Scientia Iranica, Vol. 10, No. 2, pp 186–193

SCIENTIA
IRANICA

Heat Convection on Cylinder at High Prandtl Numbers

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Natural convection flow on a vertical cylinder is considered here when the Prandtl number is large. Little work has been done in this field apart from some experimental studies which are for lower Prandtl numbers. Here, the singular perturbation technique is used to solve this problem. The method adopted is to split the flow into a thin layer close to the surface of the cylinder, surrounded by a much thicker layer where the velocity is reduced to zero. It is shown that at high Prandtl numbers, the velocity boundary layer tends to be somewhat larger due to large kinematic viscosity relative to thermal diffusivity. The motion of the outer layer, however, seems to be caused by the drag force exerted by the inner layer, not due to the buoyancy itself.

The basic properties of the flow are evaluated. The heat transfer coefficient is shown to give good prediction for all ranges of Prandtl numbers.

INTRODUCTION

This paper studies the natural convection flow along the outer surface of a vertical cylinder when the Prandtl number (Pr) is large. Materials with high Prandtl number are frequently encountered in industry. Natural cooling of industrial oil (Pr = 80000) along the outer surface of a vertical cylinder in the environment, after a quenching process, can be a practical example of this type of flow. Little work has been done in this field apart from the experiments of Libby [1] and Fujii et al. [2] and a numerical calculation made by Fujii and Uehara [3] for the case of Pr = 100. Another contribution is the investigation of Sparrow and Gregg [4] regarding this problem at moderate Prandtl numbers.

Recently, Ames [5] shows that for Pr = 1, the boundary layer governing equations can be solved in exact form. Naylor [6] presents the limiting solution of the problem of natural convection from a vertical cylinder. There are some high Prandtl number natural convection studies done by Stewartson and Jones [7] and, also, Rahimi and Gholamdokht [8], which are natural convections from a flat plate.

Considering all the above studies, it can be concluded that the solution of natural convection from a vertical cylinder for high Prandtl numbers has not been found. Here, the singular perturbation technique is used to solve the problem of natural convection from a vertical cylinder at high Prandtl numbers. The method adopted is to split the flow into a thin layer close to the surface of the cylinder (where the temperature varies), surrounded by a much thicker layer where the velocity is reduced to zero. The solution is determined in the inner region, in terms of a parameter, which is roughly equal to the ratio of the thickness of this layer to the radius of the cylinder; it is valid up to a vertical height at which this parameter is about unity. The basic properties of the flow are evaluated and it is attempted to reduce the formula for calculating the coefficient of heat transfer from the outer surface of a vertical cylinder to fluid of any Prandtl number. The influence of high ranges of Prandtl number upon heat transfer is found and shown to be in qualitative agreement with Libby [1].

PROBLEM FORMULATION

Let cylindrical coordinates (x, r) be taken whose axis is the vertical center line of the cylinder and whose origin is at the center of the base of the cylinder. Let (u, v) be the corresponding velocity components. Neglecting variable property effects other than buoyancy and adapting the well-known boundary layer approximations for steady state conditions, it is obtained that:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(r\nu) = 0, \tag{1}$$

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$$u\frac{\partial u}{\partial x}+\nu\frac{\partial u}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+g\beta_1\theta(T_w-T_\infty), \quad (2)$$

$$u\frac{\partial \theta}{\partial x}+\nu\frac{\partial \theta}{\partial r}=\frac{\nu}{r\mathrm{Pr}}\frac{\partial}{\partial r}(r\frac{\partial \theta}{\partial r}), \quad (3)$$

where ν is the kinematic viscosity, g is the acceleration due to gravity, β_1 is the coefficient of volumetric expansion, Pr is the Prandtl number and:

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}, \quad (4)$$

where T is the temperature of the fluid and T_∞, T_w are the constant temperatures of the surrounding fluid and cylinder, respectively.

Equation 1 may be eliminated by introducing a stream function, ψ, such that:

$$ur=\frac{\partial\psi}{\partial r}, \quad \nu r=-\frac{\partial\psi}{\partial x}. \quad (5)$$

The boundary conditions are: On $r=a$, the surface of the cylinder, $u=\nu=0$, and $\theta=1$. At large radial distances, u and θ tend to zero.

