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A Near Optimal Midcourse Guidance Law Based on Spherical Gravity

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In [1], an optimal midcourse guidance law for close distances, where the difference of gravity for interceptor and ballistic missile is negligible, was introduced. There, a closed form solution, based on an optimization problem, was found, with very good performance for close distances but degraded performance in real problems with unequal gravity for missile and interceptor. In this paper, the difference of gravity is taken into account by considering a spherical gravity model. A new equation to express the relative motion between missile and interceptor is used to derive a "Near Optimal Guidance Law". The results found using the new derivation are similar to those in [1] but it provides a guidance law with matrix coefficients. Next, the performance of the guidance law is improved using Kepler's algorithm. This modified approach results in an almost perfect intercept, even for large distances between missile and interceptor.

INTRODUCTION

Intercepting a ballistic missile is an important problem in air defense systems where an offensive ballistic missile (called missile) is intercepted by another defensive missile (called interceptor). Recent experience has shown that attacking ballistic missiles after re-entry is not a good strategy. It is much more effective to attack the ballistic missile before the re-entry phase, when the missile is outside the sensible atmosphere and is not maneuvering. For this, usually, an interceptor is fired toward the missile, which, after midcourse guidance, coasts ballistically toward the target and during final engagement fires a kill vehicle to achieve actual intercept.

It is clear that an optimal guidance law for the interceptor during the midcourse phase will minimize missile effort during the final intercept phase. Therefore, interceptor guidance during the midcourse phase is vital to actual intercept. In this paper, a near optimal midcourse guidance law for an interceptor is presented that uses thrust vector control during a fixed maneuvering interval. The important point to consider in this paper is spherical gravity and, based on this consideration, a near optimal guidance law is found for this problem.

The optimal guidance law for constant gravity has been derived in [1] and some non-optimal law for non-constant gravity has been introduced in [2]. The main difference between this paper and [1] is the separate consideration of gravity for the missile and interceptor. This work also differs from [2] in terms of the near optimal guidance law and clear explanation of the results obtained.

The presentation is organized as follows. In the next section, the problem is formulated using a spherical gravity model. Then, a closed form solution for a near optimal guidance law is derived, based on a linearized formula for the relative motion. The dynamical equation derived here is similar to the one given in [1] but with small differences that lead to a guidance law with matrix coefficients. This guidance law is not exact and results in small errors, due to the approximations used in deriving the relative motion. Consequently, the impact error, due to linear approximation, is removed using Kepler's algorithm. This approach leads to a perfect intercept. Finally, sample simulation results are presented and these results confirm the good performance of the improved guidance law, based on Kepler's algorithm.

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PROBLEM FORMULATION

First, the missile (M) and interceptor (I) equation of motion is described by considering the flight regime outside the atmosphere for both objects. Assuming no aerodynamic force, the equation of motion for the missile during the midcourse phase can be described as follows:

$$\dot{R}_M = V_M, \tag{1a}$$

$$\dot{V}_M = G_M. \tag{1b}$$

Here, V_M is the velocity vector, R_M is the position vector and G_M is the gravitational acceleration of the missile.

The equation of motion for the interceptor during the midcourse phase can be described as follows:

$$\dot{R}_I = V_I, \tag{2a}$$

$$\dot{V}_I = G_I + A_I. \tag{2b}$$

Here, V_I is the velocity vector, R_I is the position vector, G_I is the gravity acceleration vector and A_I is the thrust acceleration of the interceptor. The relative position and velocity of the missile is usually required because the objective is to minimize the distance between missile and interceptor. Hence, let the relative position and velocity be denoted by R and V, respectively. Using the definition of the relative quantities, one has:

$$R = R_M - R_I, \tag{3a}$$

$$V = V_M - V_I. \tag{3b}$$

Subtracting Equation 2 from Equation 1, the relative dynamic equation can be written as follows:

$$\dot{R} = V, \tag{4a}$$

$$\dot{V} = G - A_I. \tag{4b}$$

Here, G is the gravity difference and is defined as follows:

$$G = G_M - G_I. \tag{5}$$

In close distances, where R_I is close to R_M, assuming equal gravity for missile and interceptor removes the nonlinear term, G, in the relative equations of motion (Equation 4b). But, for large distances, it is more appropriate to take the gravity difference into account. Note that by assuming an inverse square gravity law, G_I and G_M can be written as follows:

$$G_I = -\frac{\mu}{\mathrm{R}_I^3} R_I, \tag{6a}$$

$$G_M = -\frac{\mu}{\mathrm{R}_M^3} R_M. \tag{6b}$$

Here, μ is the gravity constant and R represents the length of the vector R.

Now, an appropriate objective function is derived for the specific problem. The goal here is to minimize the distance of missile and interceptor at the final time, T_f, by commanding an appropriate acceleration control vector, A_I, during the finite time interval from t to t_f, where t_f denotes the thrust burnout time and t can be any time between 0 and t_f. Now, the objective function can be formulated in terms of the relative positions at final time and acceleration effort during the maneuvering interval, as follows:

$$\min J = \frac{\gamma}{2} R^T(T_f) R(T_f) + \frac{1}{2} \int_t^{t_f} A_I^T(\tau) A_I(\tau) d\tau, \tag{7a}$$

$$\dot{V} = -A_I + G_M - G_I, \quad \dot{R} = V. \tag{7b}$$

Here, T_f is the time of intercept and t_f is the time of burnout. Note that here, $T_f > t_f$, since there is a relatively long coast time after burnout and γ is the weighting factor for this optimization problem.

In order to solve the said optimal control problem, a more tractable formulation should be found for the relative gravity difference, G. For this, the physical problem is considered, where the relative gravity is due to two factors: One is the difference in height and the other is the difference in direction.

First, note that the important term in Equation 5 is the difference in direction of R_I and R_M. One can omit the nonlinear term of the relative gravity equation by setting the length of position vectors with an average quantity. Using this strategy, the nonlinear term can be replaced with a constant term, $R_{a\nu}$, so the relative gravity formula can be rewritten as follows:

$$\frac{1}{\mathrm{R}_I^3} \approx \frac{1}{\mathrm{R}_M^3} \approx \frac{1}{\mathrm{R}_{a\nu}^3}, \quad G = -\frac{\mu}{\mathrm{R}_{a\nu}^3} R. \tag{8}$$

In a real problem, where the geometric positioning of the missile and interceptor is clearly defined, the best value of $\mathrm{R}_{a\nu}$ can be computed using computer simulations. However, the authors' experience shows that the performance of the guidance law is not very sensitive to the value of $\mathrm{R}_{a\nu}$.

NEAR OPTIMAL GUIDANCE LAW

Based on formulas which were defined in the previous section, the optimal control problem of Equation 7 can be written in a new form as follows:

$$\min(J) = \frac{\gamma}{2} R^T(T_f) R(T_f) + \frac{1}{2} \int_t^{t_f} A_I^T(\tau) A_I(\tau) d\tau, \tag{9a}$$

$$\dot{V} = -A_I + E_{a\nu} R, \quad \dot{R} = V, \tag{9b}$$

where:

$$E_{a\nu} = -\mu \frac{1}{\mathrm{R}_{a\nu}^3}. \quad (9c)$$

For obtaining an ordinary optimal control problem where there is only one final time, it is necessary to omit one of the final times, T_f or t_f, from the Formula 9a. For this, the relative position vector at time T_f can be found, with respect to the relative velocity and position vector at time t_f, using a Taylor series expansion at time t_f, as follows:

$$R(T_f) = R(t_f) + \sum_{n=1}^{\infty}\left[\frac{1}{2n!}\left(\frac{d^{2n}}{dt^{2n}}R\right)_{t=t_f} * (T_f - t_f)^{2n}\right] + \sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)!}\left(\frac{d^{2n+1}}{dt^{2n+1}}R\right)_{t=t_f}(T_f - t_f)^{2n+1}\right]. \quad (10a)$$

