Darboux Integrability and Zero-Hopf Bifurcation of a Radio-Physical Oscillator System

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Abstract

The radio physical oscillator model is a nonlinear three-dimensional autonomous system of first-order ordinary differential equations with quintic polynomial. This model explains a generator of hard oscillations with coupled to a rechargeable power source and depending on four non negative parameters. The present work is divided into two parts: the first part investigates the Darboux integrability of this system. More particularly, we have determined all the invariant algebraic surfaces as well as all the exponential factors of this system. For any value of parameters, we verify that the radio physical oscillator system does not admit polynomial, Darboux and rational first integrals through the analysis of its Darboux polynomials and its exponential factors, while the second part focuses on the investigation of the periodic solutions that may bifurcate from the zero-Hopf equilibrium points using first-order averaging theory, but unfortunately, the method is unable to provide any information about the periodic solution of the system.

1. Introduction

In many dynamical systems in nature and technology, quasiperiodic oscillations are a common form of dynamical activity [1]. The persistence of quasi periodicity in conservative systems is demonstrated to be almost integrable, underscoring the significance of comprehending quasiperiodic dynamics and possibilities of their transition under parameter variation [2].
A simple minimum physical model used to study the quasiperiodic self-oscillator behavior proposed by Kuznetsov et al [3]. The physical model can be thought of as a capacitor that is charged by a voltage source and then quickly discharged through a resistor. Its dynamics are controlled by autonomous three-dimensional system of quintic differential equations:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \left(\lambda + z + x^2 - \beta x^4\right) y - \omega_0^2 x, \\ \dot{z} &= \mu - x^2, \end{aligned} \tag{1}$$

where x, y and z. are real-valued state variables, $\lambda, \omega_0, \beta, \mu$ are real parameters. Eq. (1) can be interpreted as a storage element coupled to a threshold element in the form of a generator of hard oscillations. The first two equations may describe the dynamics of a generator of hard oscillations, while the third may represent a rechargeable power source. Physically, the model consists of a capacitor that is charged by a voltage source and then rapidly discharged through a resistor. The variable x represents the dynamical state of the self-oscillatory element, while z characterizes the instantaneous state of the storage component. The parameter λ determines the strength of positive feedback, enhancing the oscillator's self-sustained behavior. The parameter μ governs the charging rate of the power source; higher values of μ lead to faster charging, thereby influencing the frequency of bursting or oscillatory behavior. The parameter ω_0 defines frequency of the self-oscillating generator, and β governs the non-linearity strength, reflecting the degree of subcriticality in the bifurcation structure [3].
When oscillatory components with incommensurate frequencies that is, frequencies that are not connected by integer ratios interact in practical environments, quasiperiodic activity naturally results. In [3], a noteworthy instance of this kind of behaviour is shown numerically. This work proposed a straightforward family of autonomous three-dimensional differential systems that function as idealized models of electronic devices with a storage component, usually a capacitor, that is quickly discharged by a threshold-triggered mechanism after being slowly charged via a resistor connected to a voltage source. Moreover, the authors of [3] observed a range of dynamical behaviours in this model through numerical simulations, most notably periodic and quasiperiodic dynamics for appropriate parameter values. In particular, Eq. (1) with $\beta = 1/2$ was introduced in [3] as one of the simplest generators of quasiperiodic oscillations.
The authors in [4], investigated torus formation in a class of three-dimensional radio-engineering oscillators with coupled slow–fast dynamics, identifying multiple routes to quasiperiodicity and highlighting a distinctive mechanism involving the saddle-node bifurcation of a stable cycle and its interaction with equilibrium points and a saddle cycle.
The phenomenon of coexisting hidden attractors is investigated in [5], along with the bifurcations through which they arise and disappear. In particular, this study investigates the emergence and bifurcation mechanisms of coexisting hidden attractors in a three-dimensional autonomous oscillator lacking equilibria, revealing the formation of multiple quasiperiodic and chaotic states through period-doubling cascades, torus bifurcations, and resonance structures, while characterizing their dynamical evolution and associated basins of attraction.

