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Research Note

Forced Vibration Analysis of Laminated Rectangular Plates Using Super Elements

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The implementation of new techniques and design of new elements have been an important issues among finite element researchers. In this regard, a designed super element has been applied to analyze a series of dynamic problems. Findings indicate that in large structure analysis the same results as the conventional method can be obtained when applying a few super elements. The time required for dynamic analysis using a super element is significantly smaller than the regular finite element. In this paper, the forced vibration of laminated composite rectangular plates is analyzed. The dynamic response of the plate, using a four-super element, is obtained. In-plane deformation, as well as bending deformation, is included in the model. The current computational model is a simple and efficient way to predict the dynamic behavior of the laminated composite plate.

INTRODUCTION

The development of composite structures has been governed by the search for a material having a higher strength to weight ratio, a lower cost of fabrication and better durability.

Recently, implementation of the finite element method in the analysis of engineering structures has been widely used. However, when applying a large number of elements and nodes in the analysis of large structures, the method becomes cumbersome.

The application of composite materials is considered as primary structural components in places where weight saving is of critical concern. All advanced structures, such as spacecraft, high-speed aircraft and naval vessels, need material properties to be designed in an optimum state.

Most structures, irrespective of their use, will be subjected to dynamic loads during their operational life. Increased use of composite laminated plates in primary structures necessitates the development of accurate theoretical models to predict their response.

Of all approximate schemes, the finite element method is the most widely used approach and has been applied with much success for the vibration analysis of plates and composite plates. Reddy et al. published a series of papers on finite element dynamic analysis of laminated composite plates [1-2]. Sinha and Mukhopadhyay [3] have developed an arbitrary shaped, higher order curved triangular stiffened shell for the static, free vibration and dynamic response analysis of plates and shells with non-uniform thickness and tapered stiffeners. This particular work was a new and valuable approach for researchers and can be used in the analysis of a variety of problems. Mukhopadhyay and Goswami [4] have used the finite element transient analysis of laminated stiffened shells with laminated stiffeners for the first time.

One of the advanced techniques recently implemented is the application of the super element method in structure analysis. Koko and Olson [5-7] and Jiang and Olson [8] have applied the super element in the nonlinear dynamic analysis of plates and shells with and without stiffeners. Vaziri et al. [9] applied the super finite element method to predict the transient response of laminated composite plates and cylindrical shells subjected to a non-penetrating impact by projectiles.

The conventional finite element method usually needs very expensive computer runtime to obtain accurate results. Hence, it is unsuitable at the preliminary design stage, where repeated calculations are often

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necessary. An efficient numerical analysis procedure that can be used as a design tool is, therefore, needed. In this paper, the super finite element method, previously proposed by Olson and Koko [5] and applied to the free vibration analysis of isotropic plate, has been extended to the forced vibration analysis of composite plates. These super elements are capable of producing sufficiently accurate results using a very coarse mesh of elements. Development of the dynamic formulation is briefly outlined as follows.

MODELING

A nine nodes super element is presented in Figure 1. The $x-y$ Cartesian coordinate system of the plate is shown in this figure, where $\xi = \frac{x}{a}, \eta = \frac{y}{b}$ and a and b are the plate length and width, respectively.

Displacement fields of the element are given by [5]:

$$u = \sum_{i=1}^{9} N_i^u u_i + \sin(2\pi\xi)\begin{Bmatrix} L_1(\eta) \\ L_2(\eta) \\ L_3(\eta) \end{Bmatrix}^T \begin{Bmatrix} u_{10} \\ u_{11} \\ u_{12} \end{Bmatrix} + \sin(4\pi\xi)\begin{Bmatrix} L_1(\eta) \\ L_2(\eta) \\ L_3(\eta) \end{Bmatrix}^T \begin{Bmatrix} u_{13} \\ u_{14} \\ u_{15} \end{Bmatrix}$$

$$\nu = \sum_{i=1}^{9} N_i^\nu \nu_i + \sin(2\pi\eta)\begin{Bmatrix} L_1(\xi) \\ L_2(\xi) \\ L_3(\xi) \end{Bmatrix}^T \begin{Bmatrix} \nu_{10} \\ \nu_{11} \\ \nu_{12} \end{Bmatrix} + \sin(4\pi\eta)\begin{Bmatrix} L_1(\xi) \\ L_2(\xi) \\ L_3(\xi) \end{Bmatrix}^T \begin{Bmatrix} \nu_{13} \\ \nu_{14} \\ \nu_{15} \end{Bmatrix}$$