FLOW IN THE INNER TEMPERATURE LAYER

The problem of natural convection from a vertical flat plate at large values of Pr discussed by Stewartson and Jones [7] and Rahimi and Gholamdokht [8], show that the flow over a flat plate consists of two regions, namely, a thin temperature region, where buoyancy is roughly balanced by viscosity and a thick momentum layer, where the temperature is approximately constant. Thus, in the limit as Pr$\rightarrow\infty$ with Gr finite, where,

$$\mathrm{Gr}=\text{Grashof number}=\mathrm{Gr}=\frac{g\beta_1(T_w-T\infty)x^3}{\nu^2}, \quad (6)$$

the temperature layer becomes vanishingly thin while the momentum region becomes infinitely thick. This suggests, in the cylindrical case, that at finite values of x for sufficiently large Pr the thickness of the temperature layer will be much smaller than the radius of the cylinder.

Using the experience in the above studies, the appropriate variables inside the inner layer in this case are:

$$X=\Big(\frac{64}{\mathrm{Gr.Pr}}\Big)^{1/4}\frac{x}{a}, \quad Y=\frac{r^2-a^2}{a^2X},$$

$$\psi=2\sqrt{2}\nu a\Big(\frac{\mathrm{Gr}}{\mathrm{Pr}^3}\Big)^{1/4}.f(X,Y). \quad (7)$$

Here, $f(X,Y)$ represents the reduced stream function to be found from the subsequent analysis. It should be noted that X gives the order of magnitude of the ratio of the thickness of the temperature layer to the radius of the cylinder. This change of variable is an adaptation of that used by Sparrow and Gregg [4] in their discussion of this problem at moderate Prandtl numbers. From Equation 7 it is readily shown that:

$$u=\frac{1}{r}\frac{\partial\psi}{\partial r}=\frac{2\nu}{x}\Big(\frac{\mathrm{Gr}}{\mathrm{Pr}}\Big)^{1/2}\frac{\partial f}{\partial Y}. \quad (8)$$

The variables in Equations 7 and 8 reduce to inner flat plate variables in the limit as $a\rightarrow\infty$.

When Equations 7 and 8 are inserted in Equations 2 and 3, the following equations are obtained:

$$(1+XY)\frac{\partial^3 f}{\partial Y^3}+X\frac{\partial^2 f}{\partial Y^2}+\theta+\frac{1}{\mathrm{Pr}}\Bigg[-X\{\frac{\partial^2 f}{\partial X\partial Y}.\frac{\partial f}{\partial Y}-\frac{\partial^2 f}{\partial Y^2}.\frac{\partial f}{\partial X}\}+3f\frac{\partial^2 f}{\partial Y^2}-2\Big(\frac{\partial f}{\partial Y}\Big)^2\Bigg]=0, \quad (9)$$

and:

$$(1+XY)\frac{\partial^2\theta}{\partial Y^2}+X\frac{\partial\theta}{\partial Y}-X(\frac{\partial f}{\partial Y}\frac{\partial\theta}{\partial X}-\frac{\partial\theta}{\partial Y}\frac{\partial f}{\partial X})+3f\frac{\partial\theta}{\partial Y}=0. \quad (10)$$

The boundary conditions on f,θ at $Y=0$ are that:

$$f=\frac{\partial f}{\partial Y}=0, \quad \theta=1.$$

FIRST-TERM SOLUTION OF INNER REGION EQUATIONS

Equations 9 and 10, along with the corresponding boundary conditions, govern the flow in the inner region. These equations are solved using perturbation techniques. Note $X\rightarrow 0$ as Pr $\rightarrow\infty$, (see Equation 7). Therefore, X can be a suitable perturbation parameter since, in this study, one is interested in solving the problem for high Prandtl numbers.

The perturbation expansions of quantities f and θ are written as:

$$f(Y,X)=f_0(Y)+Xf_1(Y)+X^2f_2(Y)+\cdots$$

$$\theta(Y,X)=\theta_0(Y)+X\theta_1(Y)+X^2\theta_2(Y)+\cdots$$

Substituting these expansions into Equations 9 and 10 and keeping the first term in expansions above gives:

$$\frac{\partial^3 f_0}{\partial Y^3}+\theta_0=0, \quad (11)$$

$$\frac{\partial^2\theta_0}{\partial Y_0^2}+3f_0\frac{\partial\theta_0}{\partial Y_0}=0, \tag{12}$$

with an error of order X, i.e. of order $(\mathrm{Pr})^{-1/4}$. Equations 11 and 12 are solved using a Runge-Kutta order of four [9], giving the following results for the solutions f_0, θ_0 and Figures 1 to 3:

