Scientia Iranica, Vol. 12, No. 4, pp 409–413

SCIENTIA
IRANICA

Research Note

Solving One Problem of Diffusion by Multiple Laplace Transforms

H.G. Hassanov[1]

In this paper, a new approach for solving partial differential equations by means of multiple Laplace transforms is developed. The theorem regarding the independence of the final image (final original) on the sequence of realizing the transforms is proved. The diffusion equation with delay is analytically exactly resolved. An algorithm of the solution is given for cases $\xi >> \gamma$ and arbitrary values of parameter γ. It has been shown what changes in solution take place for problems of diffusion with a moving boundary. The solution may be used for most problems with a delay argument.

INTRODUCTION

In [1] it has been shown that such a parameter takes on principally important significance in the course of drilling at the functioning of a stem as a drill cutting holdup at optional points of the well, z_i, in comparison with the moment of its development in the bottom hole. In other words, in solving the diffusion equation, it is necessary to take into account that the required function of the drilling mud density distribution along the well depth, $\Delta\rho(t,z)$, is, in fact, the complicated function $\Delta\rho(t-\tau,z)$, where τ is the delay time ($\tau = z/v$). A similar problem is the partial case of a more general class of equations with delay arguments. Currently existing methods of mathematical physics do not yield an exact solution to the problem. The method of multiple Laplace transforms developed herein gives an opportunity to get, at least in principle, an analytically correct solution to the problem studied. It is worthwhile to note that a review of existing literature revealed a series of papers, for example [2,3], where multidimensional Laplace transforms have already been introduced. However, in cited papers, there are a few disadvantages which do not allow practical realization of the transforms. Firstly, transforms used in [2,3] are discrete. Secondly, the obtained solutions to the considered problems are tentative and the degree of the accuracy rises by an increase of R (R is the number of artificially introduced variables). Thirdly, solved differential equations are sufficiently restricted by given conditions, so that the value of the acquired results is, essentially, limited. In this matter, the method developed by the authors in the present article is universal, since the method does not contain the above defects.

MATHEMATICAL FORMULATION OF THE PROBLEM

In this method, the Laplace transform is consecutively (two or more times) performed upon the required function by various independent variables. In other words, if each Laplace transform were marked by L_{x_1}, where x_i are independent variables included into the considered equation ($i = 1, 2\cdots$), then, the total Laplace transform can be symbolically represented as:

$$L_{\text{tot}} = L_{x_1}.L_{x_2}\cdots = \prod_{i=1}^{n} L_{x_i},$$

in which the upper limit, n, is defined by conditions of a concrete problem. As a result of such a procedure, the number of variables in each partial differential equation decreases by means of each Laplace transform.

Let one take $f(z,t)$ as a function of two real variables, z and t, when, $0 \leq z < +\infty, 0 \leq t < +\infty$, that may be integrated within any intervals in the Lebesgue meaning. Then, the expression:

$$F(\xi,s) = \int_0^\infty \int_0^\infty f(z,t)e^{-\xi z}e^{-\text{st}}dzdt,$$

will be called the double Laplace integral and the function $F(\xi,s)$ is the double Laplace image (transform)

1. *Azerbaijan State Oil Research and Project Institute (AzNSETLI), Aga-Neymatullah Str. 39, Baku, Azerbaijan.*

of the function $f(z,t)$. The magnitudes s and ξ are the time and coordinate parameters of the Laplace transform, respectively. The function:

$$G_1(\xi,t)=\int_0^\infty f(z,t)e^{-\xi z}dz, \tag{1}$$

will be called an intermediate Laplace image of the function $f(z,t)$ by coordinate, but, the function:

$$G_2(z,s)=\int_0^\infty f(z,t)e^{-\mathrm{st}}dt, \tag{2}$$

is called an intermediate Laplace image of the function $f(z,t)$ by time. Generally speaking, the intermediate images, $G_1(\xi,t)$ and $G_2(z,s)$, by various variables, are different. One can prove a very important theorem for application of the theory of multiple Laplace transforms.