Using Equation 9b, the derivatives of R can be found by noting that the thrust is zero after burnout. Hence,

$$\left[\frac{d^{2n}}{dt^{2n}}R\right]_{t=t_f} = [E_{a\nu}]^n R(t_f),$$

$$\left[\frac{d^{2n+1}}{dt^{2n+1}}R\right]_{t=t_f} = [E_{a\nu}]^n V(t_f), \quad (10b)$$

Next, the scalar functions f_1 and f_2 are defined in terms of $T_f - t_f$, using the above equation as follows:

$$R(T_f) = f_1 R(t_f) + f_2 V(t_f), \quad (11a)$$

$$f_1 = 1 + \sum_{n=1}^{\infty}\left[\frac{1}{2n!}[E_{a\nu}]^n * (T_f - t_f)^{2n}\right],$$

$$f_2 = \sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)!}[E_{a\nu}]^n (T_f - t_f)^{(2n+1)}\right]. \quad (11b)$$

Note that in [1], the formula given in Equation 11a was used but with f_1 and f_2 computed as follows:

$$f_1 = 1,$$

$$f_2 = T_f - t_f. \quad (12)$$

By defining a new variable, A^*, as follows:

$$A^* = A - E_{a\nu}R, \quad (13)$$

the optimal control problem of Equation 9 becomes identical to the optimization problem in [1]. Therefore, the optimal acceleration has the form of state feedback and can be written in terms of R and V as follows:

$$A^*_{\text{opt}} = K_R R + K_V V, \quad (14)$$

where the optimal feedback gains, K_V and K_R, are computed using the following formulas:

$$K_V(t) = [f_2 + f_1 t_b]^2 * \left[\frac{1}{\gamma} + t_b\left[f_2^2 + f_1 f_2 t_b + f_1^2 t_b^2/3\right]\right]^{-1}. \quad (15a)$$

$$K_R(t) = [f_1 f_2 + f_1^2 t_b]^2 * \left[\frac{1}{\gamma} + t_b\left[f_2^2 + f_1 f_2 t_b + f_1^2 t_b^2/3\right]\right]^{-1}. \quad (15b)$$

t_b is time to burnout, $t_b = t_f - t$.

Now, returning to the problem, the optimal acceleration, A_{opt}, can be computed by solving Equation 13 for A and substituting in Equation 14. Therefore, the optimal acceleration is:

$$A_{\text{opt}} = K_V V + (K_R + E_{a\nu})R. \quad (16)$$

The above guidance law works well when there is not too much change in the altitude of the missile and the interceptor during the coast phase after burnout. But, in actual cases where the duration of the midcourse phase is too long, with large variation in the missile and interceptor altitude, it can result in appreciable error.

To improve performance in these cases, the following formula is used for the relative motion:

$$\frac{d^2}{d_{t^2}}R = E_M R_M - E_I R_I. \quad (17a)$$

Here:

$$E_M = -\frac{\mu}{\mathrm{R}_M^3}, \qquad E_I = -\frac{\mu}{\mathrm{R}_I^3}. \quad (17b)$$

Next, Equation 17 is rewritten in such a way that the right hand side becomes linear in R. For this, first, Equation 17 is rewritten by adding and subtracting $E_I R_M$, as follows:

$$\frac{d^2}{d_{t^2}}R = E_I R + (E_M - E_I)R_M. \quad (18)$$

Now, the last term is multiplied and divided by the inner product of R and R_M and, hence, the right hand side of Equation 17 can be rewritten, as follows:

$$\frac{d^2}{d_{t^2}}R = E_I R + (E_M - E_I)\frac{R \,.\, R_M}{R \,.\, R_M}R_M. \quad (19)$$

Using algebraic vectors in Cartesian coordinates, this relation can be rewritten as follows:

$$\left[\frac{d^2}{d_{t^2}}R\right] = \left[E_I I + \frac{(E_M - E_I)}{[R]^T[R_M]}[R_M][R_M]^T\right][R], \quad (20)$$

here, $\lfloor R \rfloor$ denotes the algebraic vector of R in a Cartesian coordinate.