The study in [6] numerically investigated a four-parameter autonomous radio-physical oscillator described by a system of three first-order nonlinear ordinary differential equations. By constructing two-parameter bifurcation diagrams and characterizing the system dynamics using the Lyapunov exponent spectrum, the authors identified regions of periodic, quasiperiodic, and chaotic behavior. Their results revealed organized periodic structures embedded within chaotic domains, resembling Arnold tongues, and demonstrated the significant influence of parameter variations on the global dynamical behaviour of the system.
The study of the dynamics and synchronization properties of a quasiperiodic oscillator subjected to periodic pulse forcing under periodic, quasiperiodic, and weakly chaotic operating regimes are considered in [7]. In this work, the amplitude of the external signal, and its period of the Dirac delta-function and the number of pulses is involved.
Feedback control schemes have been applied to the system to regulate multistability and hidden dynamics. In particular, a time-varying on--off feedback strategy was investigated in [8], where it was demonstrated that, for appropriate values of the feedback strength and control period, coexisting chaotic and periodic attractors can be suppressed, leading first to a mono-stable periodic regime and subsequently to a stable equilibrium.
The verification is applied using bifurcation diagrams, Lyapunov exponents, and basin size analysis. A further special case with $\lambda = 0$ and $\beta = 1/2$ was studied in [9] in the sense of multistability involving tori, resonant cycles, and chaos.

The Darboux theory of integrability plays a crucial role in the qualitative analysis of differential systems by facilitating the identification of Darboux polynomials, exponential factors, and Darboux first integrals. Notably, if the cofactor associated with a Darboux polynomial vanishes, the polynomial itself serves as a polynomial first integral [10]. Understanding and characterizing first integrals in polynomial differential systems is widely recognized as a central and challenging problem in the qualitative theory of differential equations. In two-dimensional systems, the presence of a single first integral enables a complete classification of trajectories within the phase plane. More generally, for an n dimensional system, a comprehensive description of the phase portrait typically demands the existence of $n-1$ functionally independent first integrals. Darboux theory has been successfully applied to a wide variety of system [11–20] demonstrating its power in constructing first integrals and identifying invariant algebraic surfaces. However, no general algorithm exists for the computation of Darboux polynomials [21].
Specifically, we analyze the invariant algebraic surfaces and exponential factors for Eq. (1) to study for existence of polynomial, rational and Darboux first integrals of Eq. (1). The study of the existence of first integrals is an important problem in the qualitative theory of differential systems, because the knowledge of first integrals can help in order to simplify and understand the topological structure of their trajectories. Therefore, their presence or not can be seen as a gauge of a differential system's complexity.
The Eq. (1) has been the subject of extensive research in recent years [3–9], but we note that there is no research on the integrability of the polynomial system.
Another objective of this research is to investigate the existence of a zero-Hopf bifurcation in Eq. (1). In this study, we apply the averaging theory of first order which is a powerful tool for analyzing the behavior of periodic orbits in nonlinear dynamical systems. Many researchers have used this method to detect periodic solutions and analyzing their behavior [22–31]. However sometimes the method cannot give any knowledge about the existence or non-existence of periodic solutions [22] and [23].
We have seen that Eq. (1) is a family of three-dimensional quintic differential systems. However, integrability analysis has not been studied for Eq. (1). In addition, there is no research on the zero-Hopf bifurcation for this system. This is the main objective of the current research.
The vector field χ associated to Eq. (1) is

$$\chi = y\frac{\partial}{\partial x} + \left(\left(\lambda + z + x^2 - \beta x^4\right)y - \omega_0^2 x\right)\frac{\partial}{\partial y} + \left(\mu - x^2\right)\frac{\partial}{\partial z} \tag{2}$$

Let $U \subset \mathbb{R}^3$ be an open set. A non-constant function $f : \mathbb{R}^3 \to \mathbb{R}$ is called a first integral of a vector field χ on U if it remains constant along the trajectories of χ. That is, for any solution $\left(x(t), y(t), z(t)\right)$ of χ defined on U, the function $f\left(x(t), y(t), z(t)\right)$ remains unchanged for all values of t. This condition is equivalent to the requirement that $\chi f = 0$ on U, meaning that f is invariant under the flow of χ. The presence of a first integral in a differential system in R^3 enables the reduction of its study by one dimension. This is the main reason for seeking first integrals.
A polynomial first integral $F = F(x, y, z)$ of Eq. (1) is a function $F \in \mathrm{C}[x, y, z]$ such that

$$y\frac{\partial F}{\partial x} + \left(\left(\lambda + z + x^2 - \beta x^4\right)y - \omega_0^2 x\right)\frac{\partial F}{\partial y} + \left(\mu - x^2\right)\frac{\partial F}{\partial z} = 0 \tag{3}$$