Theorem

Independently, in the sequence of realizing integral Laplace transforms by various variables in Equations 1 and 2, the final image, $F(\xi,s)$, should be the same.

The proof of this theorem is sufficiently simple. Let two different transforms be formally introduced as:

$$K_1(\xi,s)=e^{-\xi z}e^{-\mathrm{st}}, \qquad K_2(s,\xi)=e^{-\mathrm{st}}e^{-\xi z},$$

and the final images be marked in the following way:

$$F(\xi,s)=\int_0^\infty\int_0^\infty f(z,t)K_1(\xi,s)dzdt$$

$$=\int_0^\infty\int_0^\infty f(z,t)e^{-\xi z}e^{-\mathrm{st}}dzdt,$$

and:

$$F(s,\xi)=\int_0^\infty\int_0^\infty f(z,t)K_2(s,\xi)dtdz$$

$$=\int_0^\infty\int_0^\infty f(z,t)e^{-\mathrm{st}}e^{-\xi z}dtdz.$$

The authors task is to prove the equivalence of the functions $F(\xi,s)$ and $F(s,\xi)$. Certainly, due to the permutation relations during integration, the next formula should be valid:

$$\int_0^\infty f(z,t)e^{-\mathrm{st}}dt\int_0^\infty e^{-\xi z}dz=\int_0^\infty f(z,t)e^{-\xi z}dz\int_0^\infty e^{-\mathrm{st}}dt.$$

Then, introducing the values of intermediate images from Equations 1 and 2, one gets:

$$\int_0^\infty G_2(z,s)e^{-\xi z}dz=\int_0^\infty G_1(\xi,t)e^{-\mathrm{st}}dt.$$

i.e., the identity of the final image:

$$F(\xi,s)=F(s,\xi). \tag{3}$$

In finding the function - original $f(z,t)$ from the final image $F(\xi,s)$, one needs to use the next formula of the inverse Laplace transforms, taken as a generalization of the known relationship [4]:

$$f(z,t)=\left(\frac{1}{2\pi i}\right)^2\frac{\partial}{\partial z}\frac{\partial}{\partial t}\int_{\phi-i\infty}^{\phi+i\infty}\int_{\delta-i\infty}^{\delta+i\infty}F(\xi,s)\frac{e^{\xi z}e^{\mathrm{st}}}{\xi s}d\xi ds. \tag{4}$$

For inverse transformations, one can introduce the concept of intermediate originals and consider their connection with appropriate intermediate images. Unfortunately, these interesting questions of the theory of multiple Laplace transforms are the subject of separate investigations and cannot be completely elucidated in the present paper. However, one important theorem should be kept in mind. This is the theorem about realization of the independence of the final original on the sequence of the inverse Laplace transforms. This property is important because it is necessary to make sure, after integral transforms, that the required function is univalently determined. If the function:

$$g_1(z,s)=\frac{1}{2\pi i}\frac{\partial}{\partial z}\int_{\delta-i\infty}^{\delta+i\infty}F(\xi,s)\frac{e^{\xi z}}{\xi}d\xi, \tag{5}$$

is called as the intermediate original of the image $F(\xi,s)$ by coordinate and:

$$g_2(\xi,t)=\frac{1}{2\pi i}\frac{\partial}{\partial t}\int_{\phi-i\infty}^{\phi+i\infty}F(\xi,s)\frac{e^{\mathrm{st}}}{s}ds, \tag{6}$$

the appropriate intermediate original of the same image by time, then, these originals are different, although the final original, $f(z,t)$, should be the same, independent of the sequence of realizing these transforms. The last statement is easily proved, as the above theorem, since, due to the permutation relations of the considered integrals, one has, from Equation 4, the following:

$$\frac{1}{2\pi i}\frac{\partial}{\partial t}\int_{\phi-i\infty}^{\phi+i\infty}F(\xi,s)\frac{e^{\mathrm{st}}}{s}ds.\frac{1}{2\pi i}\frac{\partial}{\partial z}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{\xi z}}{\xi}d\xi$$

$$=\frac{1}{2\pi i}\frac{\partial}{\partial z}\int_{\delta-i\infty}^{\delta+i\infty}F(\xi,s)\frac{e^{\xi z}}{\xi}d\xi.\frac{1}{2\pi i}\frac{\partial}{\partial t}\int_{\phi-i\infty}^{\phi+i\infty}\frac{e^{\mathrm{st}}}{s}ds. \tag{7}$$

In practice, the transition from the final image, $F(\xi,s)$, to the final original, $f(z,t)$, is realized by the same tables of the operational calculus. In some cases, the transition may be realized by parameter s (another parameter ξ is accepted to be permanent), then, by ξ. The choice of which transition should be firstly made is not a question of principle, but of convenience, due to Equation 7.

SOLUTION OF A DIFFUSION EQUATION WITH DELAY

For the drilling process, the diffusion equation may be represented as:

$$\frac{\partial\rho(z,t-\tau)}{\partial t}+v\frac{\partial\rho(z,t-\tau)}{\partial z}=\Delta. \tag{8}$$

Hereafter, v = velocity of liquid convection and Δ = source function, which characterizes the entrance of cuttings (any strange substance) into the well. As a rule, values of v and Δ are estimated to be constant under $\rho(z,t-\tau)$ and one means the increment of the drilling mud density with dependence on coordinate z and time t, with an account of delay τ. For simplification, let one consider the case $\tau=\text{const}$. For Equation 8, the time Laplace transform, L_t, will firstly be realized and the equation in complex plane (z,s) will be obtained as:

$$e^{-s\tau}s\rho^*(z,s)+e^{-s\tau}v\frac{d\rho^*(z,s)}{dz}=\frac{\Delta}{s}, \tag{9}$$

in deducing Equation 9, one uses the initial condition $\rho(z,t=0)=0$. Taking into consideration that $\tau=z/v$, after certain calculations, one obtains:

$$e^{-\gamma z}s\rho^*(z,s)+e^{-\gamma z}v\frac{d\rho^*(z,s)}{dz}=\frac{\Delta}{s}. \tag{10}$$

Herein, $\gamma=s/v$. The next step is to apply for Equation 10, the coordinate Laplace transform L_z. Subsequently:

$$s\frac{\xi}{\xi+\gamma}\rho^{*\circ}(\xi+\gamma,s)+v\Big\{\xi\rho^{*\circ}(\xi+\gamma,s)-\rho_\infty\frac{\chi}{s(s+\chi)}\Big\}=\frac{\Delta}{s\xi}. \tag{11}$$

In obtaining Equation 11, the following basic relationships of operational calculus:

$$\rho^{*\circ}=\rho^{*\circ}(\xi,s)=\int_0^\infty\int_0^\infty\rho(z,t)e^{-\xi z}e^{-\mathrm{st}}dzdt,$$

$$\rho^{*\circ}(\xi+\gamma,s)=\frac{\xi}{\xi+\gamma}\int_0^\infty\int_0^\infty\rho(z,t)e^{-\xi z}e^{-\gamma z}e^{-st}dzdt,$$

have been used, as well as the boundary condition, which is given in view:

$$\rho(z=0,t)=\rho_\infty(1-e^{-\chi t}), \tag{12}$$

and which corresponds to experimentally observed law. The magnitudes ρ_∞ and χ are empirically determinable. From a physics standpoint, the value of χ characterizes the relaxation of density at the point $z=0$ (bottom zone in the drilling process). In the first approximation, for solving Equation 11, let one take $\xi>>\gamma$, so that the image $\rho^{*\circ}(\xi+\gamma,s)$ could be substituted for:

$$\rho^{*\circ}(\xi+\gamma,s)\approx\rho^{*\circ}(\xi,s). \tag{13}$$

Formally, the last relationship can be obtained in the following way. First, expand the function $\rho^{*\circ}(\xi+\gamma,s)$ in a series, by degrees, of the parameter γ:

$$\rho^{*\circ}(\xi+\gamma,s)=\rho^{*\circ}(\xi,s)+\frac{\partial\rho^{*\circ}(\xi,s)}{\partial\xi}\gamma+\frac{1}{2!}\frac{\partial^2\rho^{*\circ}(\xi,s)}{\partial\xi^2}\gamma^2+\cdots. \tag{14}$$

Under infinitesimal values of γ (in the original, this corresponds to $t\to\infty$), one can neglect all the componants containing any degree of parameter γ and yield Formula 13. Inserting Formula 13 into Equation 4, one ultimately acquires:

$$s\rho^{*\circ}(\xi,s)+v\left\{\xi\rho^{*\circ}(\xi,s)-\rho_\infty\frac{\chi}{s(s+\chi)}\right\}=\frac{\Delta}{s\xi}. \tag{15}$$

From the last equation, function $\rho^{*\circ}(\xi,s)$ may be elementarily found as:

$$\rho^{*\circ}(\xi,s)=\frac{\frac{\Delta}{s\xi}+v\rho_\infty\frac{\chi}{s(s+\chi)}}{s+v\xi}. \tag{16}$$

According to the conclusions of the previous section, returning the original $\rho^{*\circ}(\xi,s)$ from Equation 16 may be realized via any method. For convenience, in this case, let one firstly realize $[L_z]^{-1}$, assuming s to be constant:

$$\rho^*(s,z)=\frac{\Delta}{s}(1-e^{-sz/v})+\frac{\rho_\infty\chi}{s+\chi}e^{-sz/v}. \tag{17}$$

Now, for the intermediate original (Equation 17), it is necessary to make the inverse Laplace transform by time $[L_t]^{-1}$. For this reason, the basic operational relations should be used, as well as the Duhamel integral. The ultimate result is written without intermediate computations as:

$$\rho(z.t)=\Delta\left(\frac{z}{v}\right)+\rho_\infty(1-e^{-\chi t}), \tag{18}$$

which is in full accordance with the boundary condition (Equation 12). An increase in the function $\rho(z,t)$ by the coordinate z, in the direct neighborhood of point $z=0$, may be explained by the convective transfer of cuttings by liquid with velocity v.

The obtained solution (Equation 18) is justified only if there is the inequality $\xi >> \gamma$. In an original plane, the condition appropriate to this inequality should be in the following form:

$$vt_0 >> z_0, \tag{19}$$

i.e., the mud density distribution (Equation 18) is valid only for the pair of value (z_0, t_0), which obey the Condition 19.

The solution of Equation 18 is a solution of the zero order of the parameter γ. Let a more general case of the problem be considered, where one could not neglect all degrees of γ. Assuming that one should take into account the componants with $j=1$ in the Expansion 14, then, one obtains:

$$\rho^{*\circ}(\xi+\gamma, s) = \rho^{*\circ}(\xi, s) + \frac{\partial \rho^{*\circ}(\xi, s)}{\partial \xi}\gamma. \tag{20}$$

Having inserted the last formula into the basic Equation 11, one obtains an ordinary differential equation of the first order relative to variable ξ, namely:

$$\frac{d\rho^{*\circ}}{d\xi}\gamma\xi\left[\frac{s}{\xi+\gamma}+v\right]+\rho^{*\circ}\xi\left[\frac{s}{\xi+\gamma}+v\right]=\frac{\Delta}{s\xi}+v\rho_\infty\frac{\chi}{s(s+\chi)}.$$

After simple changes, the last equation may be reduced to the following:

$$\frac{d\rho^{*\circ}}{d\xi} + \frac{1}{\gamma}\rho^{*\circ} = \frac{K(\xi, s)}{\xi\left[\frac{s}{\xi+\gamma}+v\right]}. \tag{21}$$