Now, assuming that the R term (which appears inside the first bracket of the right side of Equation 2), is constant and has a value of R_C, and R_I is assumed to have a value of $R_M - R_C$, the term inside the bracket becomes a function of time and the right hand side of Equation 20 becomes linear in R. Thus, one obtains:

$$\left[\frac{d^2}{d_{t^2}}R\right] = E[R], \tag{21a}$$

$$E = \left[_{\hat{E}_I} * I + \frac{(E_M - \hat{E}_I)}{[R_C]^T[R_M]}[R_M][R_M]^T\right]. \tag{21b}$$

Here:

$$\hat{E}_I = \frac{-\mu}{||R_M - R_C||^3}.$$

Note that the E matrix defined in Equation 21b is a symmetric matrix and a reasonable estimate for R_M and R_C appearing inside the bracket is the value of R_M and R at the beginning of the guidance period.

Based on the equation of motion appearing in Equation 21, the optimal solution of the last section can be modified using the following procedure. First, note that R at time T_f can be described in terms of V at time t_f and R at time t_f, using the following relation:

$$R(T_f) = F_1 R(t_f) + F_2 V(t_f), \tag{22}$$

where F_1 and F_2 are the matrix equivalent of f_1 and f_2 appearing in Equation 11b. Using a Taylor series expansion, the following relation can be derived for F_1 and F_2:

$$F_1 = I + \sum_{n=1}^{\infty}\left[\frac{1}{2n!}E^n(T_f - t_f)^{2n}\right], \tag{23a}$$

$$F_2 = \sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)!}E^n(T_f - t_f)^{(2n+1)}\right]. \tag{23b}$$

Note that these relations are, in principle, generated from Equation 11b by substituting the matrix E for the scalar E_{av}.

Note that solving the resulting optimal problem in this case is much more difficult than solving the problem in the last section. To circumvent this difficulty, the optimal guidance law of Equation 14 is used and the matrices F_1 and F_2 are simply substituted for the scalars f_1 and f_2 computed in Equation 23. Using this substitution, the following guidance law is obtained:

$$A_{\text{opt}} = K_V V + (K_R + E)R, \tag{24}$$

where the matrices K_V and K_R are computed using the following relations:

$$K_V(t) = \left[F_2 + F_1 t_b\right]^2 * \left[\frac{1}{\gamma}I + t_b\left[F_2^2 + F_1F_2t_b + F_1^2t_b^2/3\right]\right]^{-1}. \tag{25a}$$

$$K_R(t) = \left[F_1F_2 + F_1^2 t_b\right]^2 * \left[\frac{1}{\gamma}I + t_b\left[F_2^2 + F_1F_2t_b + F_1^2t_b^2/3\right]\right]^{-1}. \tag{25b}$$

Clearly, there is an approximation in deriving this guidance law. The error due to this approximation results in little error in actual intercept.

Note that the guidance law proposed here is similar to the one appearing in the last section, the main difference being in the form of the state feedback gains. Here, the gains are matrices but the gains appearing in the guidance law of the last section are scalars.

Simulation results show that the performance of this guidance law is much better than the guidance law proposed in the previous section. The impact error is usually less than 100 m for a coasting distance of 1000 km. However, the guidance law of the previous section results in impact error of about 6 km for the same problem. Of course, even this small error can be removed using a procedure based on predicting the miss distance using Kepler's equation. This procedure is outlined in the next section.

Improved Miss Calculation Formula

In the guidance law of the previous section, relative motion after interceptor burnout was computed using the approximate equation of motion given in Equation 22. The error of this equation during this free fall phase is the major cause of impact error. Basically, the guidance is trying to remove the impact error computed using approximate Equation 22 and is quite successful in doing so.

To remove impact error, the error due to the motion computation must be removed during the free fall phase. For this, the position of the missile is computed using Kepler's equation [3] and an estimated value for the impact time. A similar technique is also used to compute the interceptor position at the estimated impact time. However, for the interceptor, it is assumed that the missile burnout time is the present time during each computation cycle. Note that in order to use Kepler's equation, it is necessary to consider this assumption. Of course, the accuracy of this computation increases as the interceptor gets closer to burnout time and, basically, there will be no error at actual interceptor burnout time.