A polynomial $G \in \mathrm{C}[x, y, z]$ is called a Darboux polynomial of the vector field χ if there exists a polynomial $K \in \mathrm{C}[x, y, z]$ such that $\chi G = KG$. The function $K = K(x, y, z)$ is referred to as the cofactor of G. It can be easily shown that, for Eq. (1), the degree of the cofactor of a Darboux polynomial does not exceed 4. If $G \in \mathrm{C}[x, y, z]$ is a Darboux polynomial, then the surface defined by $G(x, y, z) = 0$ is an invariant algebraic surface for the differential Eq. (1). In other words, if an orbit intersects this surface at any point, the entire orbit remains contained within it, and any polynomial first integral corresponds to a Darboux polynomial with a cofactor equal to zero.
A Darboux polynomial associated with Eq. (1) is a polynomial $G \in \mathrm{C}[x, y, z]$ such that

$$y\frac{\partial G}{\partial x} + \left(\left(\lambda + z + x^2 - \beta x^4\right)y - {\omega_0}^2 x\right)\frac{\partial G}{\partial y} + \left(\mu - x^2\right)\frac{\partial G}{\partial z} = KG, \tag{4}$$

where $K \in \mathrm{C}[x, y, z]$ represents the cofactor of G, is a polynomial of degree at most four defined as

$$K = \sum_{i=0}^{4}\sum_{j=0}^{4-i}\sum_{k-0}^{4-i-j} a_{i,j,k} x^i y^j z^k \ , \ a_{i,j,k} \in \mathrm{C} \ . \tag{5}$$

An exponential factor $E(x, y, z)$ for Eq. (1) is a function of the form $E = exp(h_1 / h_2) \in \mathrm{C}$, where h_1. and h_2 are coprime polynomials in $\mathrm{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}]$, satisfying $\chi E = LE$ for some $L \in \mathrm{C}[x, y, z]$ with degree of polynomial L . at most four, known as the cofactor of E.. Thus, E . is an exponential factor of the Eq. (1) that satisfies

$$y\frac{\partial E}{\partial x} + \left(\left(\lambda + z + x^2 - \beta x^4\right)y - \omega_0^2 x\right)\frac{\partial E}{\partial y} + \left(\mu - x^2\right)\frac{\partial E}{\partial z} = LE \ . ,$$

For some polynomial

$$L = \sum_{i=0}^{4}\sum_{j=0}^{4-i}\sum_{k-0}^{4-i-j} b_{i,j,k} x^i y^j z^k \ , \quad b_{i,j,k} \in \mathrm{C} \ . \tag{6}$$

The Darboux first integral G for Eq. (1), whh is a first integral, is expressed as

$$F = G_1^{s_1} \cdots G_p^{s_p} E_1^{r_1} \cdots E_q^{r_q} \tag{7}$$

where $G_1, \ldots, G_p$ are Darboux polynomials and $E_1, \ldots, E_q$ are exponential factors with $\mathrm{s_i}, \mathrm{r_j} \in \mathrm{C}$, $i = 1, \ldots, p$, and $j = 1, \ldots, q$. A rational first integral of Eq. (1) is a rational function F satisfying Eq. (3).

An isolated equilibrium point of a three-dimensional autonomous differential system of is called a zero-Hopf equilibrium point if the associated Jacobian matrix has a zero eigenvalue and a pair of purely imaginary eigenvalues [22].
Darboux integrability is a method used to find first integrals of polynomial differential equations via invariant algebraic surfaces and exponential factors. Also, they are used to study center problem, limit cycles and dynamics at infinity of a system and which can help reduce the dimension of a polynomial differential system and analyze its qualitative behavior. But the Darboux transformation is used to construct exact solutions of integrable nonlinear partial differential equations. However, a Darboux transformation is a technique for producing novel solutions to integrable partial differential equations, such as the and crucial in the study of soliton solutions, Lax pairs and integrable partial differential equations. This explanation is provided so that readers can clearly see the distinction between the two concepts [32] and [33].
Previous numerical studies of the same radio-physical oscillator showed that, although the system is often chaotic, its parameter space also contains regions where the motion becomes regular and periodic. These numerical results naturally raise the question of whether the system could admit polynomial or Darboux first integrals for some parameter values, or whether periodic orbits might appear [6].
In this work, we answer these questions rigorously. We prove that the system has no polynomial or Darboux first integrals for any parameters, and we also show that first-order averaging cannot produce periodic solutions. Our results explain that the numerical periodic regions do not correspond to algebraic integrability or to periodic solutions detectable by averaging.