Herein, the function $K(\xi,s)$ has the following view:

$$K(\xi, s) = \frac{\Delta}{s\xi} + v\rho_\infty \frac{\chi}{s(s+\chi)}.$$

As known, ordinary differential equations relative to imaginary variables ξ (or s) in the Laplace transform, should be resolved just like in cases of real variables [4,5]. Taking into account that during integration by ξ the parameter s is accepted to be permanent, one can yield the following for the final image:

$$\rho^{*\circ} = e^{-\xi/\gamma}\left\{\int_C \frac{K(\xi, s)}{\xi\left[\frac{s}{\xi+\gamma}+v\right]} e^{\xi/\gamma} d\xi\right\}. \tag{22}$$

Therefore, function $K(\xi,s)$ is analytic in the simply connected region, D, of the imaginary plane (ξ,s) and C is the curve laying within the region, D. Finding the last integral is not a difficult task, since, according to the Cauchy theorem, there is a numerous set of curves C_i, which lays in the same region and possesses equal end points $(a-ib, a+ib)$, Re$\xi >$ a > 0, so that :

$$\rho^{*\circ} = e^{-\xi/\gamma}\left\{\int_C \frac{K(\xi, s)}{\xi\left[\frac{s}{\xi+\gamma}+v\right]} e^{\xi/\gamma} d\xi\right\}$$
$$= e^{-\xi/\gamma}\left\{\int_{a-i\infty}^{a+i\infty} \frac{K(\xi, s)}{\xi\left[\frac{s}{\xi+\gamma}+v\right]} e^{\xi/\gamma} d\xi\right\}. \tag{23}$$

After finding the integral in Equation 23, one has to return to the final original by means of the inverse transforms $[L_z]^{-1}$ and $[L_t]^{-1}$ in any sequence and define $\rho(z,t)$.

Hence, giving, beforehand, the degree of the parameter γ, one can find the image $\rho^{*\circ}(\xi,s)$ with desired accuracy and define the function of the mud density, $\rho(z,t)$, corresponding to each case.

SOLVING THE PROBLEM WITH A MOVING BOUNDARY

The condition $\tau =$ const. is greatly idealized. In practice, this magnitude depends upon time and such a circumstance takes place due to the shift of the boundary $z=0$ (the well bottom) while drilling. Thus, it is natural to accept $\tau = \tau(t)$. Let one elucidate as to how this aspect effects the distribution $\rho(z,t)$. The velocity of the above boundary change is taken as permanent and equal to w. In this case, the delay time, τ, is determined as:

$$\tau = \frac{z+wt}{v} = \frac{z}{v} + \frac{w}{v}t = \tau_0 + \frac{w}{v}t, \tag{24}$$

where τ_0 is the initial delay time described in the previous section. Then, a new variable, θ, is introduced and marked as $\theta = t-\tau$ and, ultimately, the following is obtained:

$$t = \frac{\theta+\tau_0}{1-\frac{w}{v}}. \tag{25}$$

In terms of the new variable, the normalization factor, e^{-st}, in the Laplace transform will be represented as:

$$e^{-st} = e^{-s\left(\frac{\theta+\tau_0}{\alpha}\right)} = e^{-s\tau_0^*}e^{-s\theta/\alpha}, \tag{26}$$

where the following abbreviations are used:

$$\tau_0^* = \frac{\tau_0}{\alpha}, \qquad \alpha = 1-\frac{w}{v}.$$

Taking into consideration the linear law (Equation 24) of the delay $\tau(t)$ dependence, for the required function

