the value of w for the two-dimensional state is:

$$w = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right),$$

$$r = ||x - y||_2,$$

and finally, the coefficient $C(x)$, depending on the location of x, might have the following values:

$$C(x) = \begin{cases} 1; & x \in \Omega \\ \frac{1}{2}; & x \in \Omega \quad \& \text{ smooth boundary} \\ \frac{\theta}{2\pi}; & x \in \Omega \quad \& \text{ non-smooth boundary} \end{cases}$$

where θ is the angle between the tangents when the boundary is broken.

As may be seen here, the left hand of Equation 2 is completely expressed on the boundary and the right hand side on the whole domain. To solve the right hand side, an equivalent integral term should be used. To do so, various methods [1,2] are applied, however, taking into account that h is harmonic and the Green theorem is used, it is found that:

$$\int_\Omega (h \quad w \quad dv) = \int_\Gamma \left(h\frac{\partial \xi}{\partial n} - \xi\frac{\partial h}{\partial n}\right) d\Gamma, \tag{4}$$

where ξ is the fundamental solution of the Laplace equation, i.e.,

$$\nabla^2\xi = w, \tag{5}$$

and the value of ξ for the two-dimensional state is:

$$\xi = \frac{r^2}{8\pi}\left[\ln\left(\frac{1}{r}\right) + 1\right]. \tag{6}$$

Thus, Equation 2, when h is harmonic, becomes:

$$C(x)u(x) + \int_\Gamma \left[u(y)\frac{\partial w}{\partial n}(x,y) - w(x,y)\frac{\partial u}{\partial n}(y)\right] d\Gamma(y)$$

$$= \int_\Gamma \left(h\frac{\partial \xi}{\partial n} - \xi\frac{\partial h}{\partial n}\right) d\Gamma(y). \tag{7}$$

WAVELETS AND MULTI-RESOLUTION ANALYSIS

The term "wavelet" was proposed by Morlet and Grossman [3,4] for square integrable functions whose translations and dilations form the bases of $L^2(R)$. Thus, studying wavelets means studying the bases of $L^2(R)$ in which all basis functions are similar and only become different by translations and changes in scale.

Orthogonal wavelets, which are exclusively applied in signal analysis, are functions that are generated as a result of their translation by integer coefficients of b with the condition that the dilations of these functions must be the same. This section is a reminder of the theoretical basis of wavelets which is called "multi-resolution analysis". Multi-resolution analysis discusses the conditions under which a complete orthonormal basis for $L^2(R)$ may be obtained through translations and dilations of a single function.

Multi-Resolution Analysis

First, it would be appropriate to define some of the terms. A "wavelet" or a "mother wavelet" of the function $\psi \in L^2(R)$ is in such a way that appropriate translations and dilations of ψ form an orthonormal basis, $L^2(R)$ [4-10]. The wavelet, ψ, is generally supposed to be a complex valued function of a real argument. However, in certain cases the focus of this paper is restricted to real-valued wavelets. The basic idea behind the analysis of wavelets is selecting $\psi, s_m > 0$ scale and translation of b_{mn} in such a way that all functions, such as $f \in L^2(R)$, shall have a series expansion as:

$$f(x) = \sum_{m,n\in z} C_{mn}\psi_{mn}(x), \tag{8}$$

where z indicates integers, and:

$$\psi_{mn}(x) = C_m\psi\left((x - b_{mn})/S_m\right),$$

C_m (which depends only on the S_m scale and not on the translation of b_{mn}) is selected in such a way that makes ψ_{mn} become orthogonal under certain appropriate conditions. Generally, C_m is equal to $S_m^{-1/2}$, thus, $||\psi_{mn}|| = ||\psi||_2$ for every m and n. The $\{\psi_{mn}\}$ set is termed as the extracted family from the wavelet ψ. The focus of this paper is restricted to the orthonormal wavelet families, i.e. wavelets that apply to the following expression because they have the widest application:

$$\langle\psi_{mn}, \psi_{kI}\rangle = \int_{-\infty}^{\infty} \psi_{mn}\psi_{kI}dx = \delta_{mk}\delta_{nl}. \tag{9}$$

Whenever it is claimed that a family is an orthonormal wavelet family, it should, naturally, be mentioned that it is a basis for $L^2(R)$. If a decomposition form is available, as in Equation 8, the $\{C_{mn}\}$ coefficients shall be called the wavelet transform of the function, f.