2. Polynomial First integral

First, we focus on studying the existence of first integrals for Eq. (1) expressed as polynomials.

Theorem 2.1. The Eq. (1) cannot possess a polynomial first integral for any value of parameters.

Proof. Assume that $F = F(x, y, z)$ is a polynomial first integral of Eq. (1), meaning that it satisfies Eq. (3). Now, we consider the following two cases.

i- If $\omega_0 = 0$. Let $F(x, y, z) = \sum_{i=0}^{n} F_i(x, y) z^i$, where each $F_i(x, y)$ is a polynomial in variables x and y. After calculating the coefficient of z^{n+1} in Eq. (3), we get $y\frac{\partial F_n(x, y)}{\partial y} = 0$, this implies that $F_n(x, y) = f_n(x)$, where $f_n(x)$ is arbitrary function only in variable x

Additionally, from calculating the coefficient of z^n in Eq. (3), we obtain

$$y\frac{\partial F_n(x, y)}{\partial x} + \left(\lambda + x^2 - \beta x^4\right)y\frac{\partial F_n(x, y)}{\partial y} + y\frac{\partial F_{n-1}(x, y)}{\partial y} = 0 ,$$

solving this equation, we get $F_{n-1}(x, y) = -y\frac{d}{dx} f_n(x) + f_{n-1}(x)$, where $f_{n-1}(x)$ is arbitrary function in variable x.

Now calculating the coefficient of z^{n-1} in Eq. (3), we obtain

$$y\left(-y\frac{d^2}{dx^2} f_n(x) + \frac{d}{dx} f_{n-1}(x)\right) - \left(\lambda + x^2 - \beta x^4\right) y \frac{d}{dx} f_n(x) + y\frac{\partial}{\partial y} F_{n-2}(x, y) + n\left(\mu - x^2\right) f_n(x) = 0 .$$

Solving the above differential equation, we obtain

$$F_{n-2}(x,y)=(\mu-x^2)n\,f_n(x)\ln(y)+\frac{1}{2}\frac{d}{dx^2}f_n(x)y^2+\frac{d}{dx}f_n(x)(\lambda+x^2-\beta x^4)y-\frac{d}{dx}f_{n-1}(x)y+f_{n-2}(x),$$

where $f_{n-2}(x)$ is arbitrary function in variable x.

Since $F_{n-2}(x,y)$ is a polynomial then $nf_n(x)$ must be zero, thus $F=F_0(x,y)$, substituting F in Eq. (3) and computing the coefficient of z in Eq. (3), we obtain $y\frac{\partial F_0(x,y)}{\partial y}=0$ so $(x,y)=f_0(x)$.

Next, calculating the coefficient z^0. gives $y\frac{\partial f_0(x)}{\partial x}=0$, this implies that $f_0(x)=C$. So, we conclude that $F_0(x,y)=C$, where C is a constant which is trivial. Then there is no polynomial first integral of Eq. (1).

ii. If $\omega\neq 0$. In order to simplify the computation, we will introduce the following weight transformants

$$x=X \quad y=Y \quad z=\varphi^{-1}Z \quad t=\varphi T.$$

Where $\varphi\in \mathrm{R}\setminus\{0\}$. Subsequently Eq. (1) transform into

$$\begin{aligned}
X'&=\varphi Y,\\
Y'&=(\lambda\varphi+Z+\varphi X^2-\varphi\beta X^4)Y-\omega_0{}^2X\varphi,,\\
Z'&=\varphi^2(\mu-X^2),
\end{aligned}\tag{8}$$

where the prime indicates the derivative with regard to the variable T. Now, we define

$$F(x,y,z)=\varphi^n f(X,Y,\varphi^{-1}Z),$$

where n represent the greatest integer such that ϕ^{-n} appear in $f(X,Y,\varphi^{-1}Z)$.

Consider $F=\sum_{i=0}^{n}\varphi^i F_i$, where each F_i is weight homogenous polynomial with weight degree $n-i$ of F and n is greatest weight degree of F with weight exponent $\alpha=(0,0,-1)$.