$\rho(z, t-\tau)$, one has the next operational relations:

$$\int_0^\infty \rho(z, t-\tau)e^{-\mathrm{st}}dt = \frac{e^{-s\tau_0^*}}{\alpha}\rho^*(z, s/\alpha),$$

$$\int_0^\infty \frac{\partial\rho(z, t-\tau)}{\partial t}e^{-\mathrm{st}}dt = \frac{e^{-s\tau_0^*}}{\alpha}s\rho^*(z, s/\alpha),$$

the initial condition will be the same as for the previous problem. After realizing L_t, there will be the following equation in view:

$$e^{-s\tau_0^*}\frac{s}{\alpha}\rho^*(z, s/\alpha) + e^{-s\tau_0^*}v\frac{d\rho^*(z, s/\alpha)}{dz} = \frac{\Delta}{s}. \quad (27)$$

Formally, the obtained Equation 27 coincides with Equation 9 for diffusion with a fixed boundary. In other words, the algorithm of the diffusion equation solution developed in the previous section will be valid for this case also. However, now, the image $\rho^*(z, s/\alpha)$ has already been dealt with. So, in solving a problem with a moving boundary by the following law:

$$z = wt,$$

one is able to, ultimately, find the function-image $\rho^{*\circ}(\xi, s/\alpha)$ and, then, by inverse Laplace transforms $[L_z]^{-1}$ and $[L_t]^{-1}$, return to the original $\rho(z, \alpha t)$, taking into account the well-known similarity theorem for the transforms.

CONCLUSION

The method of multiple Laplace transforms developed in this paper allows one to obtain an analytical solution to a whole class of problems with the delay argument. Except for the problem of cuttings distribution in the drilling of mud, which has been developed herein, this class contains, generally, all the problems of convective diffusion with the right side different from zero that characterizes the coming of the strange substance into the volume involved. Moreover, this class of problem is typical of the problems involved in the theory of automatic control, burning in rocket engines, economic and biophysical problems etc. As noted in [6], differential equations, with the delay argument, may be exactly integrated in exceptional cases only. As a rule, for their solution, one uses the numerical methods, for example [7-9]. In the light of what has been described, the method of multiple Laplace transforms providing the principal opportunity for obtaining an analytical solution to such a class of equations, acquires a special meaning. Using it, one can obtain a correct solution to the equations, which has been proved in the current paper and, undoubtedly, the development of this technique essentially expands the possibilities for the mathematical modeling of various processes.

REFERENCES

1. Hassanov, H.G. et al. "Hydraulic bases of perfection of the tailing-in while drilling process", *Annual Proceedings of AzNSETLI*, Baku: AzNSETLI Press, pp 38-50 (2000).
2. Lubbok, J. and Bansal, V. "Multidimensional Laplace transforms for solution of non-linear equations", *Proc. Inst. Electr. Eng.*, **116**(12), pp 2075-2082 (1969).
3. Smyshlyayeva, L.G., *Laplace Transforms of Functions of Multiple Variables*, Leningrad Univ. Press, Leningrad, pp 28-37 (1981).
4. Ditkin, V.A. and Prudnikov, A.P., *Integral Transformations and Operational Calculus*, Nauka, Moskva, pp 31-85 (1974).
5. Trikomi, F., *Differential Equations*, Foreign Literature press, Moskva, pp 112-137 (1962).
6. Norkin, S.B., *Differential Equations of the Second Order with Delay Argument*, Nauka, Moskva, pp 83-105 (1965).
7. Poorkarimi, H. and Wiener, J. "Bounded solutions of nonlinear parabolic equations with time delay", *15th Annual Conference on Applied Mathematics*, Electronic Journal of Differential Equations, University of Central Oklahoma, Conference 02, pp 87-91 (1999).
8. Agamaliyev, A. "On a problem of control for external-differential equation with time delay argument", Materials of Scientific Conference, *The Questions on Functional Analysis and Mathematical Physics*, Azerbaijan, pp 128-131 (1999).
9. Wiener, J. and Heller, W. "Oscillatory and periodic solutions to a diffusion equation of neutral type", *International Journal of Mathematics & Mathematical Sciences*, **22**(2), pp 313-348 (1997).