$$\frac{\partial^2 f_0}{\partial Y^2}=0.825, \quad \frac{\partial\theta_0}{\partial Y}=0.711 \quad \text{at} \quad Y=0,$$

and:

$$f_0(Y)=a_0+a_1Y, \text{ with } a_0=-0.261, \quad a_1=0.511,$$

and:

$$\theta_0\to 0 \quad \text{as} \quad Y\to\infty. \tag{13}$$

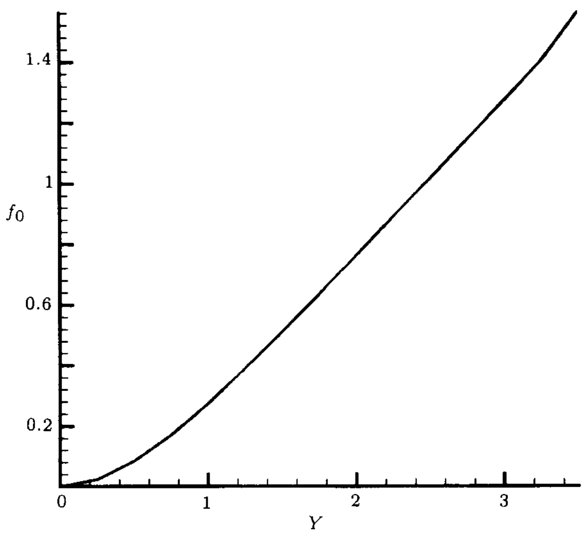

Figure 1. f_0 in terms of Y.

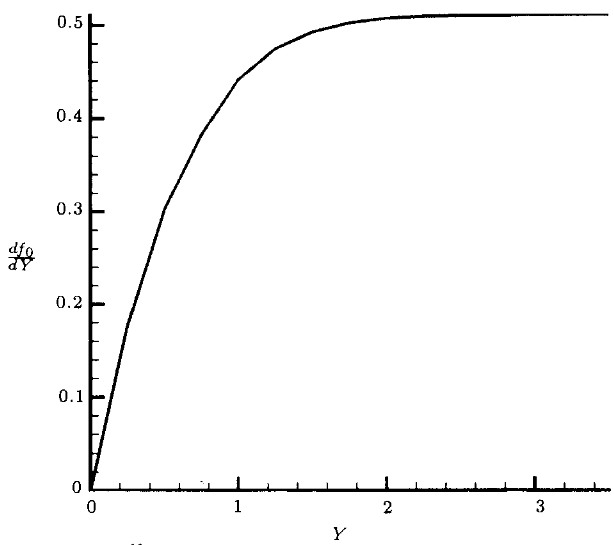

Figure 2. $\frac{df_0}{dY}$ in terms of Y.

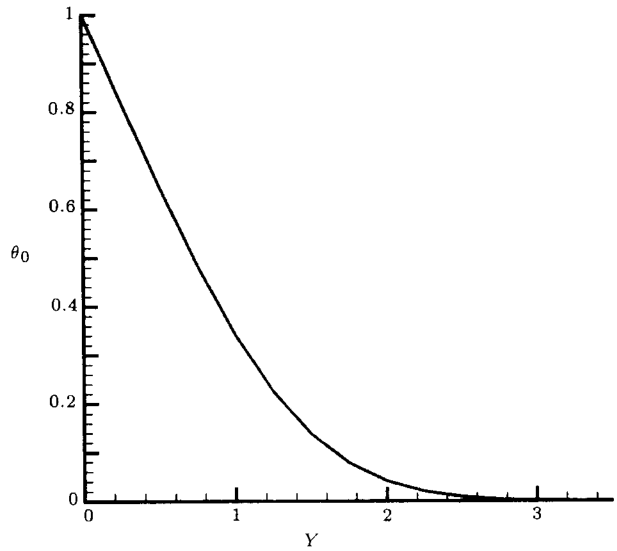

Figure 3. First term of inner temperature.

To summarize, when Pr is large, and the Grashof number is finite, the inner temperature layer forms a thin skin (whose thickness varies as $(\mathrm{Pr})^{-1/4}$) on the surface of the cylinder. From Equation 13 it can be shown that the outer surface of this skin moves with velocity $U_0(x)$, where:

$$U_0(x)=\frac{2\nu a_1}{x}\Big(\frac{\mathrm{Gr}}{\mathrm{Pr}}\Big)^{1/4}. \tag{14}$$

FLOW IN THE OUTER MOMENTUM LAYER

In the outer momentum layer, where the temperature is sensibly constant, the cylinder appears to move with velocity $U_0(x)$.