In the analysis of wavelets, the translations of b_{mn} are so adopted as to have a form such as $b_{mn} = nS_mb$, where $b = b_{01} > 0$ and S_m scales are usually selected in such a way that $m \in z$ and $S_m = 2^m$. Thus, ψ is changed to the following form:

$$\psi_{mn}(x) = 2^{-m/2}\psi(2^{-m}x - nb), \tag{10}$$

then, if there is an orthogonal family of wavelets, the wavelet transform of C_{mn} is:

$$C_{mn} = \langle \psi_{mn}, f \rangle = 2^{-m/2} \int_{-\infty}^{\infty} \psi^*(2^{-m}t - nb) f(t)dt. \tag{11}$$

Sometimes, simple symbols, such as the ones below, are used:

$$\psi(x) = S^{-1/2}\psi(S^{-1}x),$$

$$[W_S f](x) = \int_{-\infty}^{\infty} \psi_S^*(t-x) f(t)dt. \tag{12}$$

Therefore, Expression 12 is a convolution-like integral which means $W_S f(x) = f^*\psi^*(x)$ (where * indicates convolution and $\psi_S(x) = \psi_S(-x)$) with the condition that x and S are selected in such a way that can be considered continuous parameters. Most of the time, Expression 12 is called the wavelet transform of the function f. This is a simple and standard term. To avoid mistake and confusion, $W_S f$ in Expression 12 is considered as "a continuous wavelet transform with continuous parameters" and in Expression 11 as "a continuous wavelet transform with discrete parameters".

This terminology predicts the fact that a third wavelet transform should also exist, which is referred to as a discrete wavelet transform.

With the foregoing arguments in mind, if one wishes to make an orthogonal wavelet basis, b and ψ should be selected with great care. These are not free for selection. It might be strange, but it is true, that each set of ψ_{mn} is an orthogonal basis for the $L^2(R)$ space. There is a theory that can help as a guide in selecting ψ and b, which is called the multi-resolution analysis and is discussed below.

The first objective of multi-resolution analysis is expressing wavelet as $\phi \in L^2(R)$, which is called a "generating function" or a "scale function", from which a wavelet $\psi \in L^2(R)$ is extracted.

Suppose $\phi \in L^2(R)$ and $b > 0$ to be constant. Then:

$$S_0 = \text{Span}\Big\{\phi(x-nb) | n \in Z\Big\},$$

$$V_0 = \overline{S}_0,$$

$$V_m = \Big\{f(2^{-m}x) | f \in V_0\Big\},$$

where S_0 is a set of all linear combinations of ϕ translations, with integer coefficients of b, V_0 is the S_0 closure (with regard to the L^2 norm) and V_m (a set of all scaled version with 2^m factor), is an element of V_0.

Now, $\phi_{mn}(x) = 2^{-m/2}\phi(2^{-m}x - nb)$ shall be written. It can, thus, be seen that for all values of $n, \phi_{mn} \in V_m$. Suppose $\phi(x)$ has a compact support and this means that ϕ outside $[-A, A]$ is zero. Thus, $\phi(2^{-m}x)$ has the support $[-2^mA, 2^mA]$ as it changes easily (multiplying x into 2^{-m} stretches the curve ϕ with a 2^m factor) because $\phi(2^{-m}x - nb) = \phi(2^{-m}(x - n2^mb))$ has the same graph as $\phi(2^{-m}x)$, which has been translated onto the right side by 2^mnb. Finally, it can be seen that $||\phi_{mn}|| = ||\phi||$, for all integer values of m and n, because:

$$\int_{-\infty}^{\infty} |\phi_{mn}(x)|^2 dx = \int_{-\infty}^{\infty} 2^{-m}|\phi(2^{-m}x - nb)|^2 dx$$

$$= \int_{-\infty}^{\infty} |\phi(\xi)|^2 d\xi,$$

where $\xi = 2^{-m}x - nb$ and, here, the inclusion of the normalization factor of $2^{-m/2}$ in the definition of ϕ_{mn} is justified, the factor $2^{-m/2}$ is included so that the forms of all normalized scales and translated versions of ϕ have norms similar to ϕ_{mn} itself.

Thus, for every m, V_m is a closed subspace of L^2 and, for every n under the translations of 2^mnb, it is a variant. If $f(x) \in V_m$ then $f(x - 2^mnb)$ is also an element of V_m.

The above results can also show that V_m is a closure of a set of all linear combinations of the translation of $\phi_{m0} = 2^{-m/2}\phi(2^{-m}x)$ with integer coefficients of 2^mb. Now, suppose that V_0 is a closure of span $\Big\{\phi(x-nb)\Big\}, V_m$, for every integer, m is obtained from V_0 by dilation in the state mentioned above and, in addition, the conditions below are retained:

(a) $V_m \subset V_{m-1}$ for all $m \in z$,

(b) $\cap_{m \in z} V_m = \{0\}$,

(c) $\cup_{m \in z} V_m$ is dense in $L^2(R)$.

Also, there are constant values for $A > 0$ and B, which apply in the following condition to all square sumable and complex numbers (C_n):

$$A \sum_{n \in z} |C_n|^2 \leq \Big|\Big| \sum_{n \in z} C_n \phi_{0n} \Big|\Big|_2^2 \leq B \sum_{n \in z} |C_n|^2.$$

If these conditions apply, $\{V_m\}$ is called a multi-resolution analysis of L_2 and it is said that its ϕ produces the multi-resolution analysis.

In general, whenever the function ϕ is selected and the set of $\phi_{0n} = \Big\{\phi(x-n)\Big\}$ is formed, orthogonal bases are created. In case the original ϕ does not generate orthogonal bases, $\tilde{\phi}$ may be obtained from ϕ, which generates orthogonal bases and fulfils the following condition. If $\hat{\phi}$ is the Fourier transform of

ϕ, one obtains:

$$\hat{\tilde{\phi}}(\xi) = \frac{\hat{\phi}(\xi)}{\sqrt{\sum_{k\in z}\left|\hat{\phi}(\xi + 2k\pi)\right|^2}}.$$

When a set of orthogonal bases is obtained from multi-resolution analysis, wavelet orthogonal bases can be generated. Let one define W_m as $V_m^{\perp}$, i.e., an orthogonal complement. Then:

$$V_{m-1} = V_m \oplus W_m \ \& \ V_m \perp W_m,$$

where the set of $\{W_m\}$ subspaces are mutually orthogonal and $\oplus_{m\in z} W_m = L^2(R)$. It can be shown that the set $\{\psi_{0n}\} = \{\psi(x-n)\}$ is the orthogonal basis of W_0. Hence, $\{\psi_{mn}\}_{n\in z}$ is an orthogonal basis of W_m. In fact, $\{\psi_{mn}\}$ is the orthogonal basis of $L^2(R)$ and ψ is formed as follows.