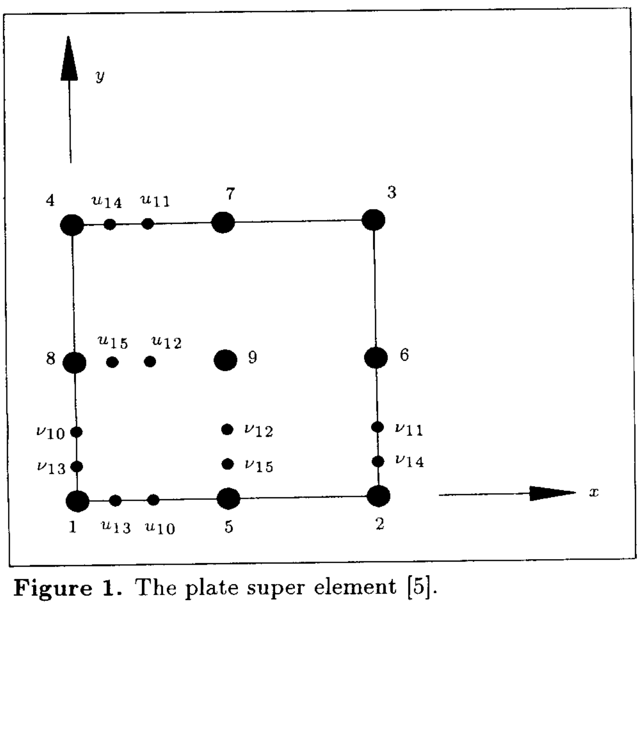

Figure 1. The plate super element [5].

$$w = \sum_{j=1}^{16} N_j^w \psi_j + \phi(\xi)\begin{Bmatrix} H_1(\eta) \\ H_2(\eta) \\ H_3(\eta) \\ H_4(\eta) \end{Bmatrix}^T \begin{Bmatrix} w_5 \\ w_{y5} \\ w_7 \\ w_{y7} \end{Bmatrix} + \phi(\eta)\begin{Bmatrix} H_1(\xi) \\ H_2(\xi) \\ H_3(\xi) \\ H_4(\xi) \end{Bmatrix}^T \begin{Bmatrix} w_8 \\ w_{x8} \\ w_6 \\ w_{x6} \end{Bmatrix} + \phi(\xi)\phi(\eta)w_9, \tag{1}$$

where u, ν and w are displacements in the x, y and z directions and N_i^u, N_i^ν and N_i^w are the super element shape functions in the x, y and z direction, respectively. Shape functions are listed in Appendix A. Here, L_i and H_i are the Lagrange and Hermit polynomials and ϕ and ψ_j are the first modal shape of the clamped beam and out of plane bending variable at the corner nodes of the super element, respectively. u_i and ν_i represent the in-plane displacements in the x and y directions. $u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ and $\nu_{10}, \nu_{11}, \nu_{12}, \nu_{13}, \nu_{14}, \nu_{15}$ are amplitudes of the sine functions of the model [5]. The total strain energy of a cross-ply symmetric composite plate is expressed as:

$$U = \frac{1}{2}\int_0^b\int_0^a [A_{11}u_{,x}^2 + 2A_{12}u_{,x}\nu_{,y} + A_{22}\nu_{,y}^2 + A_{66}(u_{,y}+\nu_{,x})^2 + D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2]dxdy = \frac{1}{2}\{q\}^T[k]\{q\}, \tag{2}$$

where the extensional stiffness of laminate $A_{ij} = A_{ji} = \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_k - z_{k-1})$, bending stiffness of laminate $D_{ij} = D_{ji} = \frac{1}{3}\sum_{k=1}^{N} \overline{Q}_{ij}^{(k)}(z_k^3 - z_{k-1}^3)$ and $\overline{Q}_{ij}^{(k)}$ are kth transformed reduced stiffness, N is the number of layers and z_k is the distance from the kth layer to the mid-plane of the plate.

The kinetic energy of the plate element can be expressed as:

$$T = \frac{\rho h}{2}\int_0^b\int_0^a [\dot{u}^2 + \dot{\nu}^2 + \dot{w}^2]dxdy = \frac{1}{2}\{\dot{q}\}^T[m]\{\dot{q}\}, \tag{3}$$

where T denotes the kinetic energy of the plate, ρ is the material density and h is the plate thickness.