In this section, the flow corresponding to such a moving cylinder is developed. It will be verified in later sections that this solution does, in fact, match the inner solution. Now, the boundary layer due to a cylinder which moves with constant velocity, has been determined by Crane [10]. The method used in [10] suggests the following change of variables when the radius of the cylinder is much thinner than the boundary layer:

$$\psi=\nu x F(\eta,\beta), \quad \eta=\frac{U_0 r^2}{4\nu x},$$

$$\beta=\ln\Big(\frac{4\nu x}{U_0 a^2}\Big)=2\ln X+\ln\mathrm{Pr}-\ln(4a_1)$$

$$=2\ln X+\ln\mathrm{Pr}-0.716. \tag{15}$$

In terms of Equation 15:

$$\frac{u}{U_0}=\frac{1}{2}\frac{\partial F}{\partial\eta}, \quad \nu=-\frac{\nu}{r}\Big[F+\frac{1}{2}\frac{\partial F}{\partial\beta}-\frac{1}{2}\eta\frac{\partial F}{\partial\eta}\Big], \tag{16}$$

and Equation 2, with $\theta = 0$, reduces to:

$$\frac{\partial}{\partial\eta}(\eta\frac{\partial^2 F}{\partial\eta^2}) + \frac{1}{2}F\frac{\partial^2 F}{\partial\eta^2} - \frac{1}{4}\left(\frac{\partial F}{\partial\eta}\right)^2 - \frac{1}{4}\frac{\partial^2 F}{\partial\beta\partial\eta}\frac{\partial F}{\partial\eta} + \frac{1}{4}\frac{\partial F}{\partial\beta}\frac{\partial^2 F}{\partial\eta^2} = 0. \quad (17)$$

The boundary conditions are: On the surface of the moving cylinder, which may be taken to a good approximation at $r = a$, i.e. $\eta = e^{-\beta}$,

$$\frac{\partial F}{\partial\eta} = 2 \quad \text{and} \quad F + \frac{1}{2}\frac{\partial F}{\partial\beta} = e^{-\beta}, \quad (18)$$

while $\frac{\partial F}{\partial\eta} \to 0$ as $\eta \to \infty$.

It will be assumed that, in the limit as $\beta \to \infty$ (which occurs when $\Pr \to \infty$) the terms in $\frac{\partial}{\partial\beta}$ tend to zero. Then, for large values of β, Equation 17 becomes approximately:

$$\frac{\partial}{\partial\eta}\left(\eta\frac{\partial^2 F}{\partial\eta^2}\right) + \frac{1}{2}F\frac{\partial^2 F}{\partial\eta^2} - \frac{1}{4}\left(\frac{\partial F}{\partial\eta}\right)^2 = 0. \quad (19)$$

Equation 19 has the property that, if $F_0(\eta)$ is a solution, so is $F_0(z)$, where $z = \eta/\gamma$ and γ is any positive function of β. Equation 19 then becomes:

$$\frac{d}{dz}\left(z\frac{d^2 F_0}{dz^2}\right) + \frac{1}{2}F_0\frac{d^2 F_0}{dz^2} - \frac{1}{4}\left(\frac{dF_0}{dz}\right)^2 = 0. \quad (20)$$

A standard solution of Equation 20 was calculated subject to the following conditions:

$$\frac{dF_0}{dz} \to 0 \quad \text{as} \quad z \to \infty,$$

$$F_0 \to 0 \quad \text{as} \quad z \to 0.$$

When z is large:

$$F_0 = 2(D+1) + \sum_{n=1}^{\infty}\frac{A_n}{Z^{nD}}, \quad (21)$$

where D is a constant to be determined. The first few terms of Equation 21 and its derivatives provide starting values (at a sufficiently large value of z) for the numerical integration of Equation 20 in the direction of decreazing z (see Table 1). When z is small it is found that, when $D = 0.539$,

$$F_0 \approx Az(\ln z + B - 1) + O(z^2(\ln z)^2),$$

$$\frac{dF_0}{dz} \approx A(\ln z + B) + O(z),$$

where $A = -30.90$, $B = 4.61$. Now when z is small:

$$\frac{\partial F}{\partial\eta} = \frac{1}{\gamma}\frac{dF_0}{dz} = \frac{A(\ln z + B)}{\gamma} + O(z).$$

Table 1. Numerical solution of f_2 and θ_2.