After computing the missile and interceptor position at the estimated impact time, miss distance, which is the difference between these two position values, is computed using the following simple relation:

$$R_{\text{miss}} = R_M(T_f) - R_I(T_f). \tag{26}$$

Now, there are two options for removing the miss distance. The first technique is to multiply the computed miss distance by an appropriate gain and command this as an acceleration to the interceptor. For the gain in this case, the gains computed in [1] can be used. A procedure similar to this is outlined in [2] but the gain is not optimal. Note that the computed acceleration, here, is not necessarily optimal by any means, because the gains computed in [1] are based on a flat earth approximation.

In the second technique, the miss distance, R_{miss}, is used to compute a residual acceleration command that is added to the acceleration command computed, using the near optimal guidance law proposed in the previous section. This residual acceleration is computed as follows. First, an estimated miss distance is computed using the approximate Relation 22 based on the Taylor series expansion at the present time. Let this miss distance be denoted by $\hat{R}_{\text{miss}}$. Next, the true miss distance is computed using Kepler's algorithm, denoted as R_{miss}. So, the error is the difference of estimated and true miss distance, which can be used for removing the miss completely, as follows:

$$\Delta R_{\text{miss}} = R_{\text{miss}} - \hat{R}_{\text{miss}}. \tag{27}$$

Now, the residual acceleration is computed by multiplying the residual miss, ΔR_{miss}, by gain values introduced in [1], as follows:

$$\Delta A_I = \frac{t_g}{\frac{1}{\gamma} + t_b(t_g^* t_g + t_g^* t_b + t_b^* t_b/3)} \Delta R_{\text{miss}}. \tag{28}$$

Here, t_g is time to go until intercept and is defined as, $t_g = T_f - t$.

It is very interesting that by using this technique, the error of the guidance law can be removed completely and, also, a near optimal profile for the acceleration command can be achieved.

SIMULATION RESULTS

The example of [2] is used as a bench mark for comparing the performance of different guidance laws proposed in this paper.

Let it be assumed that the initial position and velocity of the missile and interceptor are as follows:

$$R_M(t=0) = (1000000, 0, 6800000)[\text{m}],$$

$$V_M(t=0) = (-4000, 0, 200)[\text{m/sec}],$$

$$R_I(t=0) = (0, 0, 6500000)[\text{m}],$$

$$V_I(t=0) = (600, 0, 400)[\text{m/sec}].$$

Also, set the weighting factor γ, appearing in Equation 9, to 100 and assume the nominal burnout time, t_f, and impact time, T_f, are 25 and 150 seconds, respectively.

A simple computation shows that the initial distance between missile and interceptor is about 1000 km, which is quite large. In this section, the performance of three guidance laws are to be compared under these conditions.

First, the optimal guidance law in [1] is applied to the problem in hand. In this law, it is supposed that the effect of gravity is constant for a ballistic missile and interceptor.

Figure 1 illustrates the relative distance between ballistic missile and interceptor. The miss distance, R_{miss}, is around 6000 meters in this case. This large error is mainly due to gravity variation during the interceptor trajectory. Also, in Figure 2, the profile of acceleration is shown for this example.

The second simulation uses the near optimal guidance law with matrix gains. In this guidance law, the effects of gravity variation are considered in the relative equation of motion. Therefore, an improved performance is expected, which can be seen in Figures 3 and 4.