According to the definition of the function ϕ and multi-resolution analysis, there are D_n values where:

$$\phi(x) = \sum_n D_n \phi(2x - n),$$

thus:

$$\psi(x) = \sum_n (-1)^n D_{1-n}\phi(2x - n)$$

As can be seen, the coefficients C_n and ψ are obtained. For the orthogonal basis ϕ to be normalized, one should have:

$$\int \phi dx = +1.$$

WAVELETS-BOUNDARY INTEGRAL METHOD AND RESULTS

In this section, the boundary integral and wavelets are combined and the result of this combination shall be known as "Wavelet Boundary Integral Method" (WBIM). The result indicates the accuracy of this method in solving related problems. The problem described in this article is finding the normal derivative of the stress function for the torsion of a prismatic element, where the cross-section curve is singly connected. As seen from the results, this method can serve as an appropriate substitution for precise solutions whenever such solutions cannot be achieved.

Definition of the Problem

As mentioned before, the intention here is to solve the Poission equation to obtain the normal directional derivative of stress function for sections whose curves are of a single criterion type. Like the usual solutions using the boundary element method, neither constant linear elements nor any quadratic elements are used. Rather, here, the boundary is considered as an element and the stress function is regarded as a wavelet dilation of a generating function or the ϕ scale so that:

$$u(x) = \sum_{i=1}^{N} (C_m)_i \phi(m, x_i, x).$$

Here, the generating function used is Battle-Lemarie and the wavelet are of a compactly support type. If the above expression is used in the boundary integral equation (Equation 7), and this equation is used for N points on the boundary, N number of independent equations would be obtained, through which the wavelet coefficients may be calculated.

$$C(x)\sum_{i=1}^{N}(C_m)_i\phi(m,x,x_i)$$
$$+\int_\Gamma\left[\frac{\partial w}{\partial n}(x,y)\sum_{i=1}^{N}(C_m)_i\phi(m,y,y_i)\right.$$
$$\left.- w(x,y)\sum_{i=1}^{N}(C_m)_i\frac{\partial\phi}{\partial n}(m_I y_I y_i)\right]d\Gamma(y)$$
$$=\int_\Gamma\left(h\frac{\partial\xi}{\partial n} - \xi\frac{\partial h}{\partial n}\right)d\Gamma(y).$$

Then:

$$\sum_{i=1}^{N}(C_m)_i\left\{C(x)\phi(m,x,x_i)\right.$$
$$+\int_\Gamma\left[\frac{\partial w}{\partial n}(x,y)\phi(m,y,y_i)\right.$$
$$\left.\left.- w(x,y)\frac{\partial\phi}{\partial n}(m,y,y_i)\right]d\Gamma(y)\right\}$$
$$=\int_\Gamma\left(h\frac{\partial\xi}{\partial n} - \xi\frac{\partial h}{\partial n}\right)d\Gamma(y), \qquad (13)$$

by defining matrixes H, B and C one has:

$$H = C(x)\phi(m,x,x_i) + \int_\Gamma\left[\frac{\partial w}{\partial n}(x,y)\phi(m,y,y_i)\right.$$
$$\left.- w(x,y)\frac{\partial\phi}{\partial n}(m,y,y_i)\right]d\Gamma(y),$$

$$B = \int_\Gamma\left(h\frac{\partial\xi}{\partial n} - \xi\frac{\partial h}{\partial n}\right)d\Gamma(y),$$

$$C = (C_m)_i.$$

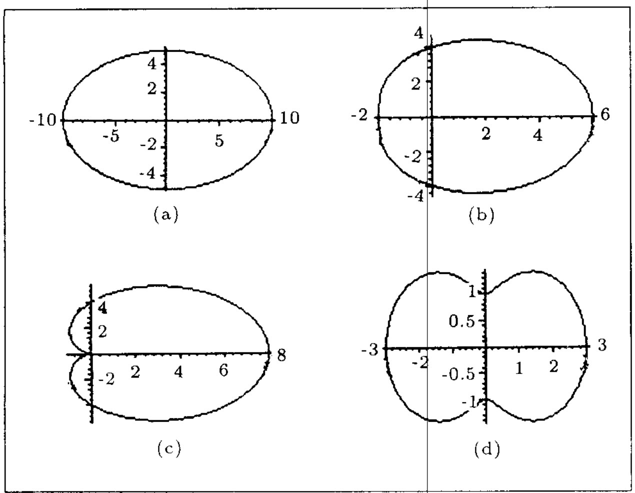

Figure 1. Cross sections of prismatic bars.