$\log z$	F_0	$\frac{dF_0}{dz}$	$\frac{d^2F_0}{dz^2}$	F_1	$\frac{dF_1}{dz}$	$\frac{d^2F_1}{dz^2}$
-12	0.001	227	-30.8	0	20.0	-0.082
-10	0.009	166	-30.3	0.001	19.5	-0.416
-8	0.046	107	-28.5	0.006	17.8	-1.49
-6	0.199	54.4	-22.9	0.038	12.8	-3.51
-4	0.659	18.4	-12.5	0.160	4.96	-3.64
-2	1.49	3.36	-3.42	0.343	0.50	-0.85
0	2.30	0.326	-0.422	0.378	-0.026	-0.003
2	2.77	0.021	-0.030	0.297	-0.006	0.007
4	2.97	0.001	-0.002	0.224	-0.0005	0.0006
6	3.04	0.0001	-0.0001	0.184	-0.00003	0.00005
8	3.07	0.000002	-0.000004	0.167	-0.000005	0.000003

It follows that the first part of Condition 18 will be satisfied, to within a fractional error of order $e^{-\beta}$ (i.e. $\Pr^{-1/2}$) provided that:

$$\beta = \beta^* - \ln\beta^* + \beta - \ln(-\frac{1}{2}A), \quad (22)$$

where $\beta^* = \frac{-2\gamma}{A}$.

Thus, a first approximation to the outer velocity profile is:

$$\frac{u}{U_0} = \frac{1}{\gamma}\frac{dF_0}{dz}.$$

HIGHER APPROXIMATIONS TO THE OUTER SOLUTION

The error due to the neglected terms in $\frac{\partial}{\partial\beta}$ in Equation 17 is readily shown to be of the order $(\beta^*)^{-1}$.

This error may be corrected by expanding F in the series:

$$F = F_0(z) + \frac{1}{\beta^*}F_1(z) + \cdots \quad (23)$$

The equation for F_1 is found by substituting Equation 23 in Equation 17 and equating terms of order $(\beta^*)^{-1}$. Then:

$$z\frac{d^3F_1}{dz^3} + \frac{d^2F_1}{dz^2} + \frac{1}{2}F_0\frac{d^2F_1}{dz^2} + \frac{1}{2}F_1\frac{d^2F_0}{dz^2} - \frac{1}{2}\frac{dF_0}{dz}\frac{dF_1}{dz} + \frac{1}{4}\left(\frac{dF_0}{dz}\right)^2 = 0. \quad (24)$$

The solution of Equation 24 for which:

$$(F_1)_0 = \left(\frac{d^2F_1}{dz^2}\right)_0 = 0, \quad \left(\frac{dF_1}{dz}\right)_\infty = 0,$$

is given in Table 1. Note the values:

$$\left.\frac{dF_1}{dz}\right|_{t=0} = \bar{c}_1 = 20.0, \quad F_1(\infty) = 0.154.$$

Now, at $\eta = e^{-\beta}$:

$$\frac{u}{U_0} = 1 = \frac{1}{2\gamma}\left[A(-\beta - \ln\gamma + B) + \frac{\overline{C}_1}{\beta^*}\right].$$

It follows that Definition 22 of β^* must be amended to:

$$\beta = \beta^* - \ln\beta^* + B - \ln(-\frac{1}{2}A) + \frac{\bar{c}_1}{A\beta^*},$$

or:

$$\beta = \beta^* - \ln\beta^* + 1.89 - \frac{0.65}{\beta^*}. \tag{25}$$

To summarize, the outer velocity profile is:

$$\frac{u}{U_0} = -\frac{1}{A\beta^*}\left[\frac{dF_0}{dz} + \frac{1}{\beta^*}\frac{dF_1}{dz} + O\Big(\frac{1}{\beta^{*2}}\Big)\right]. \tag{26}$$

This formula gives u/U_0 to within a few percent when β^* or β is greater than 5. This will be roughly true for $X > 0.1$ when $\Pr > 10^4$.

HIGHER APPROXIMATIONS TO THE INNER SOLUTION

The error in the approximate inner solution (Equation 13) has two sources, firstly that Equation 13 is a solution of the truncated Equations 11 and 12, in which terms of order X have been neglected and, secondly, that the outer solution (Equation 26) does not exactly match the inner solution (Equation 13).