The governing equation of motion for the undamped forced vibration can be expressed as:

$$[m]\{\ddot{q}\} + [k]\{q\} = \{F(t)\}, \tag{4}$$

where $[m]$ and $[k]$ are the assembled mass and stiffness matrix and $\{\ddot{q}\}$ and $\{q\}$ are the global acceleration and

Table 1. $\overline{\omega}$ comparison for $ss-1$ symmetric cross-ply $[0°/90°/0°]$ square plate.

Method / Mode	CLPT [1] Without RI	CLPT [1] With RI	FSDT [1] Without RI $k=1$	FSDT [1] Without RI $k=5/6$	FSDT [1] With RI $k=1$	FSDT [1] With RI $k=5/6$	ANSYS	Present
1	15.228	15.227	15.192	15.185	15.191	15.183	15.184	15.468
2	22.877	22.873	22.831	22.821	22.827	22.817	22.822	23.46

RI: Rotary Inertia; CLPT: Classical Laminated Plate Theory; k: shear correction factor,

FSDT: First Order Shear Deformation Theory.

displacement, respectively. $\{F(t)\}$ is the external time varying forcing function, $\{F(t)\} = \{F_0\}\sin\omega t$, where F_0 is the force amplitude and ω is the external force frequency.

NUMERICAL EXAMPLES

This method is applied to solve the static, free and forced vibration of isotropic, orthotropic and laminated composite plates. In the following examples, a four-super element is used to analyze the dynamic response of a series of plates and the results are compared with a 400 conventional element (SHELL63- SHELL93), using "ANSYS" commercial software.

Free Vibration of a Simply Supported [0°/90°/0°] Laminated Composite Square Plate

Considering a simply supported laminated composite plate, a free vibration analysis of ss-1 simply supported symmetric cross-ply $[0°/90°/0°]$ is presented and the results are compared with the literature.

The material properties of a laminated composite plate are as follows:

$$\underline{\text{Material \#1}}: E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2,$$

$$G_{23} = 0.2E_2, \quad \nu_{12} = 0.25,$$

The non-dimensional natural frequency is given as [1]:

$$\overline{\omega} = \frac{\omega_n a^2}{h}\sqrt{\frac{\rho}{E_2}},$$

where $\overline{\omega}$ is the non-dimensional natural frequency, ω_n is the natural frequency and ρ is the density of the plate. Side to thickness ratio is considered as $\frac{a}{h} = 100$.

Layers are assumed to have the same thickness and the results are presented in Table 1.

Comparing the findings of ANSYS and the classical laminated plate theory indicates a 1.6 % error in the fundamental frequency, with respect to CLPT and a 1.9 % error, with respect to ANSYS.

Table 2. Comparison between natural frequency $[0°/90°/0°/90°/0°]_s$, simply supported square plate.

	Super Element [9]	CLPT [10]	FSDT [9]	ANSYS	Present
Natural Frequency (Hz)	303.116	303.08	302.698	302.8	303.101

Free Vibration of Simply Supported $[0°/90°/0°/90°/0°]_s$ Laminated Composite Square Plate

Consider a ten-layer simply supported laminated square plate with the following sequences $[0°/90°/0°/90°/0°]_s$ and a side length of 0.2 m with a thickness of 0.00269 m.

The material is considered to be T300/934 CFRP with the mechanical properties as:

$$\underline{\text{Material \#2}}: E_1 = 120 \text{ GPa}, \quad E_2 = 7.9 \text{ GPa},$$

$$G_{12} = 5.5 \text{ GPa}, \quad \rho = 1580 \text{ kg/m}^3,$$

$$\nu_{12} = 0.33.$$

Vaziri et al. [9] have implemented the principle of virtual work to solve this problem directly. In the present study, Hamilton's principle, along with the super element, is applied to obtain the stiffness and mass matrices.

Comparison between various methods is presented in Table 2. The results in Table 2 indicate good agreement between the present methods with the analytical solution.

Static Analysis of Symmetric Laminated Square Simply Supported Plate Under Point Force at the Center

Here, a square $[0°/90°/0°]$ symmetric, cross-ply laminated plate is considered. Assume a point force, F_0, is applied at the center of the plate which is made of Material # 1.

Table 3. Non-dimensional static deflection of simply supported [0°/90°/0°] of square plate under point load at the center of plate.