Consider $F(X,Y,Z)$ is polynomial first integral of the system (8), that is

$$\varphi Y\frac{\partial\sum_{i=0}^{n}\varphi^i F_i}{\partial X}+\left((\lambda\varphi+Z+\varphi X^2-\varphi\beta X^4)Y-\omega_0{}^2X\varphi\right)\frac{\partial\sum_{i=0}^{n}\varphi^i F_i}{\partial Y}+\varphi^2(\mu-X^2)\frac{\partial\sum_{i=0}^{n}\varphi^i F_i}{\partial Z}=0.\tag{9}$$

From calculating φ^0s coefficients in Eq. (9), we get $YZ\frac{\partial F_0}{\partial x}=0$, hence $F_0=f_0(X,Z)$, where $f_0(X,Z)$ is a function of variables X, Z.

From calculating φ^1s coefficients in Eq. (9), we get $Y\frac{\partial}{\partial X}G_0(X,Z)+YZ\left(\frac{\partial}{\partial Y}F_1\right)=0$.

Solvig it, yields $F_1=-\frac{Y\frac{\partial}{\partial X}f_0(X,Z)}{Z}+f_1(X,Z)$, where $f_1(X,Z)$ is any function in variables X, Z.

From calculating φ^2s coefficients in Eq. (9), we get

$$Y\frac{\partial F_1}{\partial X}+(-Y\beta X^4+YX^2-\omega_0^2X+\lambda Y)\frac{\partial F_1}{\partial Y}+YZ\frac{\partial F_2}{\partial Y}+(\mu-X^2)\frac{\partial F_0}{\partial Z}=0.\tag{10}$$

Using the expression of F_0 and F_1 and solving Eq. (10) gives

$$F_2=\left(-\frac{X\omega_0^2\frac{\partial}{\partial X}f_0(X,Z)}{Z^2}+\frac{X^2\frac{\partial}{\partial Z}f_0(X,Z)}{Z}-\frac{\mu\frac{\partial}{\partial Z}f_0(X,Z)}{Z}\right)ln(Y)+\frac{Y^2\frac{\partial^2}{\partial X^2}f_0(X,Z)}{2Z^2}$$
$$-\frac{Y\beta X^4\frac{\partial}{\partial X}f_0(X,Z)}{Z^2}+\frac{YX^2\frac{\partial}{\partial X}f_0(X,Z)}{Z^2}-\frac{Y\frac{\partial}{\partial X}f_1(X,Z)}{Z}+\frac{\lambda Y\frac{\partial}{\partial X}f_0(X,Z)}{Z^2}+f_2(X,Z),$$

where $f_2(X,Z)$ is an arbitrary function in variables X and Z.

Since F_2 must be polynomial, we get $-\frac{X\omega_0^2\frac{\partial}{\partial X}f_0(X,Z)}{Z^2}+\frac{X^2\frac{\partial}{\partial Z}f_0(X,Z)}{Z}-\frac{\mu\frac{\partial}{\partial Z}f_0(X,Z)}{Z}=0$.

Solving it gets $f_0(X,Z)=g_0\left(Ze^{\left(\frac{X^2}{2\omega_0^2}\right)}X^{-\frac{\mu}{\omega_0^2}}\right)$.

This is contradiction with the fact that $f_0(X,Z)$ is a polynomial. This concludes the proof of the theorem.

3 Darboux Polynomials with nonzero cofactor

The main topic of this section is the study of the existence of Darboux polynomials of Eq. (1).

Theorem 3.1 The only irreducible Darboux polynomial of the Eq. (1) is y if and only if $\omega_0 = 0$. Moreover, a Darboux polynomial is y with cofactor $\left(\lambda + z + x^2 - \beta x^4\right)$.

To prove Theorem 3.1, we recall and prove a few auxiliary results.

Lemma 3.2 [15] Let f be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathrm{C}[x, y, z]$. Then f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s} \alpha_j K_j$.