The first source of error may be approximately corrected by writing:

$$f = f_0 + Xf_1, \quad \theta = \theta_0 + X\theta_1, \tag{27}$$

when Equation 27 is inserted in Equations 9 and 10 and terms of order X equated, it is found that:

$$\frac{d^3f_1}{dY^3} + \theta_1 + Y\frac{d^3f_0}{dY^3} + \frac{d^2f_0}{dY^2} = 0, \tag{28}$$

$$\frac{d^2\theta_1}{dY^2} + 3f_0\frac{d\theta_1}{dY} - \frac{df_0}{dY}\theta_1 + 4f_1\frac{d\theta_0}{dY} + \frac{d\theta_0}{dY} + Y\frac{d^2\theta_0}{dY^2} = 0. \tag{29}$$

Table 2 gives the solution to Equations 28 and 29 which satisfies the conditions: On $Y = 0$:

$$f_1 = \frac{df_1}{dY} = \theta_1 = 0,$$

and at $Y = \infty$,

$$\frac{df_1}{dY} = 0, \quad \theta_1 = 0.$$

Table 2. Numeria solution of F_0.

Y	f_o	$\frac{df_0}{dY}$	θ_0	f_1	$\frac{df_1}{dY}$	θ_1	f_2	$\frac{df_2}{dY}$	θ_2
0	0	0	1.0	0	0	0	0	0	0
0.25	0.023	0.177	0.822	0	0.01	-0.04	-0.01	-0.07	0.04
0.50	0.084	0.302	0.649	0	0	-0.05	-0.03	-0.14	0.08
0.75	0.171	0.387	0.485	0	-0.01	-0.01	-0.08	-0.21	0.12
1.0	0.275	0.441	0.342	0	-0.02	0.04	-0.14	-0.29	0.14
1.25	0.390	0.474	0.225	-0.01	-0.03	0.09	-0.22	-0.38	0.15
1.50	0.511	0.492	0.138	-0.02	-0.02	0.11	-0.33	-0.48	0.13
1.75	0.635	0.502	0.078	-0.02	-0.01	0.12	-0.46	-0.59	0.11
2.0	0.761	0.507	0.041	-0.02	-0.01	0.10	-0.62	-0.70	0.08
2.25	0.890	0.509	0.020	-0.02	0	0.07	-0.81	-0.82	0.05
2.50	1.016	0.510	0.009	-0.02	0	0.04	-1.04	-0.94	0.03
2.75	1.143	0.510	0.003	-0.02	0	0.02	-1.29	-1.07	0.02
3.00	1.271	0.511	0.001	-0.02	0	0.01	-1.57	-1.19	0.01
3.25	1.399	0.511	0	-0.02	0	0.01	-1.88	-1.32	0
3.50	1.526	0.511	0	-0.02	0	0	-2.23	-1.45	0

MATCHING PROCEDURE OF THE INNER AND OUTER REGIONS

In this section, the inner temperature region and outer momentum regions are matched asymptotically, see Nayfeh [11], in order to get the velocity profile. Following Van Dyke [12], the inner flow is written in terms of the outer variable and is equated to the outer flow written in terms of the inner variable. The first part is called the inner limit of outer flow and the second part is called the outer limit of inner flow. The inner limit of the outer flow Equation 26 is:

$$u = \frac{U_0}{2}\frac{\partial F}{\partial\eta} = U_0\left[1 - \frac{1}{\beta^*}\ln(1 + XY)\right]$$

$$= U_0\left[1 - \frac{XY}{\beta^*} + O(X^2)\right],$$

when the outer solution is expressed in terms of X, Y. Now the outer limit of the inner flow is:

$$u = U_0.$$

It follows that a correction term of order X/β^* must be added to Equation 27, i.e.:

$$f(X,Y) = f_0(Y) + Xf_1(Y) + \frac{X}{\beta^*}f_2(Y) + \cdots$$

$$\theta(X,Y) = \theta_0(Y) + X\theta_1(Y) + \frac{X}{\beta^*}\theta_2(Y) + \cdots$$