By solving this set of N linear equations, the C_m coefficients may be obtained. Now, the first part of the problem is over and its derivative toward the normal direction is on the boundary. Thus, through Equation 7, the stress function may be calculated at any point of the field.

Differential Equation of the Torsion of a Prismatic Bar and Its Boundary Integral Form

The equations of the torsion of a prismatic bar, based on the stress function, is:

$$\nabla^2 u = F,$$

$$u = 0 \text{ on } \Gamma, \qquad (14)$$

where $F = -2G\beta$ and is a constant number [11]. The integral form of Equation 14 in view of Equation 13 is as follows:

$$H = -\int_\Gamma w(x,y)\frac{\partial \phi}{\partial n}(m,y,y_i)d\Gamma(y),$$

$$B = F\int_\Gamma \frac{\partial \xi}{\partial n}d\Gamma(y).$$

Results

In Figure 1 four sections have been considered out of which only the ellipse has an analytical solution (Figure 2). Distribution of directional derivative by the new method is shown in Figures 3 to 6. Also, the comparison of the analytical solution and new solution for the eliptic cross-section is demonstrated in Figure 7.

CONCLUSION

In this paper, it has been shown how to approximate solution of a partial differential equation with wavelets and boundary element methods. Moreover, Wavelet Boundary Integral Method (WBIM) has high accuracy as well as obviating singularity of fundamental solution. In addition, normal partial derivative of solution has been found with Dirichlet constant boundary condition and over cross-sections which their curves are expressed with one criterion. For further research Nueman and multi-boundary conditions must be included.

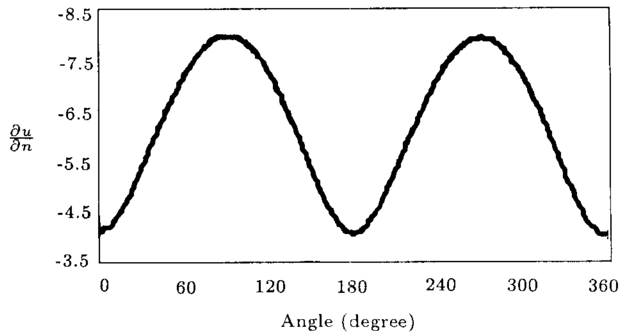

Figure 2. Distribution of directional derivative of stress function over elliptic cross-section boundary (Figure 1a) by analytical method.

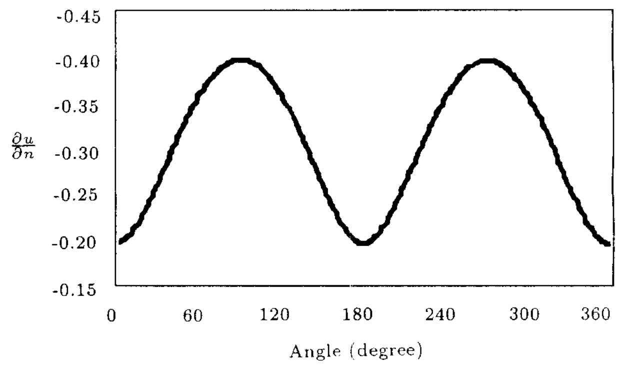

Figure 3. Distribution of directional derivative of stress function over elliptic cross-section boundary by WBIM.