Figure 3 shows the relative distance in this case. As is clear, R_{miss}, in this case, is reduced from 6000 m

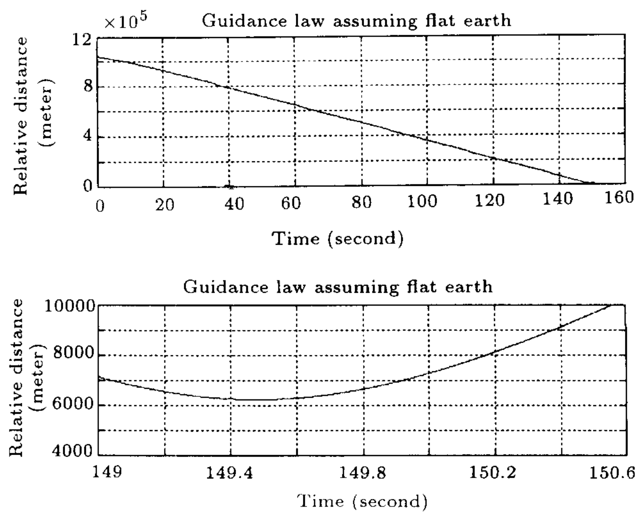

Figure 1. Relative distance using guidance law and assuming flat earth.

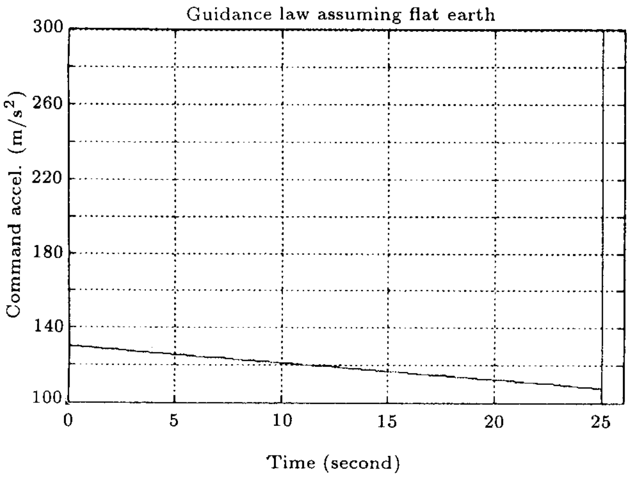

Figure 2. Thrust acceleration using guidance law and assuming flat earth.

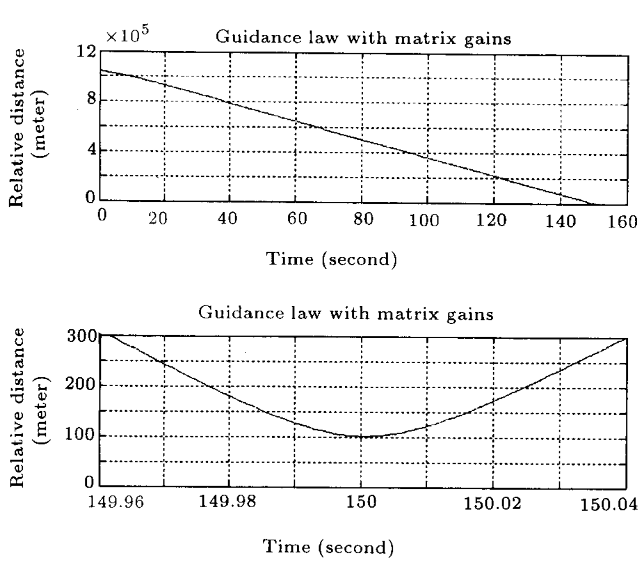

Figure 3. Relative distance using guidance law with matrix gains.

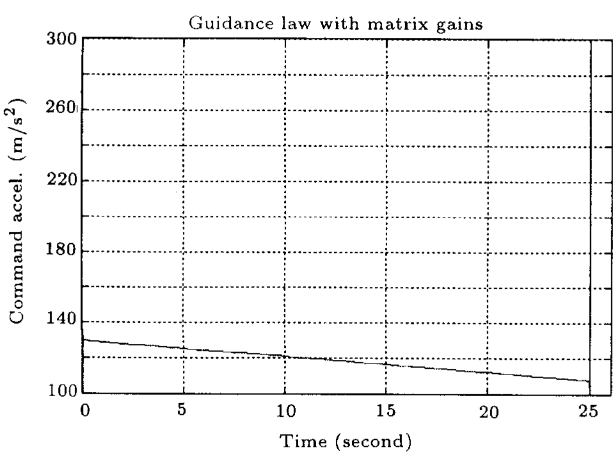

Figure 4. Thrust acceleration using guidance law with matrix gains.

to less than 100 m, computed in the first simulation. Also, Figure 4 shows the profile of acceleration in this example.