Lemma 3.3 For $\beta \neq 0$, if g is an irreducible Darboux polynomial for Eq. (1) with cofactor

$$K_\phi = \sum_{i=0}^{4} \sum_{j=0}^{4-i} \sum_{k-0}^{4-i-j} a_{i,j,k} x^i y^j z^k \tag{11}$$

$$\begin{aligned} K_\phi &= 2a_{400}x^4 + 2a_{200}x^2 + 2a_{201}x^2 z + 2a_{202}x^2 z^2 + 2a_{310}x^3 y + 2a_{220}x^2 y^2 + 2a_{110}xy + 2a_{111}xyz + 2a_{112}xyz^2 + 2a_{130}xy^3 \\ &+ 2a_{000} + 2a_{001}z + 2a_{002}z^2 + 2a_{003}z^3 + 2a_{004}z^4 + 2a_{020}y^2 + 2a_{021}y^2 z + 2a_{022}y^2 z^2 + 2a_{040}y^4 \end{aligned} \tag{12}$$

Proof. Let $\phi : \mathrm{C}[x, y, z] \to \mathrm{C}[x, y, z]$ be the automorphism, defined by $\phi(-x, -y, z) = (x, y, z)$. Since Eq. (1) is invariant under ϕ, the function ϕg is a Darboux polynomial of Eq. (1) with cofactor $\phi(K)$. Moreover, by Lemma 3.2 $g \cdot \phi g$ is also a Darboux polynomial of Eq. (1) with cofactor $K + \phi(K)$. Therefore, again by Lemma 3.2 the cofactor of G is the $K_\phi = K + \phi(K)$ which is given in Eq. (12).

Lemma 3.4 For $\beta \neq 0$, the coefficients a_{310}, a_{220}, a_{130}, a_{112}, a_{040}, a_{022}, a_{202}, a_{004}, a_{201}, a_{003}, a_{002} of cofactor for Darboux polynomial of Eq. (1) are zero. Moreover, the coefficients $a_{400} = -\dfrac{\beta \cdot m}{2}$, $a_{200} = a_{001} = \dfrac{1}{2}m$, where $m \in \mathrm{N} \cup \{0\}$.

Proof. Consider G a Darboux polynomial of Eq. (1) has the form $G = \sum_{i=0}^{m} G_i(x, y, z)$, where each $G_i(x, y, z)$ is a homogenous polynomial of degree i, and $G_n(x, y, z) \neq 0$.

From the definition of Darboux polynomial, we obtain

$$y\frac{\partial G}{\partial x} + \left(\left(\lambda + z + x^2 - \beta x^4\right)y - \omega_0^2 x\right)\frac{\partial G}{\partial y} + \left(\mu - x^2\right)\frac{\partial G}{\partial z} = K_\phi G. \tag{13}$$

First, calculate homogenous terms with degree $n+4$ in Eq. (13), we get

$$-\beta y x^4 \frac{\partial}{\partial y} G_n(x, y, z) = \left(2x^4 a_{400} + 2x^3 y a_{310} + 2x^2 y^2 a_{220} + 2x^2 z^2 a_{202} + 2xy^3 a_{130} + 2xyz^2 a_{112} + 2y^4 a_{040} + 2y^2 z^2 a_{022} + 2z^4 a_{004}\right) G_n(x, y, z).$$

Therefore $G_n(x, y, z) = y^{-\frac{2\left(x^4 a_{400} + x^2 z^2 a_{202} + z^4 a_{004}\right)}{\beta x^4}} g_n(x, z) e^{-\frac{1}{6}\frac{y\left(12x^3 a_{310} + 6x^2 y a_{220} + 4xy^2 a_{130} + 12xz^2 a_{112} + 3y^3 a_{040} + 6yz^2 a_{022}\right)}{\beta x^4}}$,

where $g_n(x, z)$ is arbitrary function in x and y. In order $G_n(x, y, z)$ to be polynomial we must have

$a_{310} = a_{220} = a_{130} = a_{112} = a_{040} = a_{022} = a_{202} = a_{004} = 0,$ and $a_{400} = -\dfrac{\beta \cdot m}{2}$,where $\beta \neq 0$.

Therefore, $G_n(x, y, z) = y^m g_n(x, z)$.

Next, proceed to calculate terms with degree $n+3$ in Eq. (13), we get

$$-\beta x^4 y \frac{\partial}{\partial y} G_{n-1}(x, y, z) - 2y^m g_n(x, z)\left(x^2 z a_{201} + xyz a_{111} + y^2 z a_{021} + z^3 a_{003}\right) + G_{n-1}(x, y, z)\beta m x^4 = 0,$$

that is $G_{n-1}(x, y, z) = -\dfrac{g_n(x, z) z\left(2x^2 a_{201} + 2z^2 a_{003}\right) y^m \ln(y)}{\beta x^4} + y^m g_{n-1}(x, z) - \dfrac{g_n(x, z) z\left(2xy a_{111} + y^2 a_{021}\right) y^m}{\beta x^4}.$