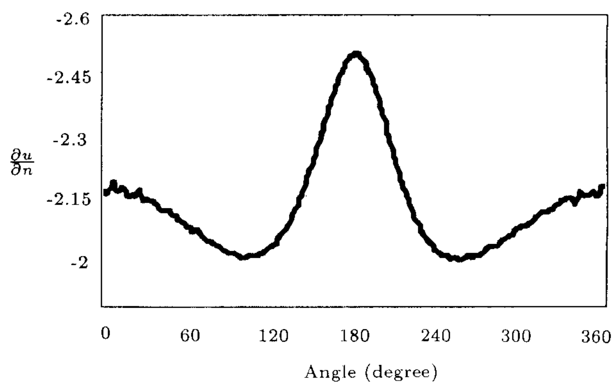

Figure 4. Distribution of directional derivative of stress function over cross-section boundary (Figure 1b) by WBIM.

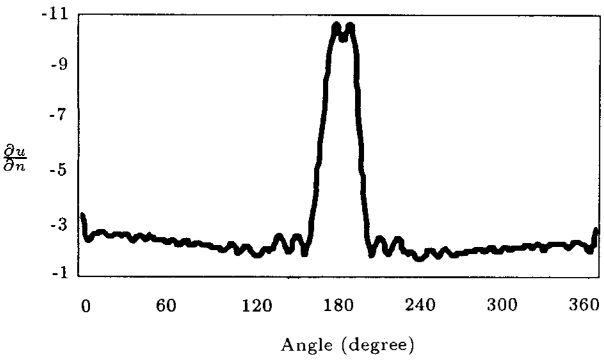

Figure 5. Distribution of directional derivative of stress function over cross-section boundary (Figure 1c) by WBIM.

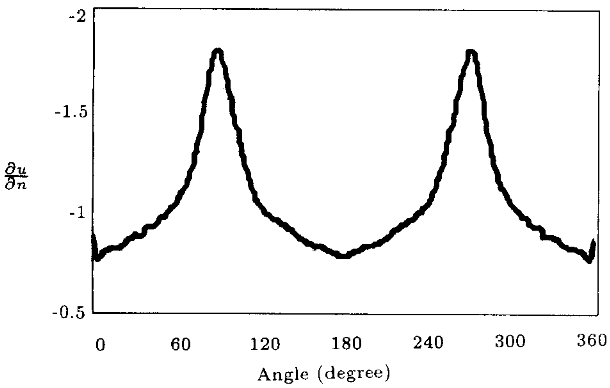

Figure 6. Distribution of directional derivative of stress function over cross-section boundary (Figure 1d) by WBIM.

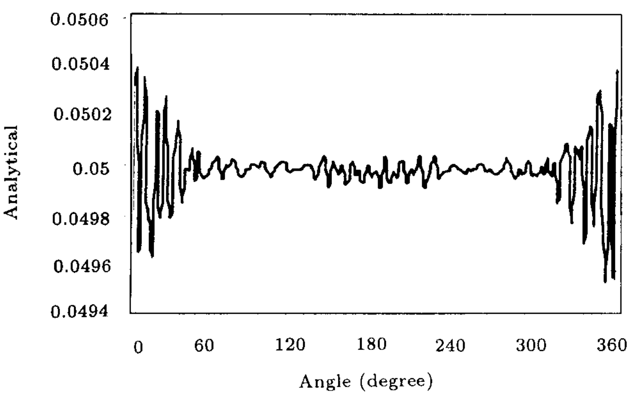

Figure 7. Ratio of analytical solution to WBIM solution.

NOMENCLATURE

$C(x)$	coefficient of point location
w	Green function
ξ	Green function
ψ	wavelet function
ψ_{mn}	wavelet function with translation and dilation
ϕ	scale or generating function
ϕ_{mn}	scale function with translation and dilation
$\hat{\phi}$	fourier transform of scale function
C_{mn}	wavelet transform
$(C_m)_i$	scale function coefficient for approximation
S_m	dilation
b_{mn}	translation
D_n	wavelet expansion coefficient
r	distance between two points
G	shear modulus (Gpa)
β	angle of rotation per length

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