Third, Kepler's equation is used to compute the real miss and remove error using the procedure outlined previously. Since, in this case, gravity effects are taken into account completely, an even better performance is expected.

Figure 5 shows the relative distance between ballistic missile and interceptor in this case, as well as the profile of acceleration in Figure 6. It can be seen that, here, R_{miss} is reduced to less than 1 meter and has a better acceleration profile. This is a very good result. Therefore, by applying this new guidance law, an almost perfect intercept can be achieved with less effort in burnout time.

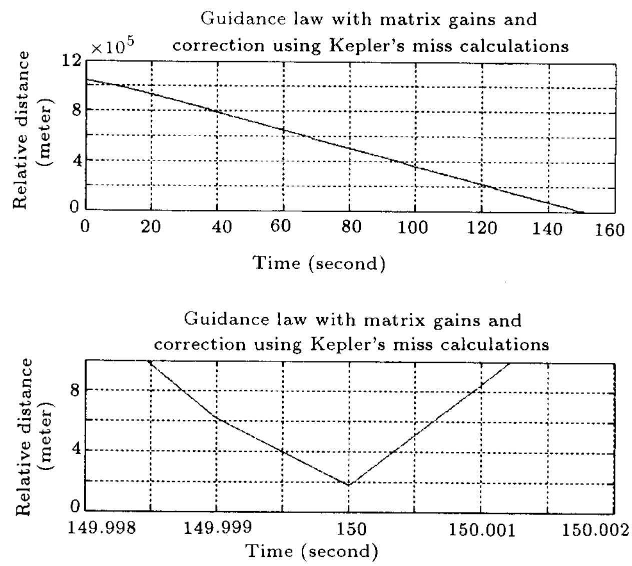

Figure 5. Relative distance using matrix guidance gains and Kepler's miss calculation.

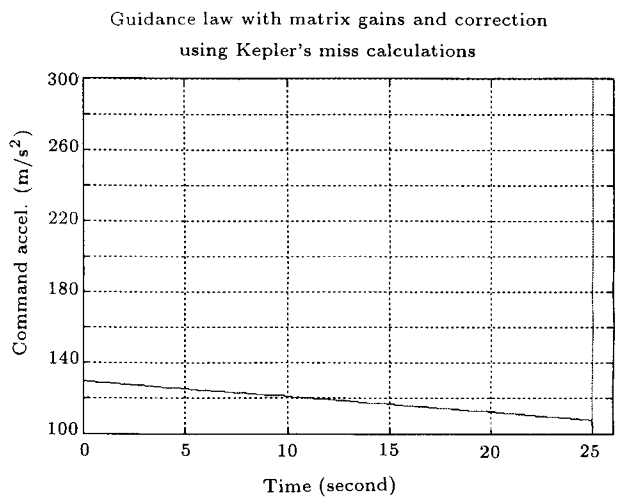

Figure 6. Thrust acceleration using matrix guidance gains and Kepler's miss calculation.

CONCLUSIONS

The simulation results illustrate the excellent capability of "Near Optimal Guidance Law" and "Improved Miss Calculation Formula" in reducing the relative distance between a ballistic missile and an interceptor at impact time. The improved 'miss' calculation can also be used for the optimal guidance law of [1] and will reduce the 6000 m 'miss' to less than 1 meter, the same as the result obtained for the near optimal guidance law that reduces 100 m 'miss' to less than 1 meter.

Therefore, in cases where one can compute the 'miss' distance using Kepler's algorithm, there is not too much difference between the guidance law proposed here and the one given in [1]. But, in cases where one cannot directly compute the 'miss' distance, the guidance law, proposed here, has superior performance.

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