For $G_{n-1}(x, y, z)$ to be polynomial we must have $a_{201} = a_{003} = 0$. Then $G_{n-1}(x, y, z)$ takes the form

$$G_{n-1} = y^m g_{n-1}(x, z) - \frac{g_n(x, z) z\left(2xy a_{111} + y^2 a_{021}\right) y^m}{\beta x^4}.$$

Additionally, calculate terms with degree $n+2$ in Eq. (13), we get

$$x^2 y \frac{\partial g_n}{\partial y} - \beta x^4 y \frac{\partial g_{n-2}}{\partial y} - 2g_n\left(a_{200}x^2 + a_{002}z^2 + a_{110}xy + a_{020}y^2\right) - 2G_{n-1}\left(a_{111}xyz + a_{021}y^2 z\right) - 2G_{n-2}(x, y, z) a_{400} x^4 = 0.$$

Solving the above partial differential equation, we get

$$G_{n-2}(x,y,z)=\left(\frac{1}{2}\frac{y^4z^2a_{021}^2}{\beta^2x^8}+\frac{2y^3z^2a_{021}a_{111}}{\beta^2x^7}-\frac{y^2a_{020}}{\beta x^4}+\frac{2y^2z^2a_{111}^2}{\beta^2x^6}-\frac{2ya_{110}}{\beta x^3}\right)y^m+\left(\frac{m}{\beta x^2}-\frac{2a_{200}}{\beta x^2}-\frac{2z^2a_{002}}{\beta x^4}\right)y^m\ln(y)$$

$$-g_n(x,z)+\left(-\frac{g_{n-1}y^2za_{021}}{\beta x^4}-\frac{2g_{n-1}(x,z)yza_{111}}{\beta x^3}+g_{n-2}(x,z)\right)y^m,$$

where $g_{n-2}(x,z)$ is arbitrary function in x and z.

For $G_{n-2}(x,y,z)$ to be polynomial we must have $a_{002}=0$ and $a_{200}=\frac{1}{2}m$. So $G_{n-2}(x,y,z)$ becomes

$$G_{n-2}(x,y,z)=\left(+\left(\frac{1}{2}\frac{y^4z^2a_{021}^2}{\beta^2x^8}+\frac{2y^3z^2a_{021}a_{111}}{\beta^2x^7}-\frac{y^2a_{020}}{\beta x^4}+\frac{2y^2z^2a_{111}^2}{\beta^2x^6}-\frac{2ya_{110}}{\beta x^3}\right)y^m\right)-g_n(x,z)+\left(-\frac{g_{n-1}y^2za_{021}}{\beta x^4}-\frac{2g_{n-1}(x,z)yza_{111}}{\beta x^3}+g_{n-2}(x,z\right.$$

.

Finally, calculate terms with degree $n+1$ in Eq. (13), we get

$$zy\frac{\partial G_n}{\partial y}+x^2y\frac{\partial G_{n-1}}{\partial y}-\beta x^4y\frac{\partial G_{n-3}}{\partial y}-x^2\frac{\partial G_n}{\partial z}-2G_n(a_{001}z)-2G_{n-1}(a_{200}x^2+a_{110}xy+a_{020}y^2)-2G_{n-2}(a_{111}xyz+a_{021}y^2z)-2a_{400}G_{n-3}(x,y,z)x^4=$$

Solving this partial differential equation, we get $G_{n-3}(x,y,z)$ as presented in Eq (51) in the Appendix.

Imposing that $G_{n-3}(x,y,z)$ must be polynomial, we obtain that

$$x^2\frac{\partial}{\partial z}g_n(x,z)-zmg_n(x,z)+2g_n(x,z)a_{001}z=0. \tag{14}$$

Solving this differential equation, we obtain $g_n(x,z)=g_n(x)e^{\frac{1}{2}\frac{z^2(m-2a_{001})}{x^2}}$, where $g_n(x)$ is a function of x. For $g_n(x,z)$ to be polynomial, we must have either $g_n(x)=0$ or $a_{001}=\frac{1}{2}m$.

Supposing $g_n(x)=0$ and $a_{001}\neq\frac{1}{2}m$ this would lead to $G_n(x,y,z)=0$, which contradicts the truth that $G_n(x,y,z)$ is Darboux polynomial, implies that $a_{001}=\frac{1}{2}m$.