Substituting these expansions into Equations 9 and 10 gives the following equations for $f_2(Y)$ and $\theta_2(Y)$:

$$\begin{cases} \frac{\partial^3 f_2}{\partial Y^3} + \theta_2 = 0 \\ \frac{\partial^2 \theta_2}{\partial Y^2} - \theta_2 \frac{\partial f_0}{\partial Y} + \frac{\partial \theta_0}{\partial Y} f_2 + 3 f_0 \frac{\partial \theta_2}{\partial Y} + 3 f_2 \frac{\partial \theta_0}{\partial Y} = 0 \end{cases}$$

where, for $Y = \infty$:

$$\frac{df_2}{dy} \approx b_1 - a_1 Y, \quad \theta_2 \approx 0,$$

and for $Y = 0$:

$$f_2 = \frac{df_2}{dY} = \theta_2 = 0.$$

Solution to these equations are given in Table 2. From this integration it is found that:

$$b_1 = 0.34.$$

It should be noted that the term in b_1 is unmatched in the outer flow. This defect may be corrected by amending the value of $U_0(x)$ to:

$$U_0(x) = \frac{2\nu}{x}\left(\frac{\mathrm{Gr}}{\mathrm{Pr}}\right)^{1/2}\left[a_1 + \frac{b_1 X}{\beta^*}\right]. \tag{30}$$

BOUNDARY LAYER PROPERTIES

In this section, some of the overall properties of the flow are brought together.

Maximum velocity $= U_0(x)$

$$= \frac{2\nu}{x}\left(\frac{\mathrm{Gr}}{\mathrm{Pr}}\right)^{1/2}\left[a_1 + \frac{b_1 X}{\beta^*} + O(X^2)\right].$$

Rate at which mass is the entrained/unit length of the cylinder:

$$\overset{\circ}{m} = 2\pi\rho\nu x F(\infty)$$

$$= 2\pi\rho\nu x\left[F_0(\infty) + \frac{1}{\beta^*}F_1(\infty) + O\left(\frac{1}{\beta^{*2}}\right)\right]$$

$$= 2\pi\rho\nu x\left[3.078 + \frac{0.154}{\beta^*}\right]. \tag{31}$$

Heat transfer, in this case expressed in terms of the Nusselt number, Nu, defined by:

$$\mathrm{Nu} = -2\pi a\left(\frac{\partial\theta}{\partial r}\right)_{r=a},$$

is most conveniently expressed in terms of the dimensionless number, Nu, defined by:

$$\overline{\mathrm{Nu}} = \frac{x\ \mathrm{Nu}}{2\pi a(\mathrm{Gr.Pr})^{1/4}}$$

$$= -\frac{1}{\sqrt{2}}\left[\frac{d\theta_0}{dY} + X\frac{d\theta_1}{dY} + \frac{X}{\beta^*}\frac{d\theta_2}{dY} + O(X^2)\right]_{Y=0},$$

or:

$$\overline{\mathrm{Nu}} = 0.503 + 0.181X - 0.123\frac{X}{\beta^*}. \tag{32}$$

Note that X, β^* are defined by Equations 7 and 25 respectively.

DISCUSSION

The most important of the above properties is Equation 32, for which it should be noted that the term in X/β^* is negligible for $\mathrm{Pr} > 10^4$ and $X < 1$. It follows that Equation 32 is now sensibly:

$$\overline{\mathrm{Nu}} = 0.503 + 0.186X + O(X^2). \tag{33}$$

It is interesting to compare Equation 33 with the numerical calculations of Fujii and Uehara [3], for $\mathrm{Pr} = 100$, whose results, when expressed in the notation of this paper, give:

$$\overline{\mathrm{Nu}} = 0.489 + 0.051X - 0.003X^2. \tag{34}$$

While the agreement between Equations 33 and 34 cannot be expected to be good, Equation 34 indicates that the error in Equation 33 should be less than one percent even when $X = 1$.

The experiments of Libby [1] were performed on a cylinder of radius 1.85 cm at a vertical height of 7 cm in the ranges of $10^4 < \mathrm{Pr} < 10^6$, $10^4 < \mathrm{Pr.Gr} < 10^8$.

This gives a range of values of X from 0.1 to 1 and of β from 4 to about 12, in which Equation 33 is valid. Libby's results show a linear dependence of Nu on $(\mathrm{Gr.Pr})^{1/4}$ giving a value of $\mathrm{Nu} = 0.47$.

It is surprising that no curvature effects were observed in view of the range of values of X. However, the experimental values for wall temperature gradient were obtained from the temperature profile. Now, consideration of Table 1 shows that the average value of θ_1 in the range $0 < Y < 0.75$, which corresponds roughly to the linear part of the temperature profile, is about 0.2. It follows that the average effect of curvature on the temperature profile in this range is, at most, about 2% (when $X = 1$). Thus, curvature will have no sensible effect on values of wall temperature gradient derived from points on the experimental temperature.