Method	$\hat{w}$
Present	2.0694 (-2.6%)
ANSYS	2.1664 (+1.9%)
CLPT [1]	2.1257

The non-dimensional deflection of the plate center can be defined as [1]:

$$\hat{w} = w\frac{E_2 h^3}{F_0 a^2},$$

where $\hat{w}$ is the non-dimensional deflection of the plate and h and a are the plate thickness and side length of the plate, respectively. In Table 3 the non-dimensional deflection of the plate at the center using the super element, ANSYS and Navier's solution (CLPT) are presented.

The results indicate that a good agreement can be obtained using four-super elements, in comparison with 400 conventional elements and Navier's solution.

Forced Vibration of Clamped Isotropic Square Plate

Consider a single layer square isotropic plate with a point force of $F_0 \sin\omega_t$ at the center. The plate response under force F_0 is calculated in terms of applied frequency. The main idea was used for simple system by Thomson [11].

The non-dimensional amplitude of the plate center versus the non-dimensional frequency is shown in Figure 2. Fifty points are implemented to draw the curve.

In this figure, the non-dimensional parameters are $\overline{w} = \frac{w_d}{w_s}$ and $\overline{\omega} = \frac{\omega}{\omega_n}$ where:

- w_d: dynamic response of the plate at the center,
- w_s: static deflection of the plate center under point load F_0 at the plate center,
- ω: exciting frequency,
- ω_n: fundamental natural frequency

Forced Vibration of Simply Supported Isotropic Square Plate

Consider a single layer square isotropic plate under a point load $F_0 \sin\omega t$ at the center. The results are presented in Figure 3. Fifty points are implemented to draw the curve.

As expected at $\omega = \omega_n$ or $\overline{\omega} = 1$, resonance occurs and amplitude tends towards an unlimited value.

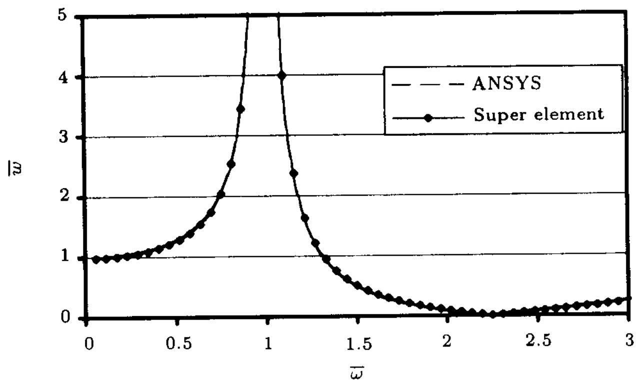

Figure 2. Non-dimensional forced response of square isotropic clamped plate.

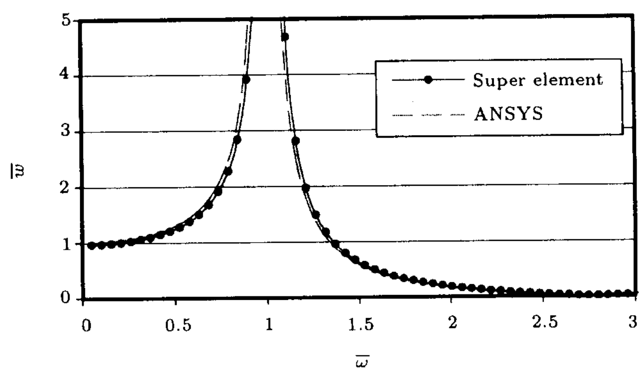

Figure 3. Non-dimensional forced response of square isotropic simply supported plate.

Forced Vibration of Simply Supported Orthotropic Square Plate

Consider a single layer square orthotropic plate with the following properties:

$$\underline{\text{Material \#3}}: \frac{D_{11}}{D} = 0.5 \quad \text{and} \quad \frac{D_{22}}{D} = 2,$$

where $D = D_{12} + 2D_{66}$. In the orthotropic plate, extensional matrix $[A]$ and bending matrix $[D]$ can be summarized as follows:

$$A_{11} = h\frac{E_1}{1-\nu_{12}\nu_{21}} \quad A_{12} = h\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$$

$$A_{22} = h\frac{E_2}{1-\nu_{12}\nu_{21}} \quad A_{66} = hG_{12}$$

$$D_{11} = \frac{h^3}{12}\frac{E_1}{1-\nu_{12}\nu_{21}} \quad D_{12} = \frac{h^3}{12}\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$$

$$D_{22} = \frac{h^3}{12}\frac{E_2}{1-\nu_{12}\nu_{21}} \quad D_{66} = \frac{h^3}{12}G_{12}.$$

