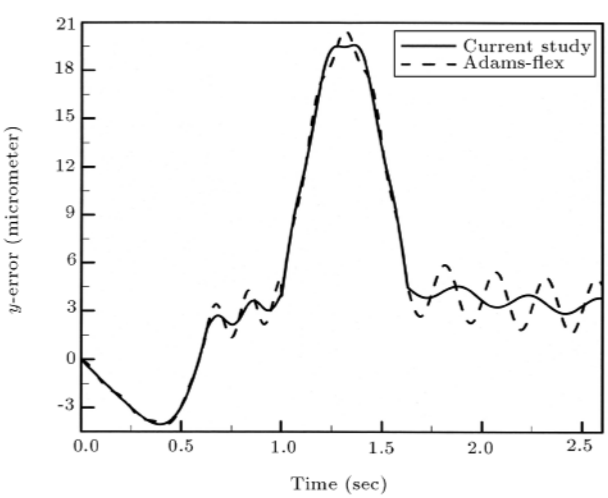

Figure 8. Dynamic error of probe in y-direction for semi-sinusoidal acceleration and deceleration.

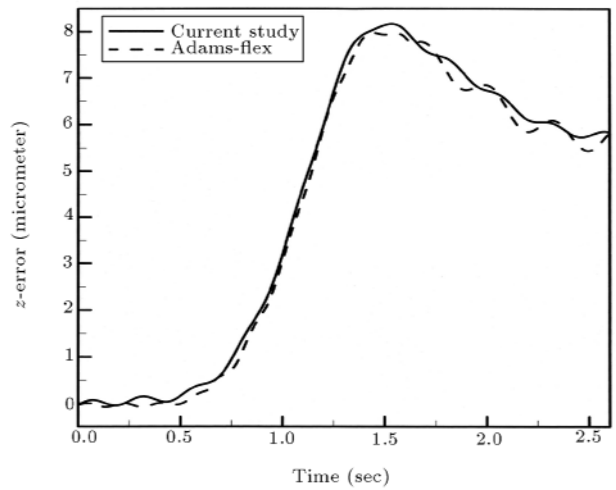

Figure 9. Dynamic error of probe in z-direction for semi-sinusoidal acceleration and deceleration.

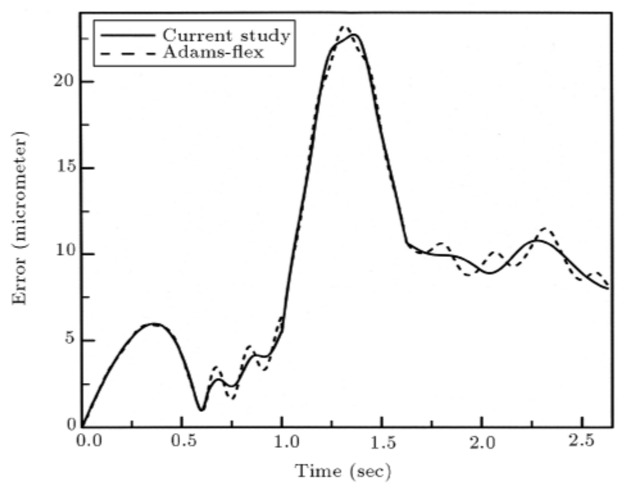

Figure 10. Magnitude of dynamic error of probe for semi-sinusoidal acceleration and deceleration.

The results of the authors' analytical modeling indicate good agreement and high accuracy in comparison with the results found by dynamic modeling software.

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