For completeness, the values of f and θ are shown in Figures 4 and 5 for $X = 0.1$ and $X = 0.5$. These

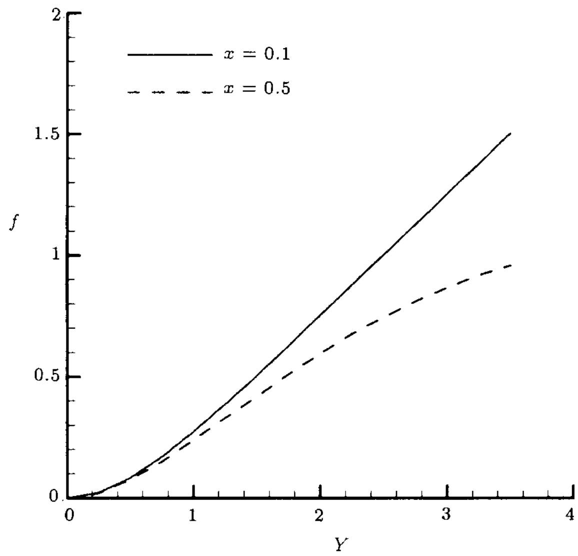

Figure 4. Two-term approximation of f.

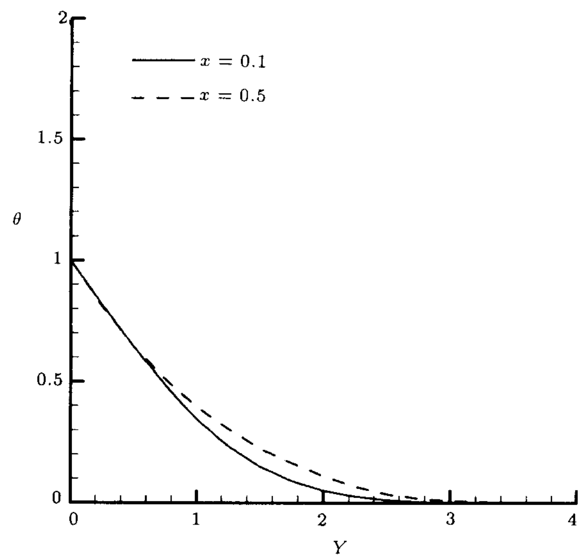

Figure 5. Two-term approximation of θ.

are the corrected values of these quantities. Using this corrected value of f, the composite value of velocity component, u, is calculated and shown in Figure 6 for $X = 0.1$ and $X = 0.5$. As expected, the maximum velocity is increased as $\Pr \to \infty$.

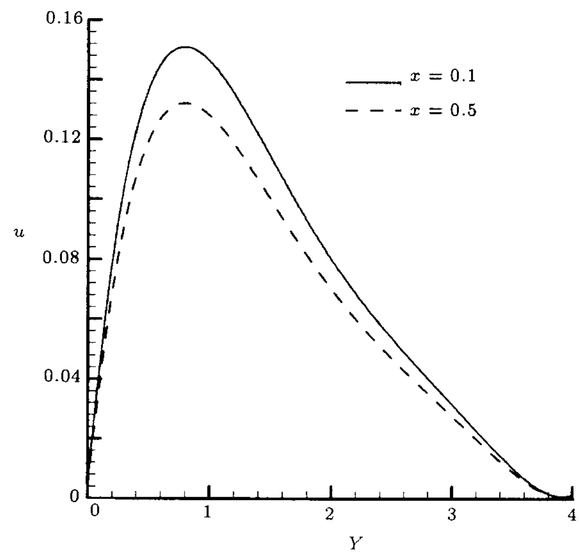

Figure 6. Velocity component for $\Pr = 10^4$.

NOMENCLATURE

a	radius of cylinder
a_1, b_1	constants
A, B, C, D	constants
f	function
F	function
g	gravity
Gr	Grashof number
Nu	Nusselt number
Pr	Prandtl number
T	temperature
u, ν	velocity components
U_0	outer layer velocity
x, r	cylindrical coord
X, Y	inner layer variables
z	outer layer variable

Greek

β_1	volumetric expansion
β	outer layer variable
γ	function
η	similarity variable
θ	dimensionless temp
ν	kinematic viscosity
ψ	stream function

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