The non-dimensional dynamic response of the plate is shown in Figure 4. Comparison between ANSYS results and findings using the super element

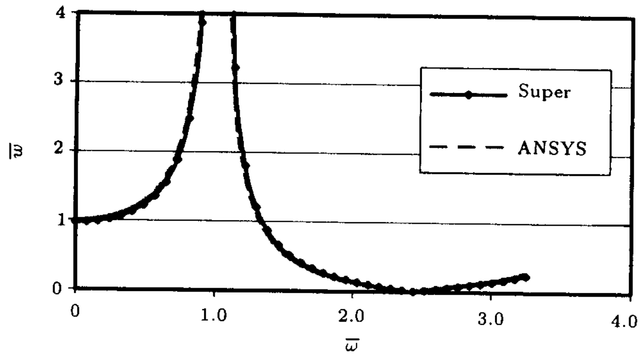

Figure 4. Non-dimensional forced response of square orthotropic simply supported plate.

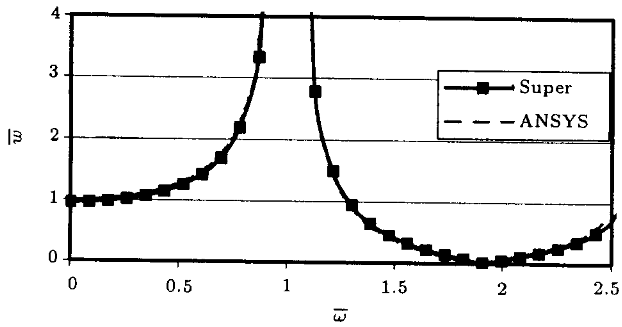

Figure 5. Non-dimensional forced response of clamped orthotropic rectangular plate.

indicate an error of less than 2% at the most frequency domain. Forty points are implemented to draw the curve.

Forced Vibration of Clamped Orthotropic Rectangular Plate under Point Force $F_0 \sin \omega t$ at the Center

Consider a clamped single layer orthotropic rectangular plate with properties of Material # 3 and aspect ratio $1.2(\frac{a}{b} = 1.2)$. The non-dimensional dynamic response of the plate is given in Figure 5. Thirty points are implemented to draw the curve.

A comparison between a clamped orthotropic rectangular plate and a clamped orthotropic square plate indicates that by increasing the aspect ratio of the plate, the dynamic stiffness of the plate will decrease.

Forced Vibration of Clamped Laminated Composite Square Plate

Consider a symmetric cross-ply $[0°/90°]_s$ laminated composite clamped plate with properties of Material # 2. In Figure 6, the dynamic response of the plate is shown. Fifty points are implemented to draw the curve.

Results indicate that the general behavior of the

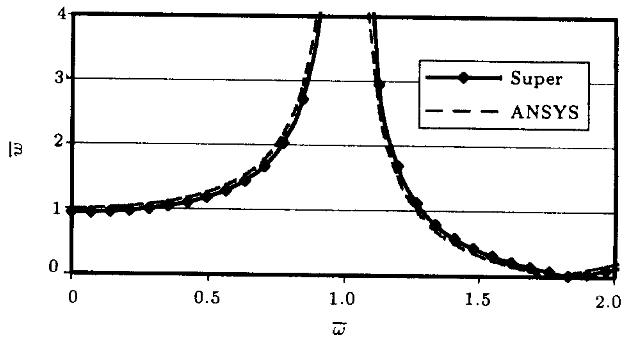

Figure 6. Non-dimensional forced response of clamped laminated composite square plate.

laminated composites is similar to the single layer with the larger bandwidth.

CONCLUSION

Finite element vibration analysis of isotropic, orthotropic and symmetric, cross-ply, laminated, composite plates is considered. Free vibration, static and dynamic analysis of plates are discussed in a non-dimensional form.

Results indicate that the application of the super element in the analysis of laminated composite plates gives the same results in dynamic response with a run-time of less than one-third in comparison with the conventional method.

The maximum amplitude response of plates under static and dynamic load with the same load amplitude can change drastically, depending on the non-dimensional frequency.

Also, findings indicate that the super element is a very good tool in the analysis of dynamic problems with smaller computational time, in comparison to conventional finite element methods with sufficiently accurate results.

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APPENDIX A

- The quadratic Lagrange polynomials:

$$L_1(\xi) = 2\xi^2 - 3\xi + 1,$$

$$L_2(\xi) = 2\xi^2 - \xi,$$

$$L_3(\xi) = 4(\xi - \xi^2).$$

- The Hermitian polynomials:

$$H_1(\xi) = 1 - 3\xi^2 + 2\xi^3,$$

$$H_2(\xi) = a(\xi - 2\xi^2 + \xi^3),$$

$$H_3(\xi) = 3\xi^2 - 2\xi^3,$$

$$H_4(\xi) = a(-\xi^2 + \xi^3).$$

- The clamped beam vibration mode:

$$\phi(\xi) = [\alpha(\sinh\beta\xi - \sin\beta\xi) + (\cosh\beta\xi - \cos\beta\xi)]/A,$$

where $\beta = 4.7300407448$,

$$A = \alpha(\sinh 0.5\beta - \sin 0.5\beta) + (\cosh 0.5\beta - \cos 0.5\beta),$$

$$\alpha = \frac{\cos\beta - \cosh\beta}{\sinh\beta - \sin\beta}.$$

- The shape function of super element:

$$N_1^u = L_1(\xi)L_1(\eta), \quad N_2^u = L_2(\xi)L_1(\eta),$$

$$N_3^u = L_2(\xi)L_2(\eta), \quad N_4^u = L_1(\xi)L_2(\eta),$$

$$N_5^u = L_3(\xi)L_1(\eta), \quad N_6^u = L_2(\xi)L_3(\eta),$$

$$N_7^u = L_3(\xi)L_2(\eta), \quad N_8^u = L_1(\xi)L_3(\eta),$$

$$N_9^u = L_3(\xi)L_3(\eta), \quad N_{10}^u = \sin(2\pi\xi)L_1(\eta),$$

$$N_{11}^u = \sin(2\pi\xi)L_2(\eta), \quad N_{12}^u = \sin(2\pi\xi)L_3(\eta),$$

$$N_{13}^u = \sin(4\pi\xi)L_1(\eta), \quad N_{14}^u = \sin(4\pi\xi)L_2(\eta),$$

$$N_{15}^u = \sin(4\pi\xi)L_3(\eta),$$

$$N_i^\nu = N_i^u \quad \text{for} \quad i \le 9.$$

$$N_{10}^\nu = L_1(\xi)\sin(2\pi\eta), \quad N_{11}^\nu = L_2(\xi)\sin(2\pi\eta),$$

$$N_{12}^\nu = L_3(\xi)\sin(2\pi\eta), \quad N_{13}^\nu = L_1(\xi)\sin(4\pi\eta),$$

$$N_{14}^\nu = L_2(\xi)\sin(4\pi\eta), \quad N_{15}^\nu = L_3(\xi)\sin(2\pi\eta),$$

$$N_1^w = H_1(\xi)H_1(\eta), \quad N_2^w = H_2(\xi)H_1(\eta),$$

$$N_3^w = H_1(\xi)H_2(\eta), \quad N_4^w = H_2(\xi)H_2(\eta),$$

$$N_5^w = H_3(\xi)H_1(\eta), \quad N_6^w = H_4(\xi)H_1(\eta),$$

$$N_7^w = H_3(\xi)H_2(\eta), \quad N_8^w = H_4(\xi)H_2(\eta),$$

$$N_9^w = H_3(\xi)H_3(\eta), \quad N_{10}^w = H_4(\xi)H_3(\eta),$$

$$N_{11}^w = H_3(\xi)H_4(\eta), \quad N_{12}^w = H_4(\xi)H_4(\eta),$$

$$N_{13}^w = H_1(\xi)H_3(\eta), \quad N_{14}^w = H_2(\xi)H_3(\eta),$$

$$N_{15}^w = H_1(\xi)H_4(\eta), \quad N_{16}^w = H_2(\xi)H_4(\eta),$$

$$N_{17}^w = \phi(\xi)H_1(\eta), \quad N_{18}^w = \phi(\xi)H_2(\eta),$$

$$N_{19}^w = H_3(\xi)\phi(\eta), \quad N_{20}^w = H_4(\xi)\phi(\eta),$$

$$N_{21}^w = \phi(\xi)H_3(\eta), \quad N_{22}^w = \phi(\xi)H_4(\eta),$$

$$N_{23}^w = H_1(\xi)\phi(\eta), \quad N_{24}^w = H_2(\xi)\phi(\eta),$$

$$N_{25}^w = \phi(\xi)\phi(\eta).$$