Lemma 3.5 For $\beta\neq0$, the coefficients a_{020}, a_{021}, a_{110} and a_{111} of cofactor for Darboux polynomial of Eq. (1) are zero.

Proof. Consider $G=\sum_{i=1}^{n}g_i(x,z)y^i$ where each $g_i(x,z)$ for $i=0,\dots,n$ is polynomials in variables x and z only. Then we have

$$y\frac{\partial G}{\partial x}+\left((\lambda+z+x^2-\beta x^4)y-\omega_0^2x\right)\frac{\partial G}{\partial y}+(\mu-x^2)\frac{\partial G}{\partial z}=(-\beta mx^4+mx^2+2a_{110}xy+2a_{111}xyz+2a_{112}xyz^2+2a_{000}$$
$$+mz+2a_{020}y^2+2a_{021}y^2z+2a_{022}y^2z^2)G.$$
(15)

First calculate y^{n+2} coefficient in Eq. (15), we obtain $2(a_{020}+a_{021}z)g_n(x,z)=0$.

The equation holds if either $a_{020}=a_{021}=0$, or $g_n(x,z)=0$.

Suppose $g_n(x,z)=0$ and $a_{020}\neq0, a_{021}\neq0$ implies $G=g_0(x,z)$, substituting in Eq. (13), we get

$$y\frac{\partial g_0(x,z)}{\partial x}+(\mu-x^2)\frac{\partial g_0(x,z)}{\partial z}=\left(-m\beta x^4+m\,x^2+2a_{110}xy+2a_{111}xyz+2a_{000}+mz+2a_{020}y^2+2a_{021}y^2z\right)g_0(x,z), \tag{16}$$

Calculating the y^2 coefficient in Eq. (16), we obtain $(2a_{020}+2a_{021}z)g_0(x,z)=0$, which implies $g_0(x,z)=0$. This would lead to $G=0$, contradicting the truth that G is a Darboux polynomial. Therefore, we must have $a_{020}=a_{021}=0$.

Now, calculating the coefficient of y^{n+1} in Eq. (15), we obtain $\frac{\partial g_n(x,z)}{\partial x}=(2a_{110}x+2a_{111}xz)g_n$.

So $g_n(x,z)=g_n(z)e^{x^2(a_{111}z+a_{110})}$, where $g_n(z)$ is arbitrary function in z only.

In order $g_n(x,z)$ be polynomial $g_n(z)=0$ or $a_{111}=a_{110}=0$.

Consider $g_n(z)=0$, where a_{111} *and* a_{110} are not zero, which indicates that $g_n(x,z)=0$, implying $G=g_0(x,z)$, therefore we obtain

$$y\frac{\partial g_0}{\partial x}+(\mu-x^2)\frac{\partial}{\partial z}=\left(-m\beta x^4+mx^2+2a_{110}xy+2a_{111}xyz+2a_{000}\right)g_0. \tag{17}$$

By calculating $y's$ coefficient in Eq. (17), we obtain $y\frac{\partial g_0(x,z)}{\partial x}=(2a_{110}xy+2a_{111}xyz)g_0(x,z)=0$,

so $g_0(x,z)=\breve{g}_0(z)e^{x^2(a_{111}z+a_{110})}$. Then $g_0(x,z)$ will be polynomial only when $\breve{g}_0(z)=0$, which indicates $g_0(x,z)=0$, implying $G=0$. This contradicts to the truth that G is Darboux polynomial, implies that $a_{111}=a_{110}=0$. This concludes the proof of lemma.

Lemma 3.6 For $\beta\neq 0,t$ he only irreducible Darboux polynomial of the Eq. (1) is y if and only if $\omega_0=0$. Moreover, a Darboux polynomial is y with cofactor $(\lambda+z+x^2-\beta x^4)$.

Proof Consider $G=\sum_{i=1}^{n}g_i(x,y)z^i$, where each $g_i(x,y)$ is polynomial in only variables x and y.

Substituting G in Eq. (13), results will be

$$\begin{aligned}&y\frac{\partial\sum_{i=1}^{n}g_i(x,y)z^i}{\partial x}+\left(\left(\lambda+x^2+z-\beta x^4\right)y-\omega_0^2x\right)\frac{\partial\sum_{i=1}^{n}g_i(x,y)z^i}{\partial y}+\left(\mu-x^2\right)\frac{\partial\sum_{i=1}^{n}g_i(x,y)z^i}{\partial z}\\&=\left(-m\beta x^4+m\,x^2+2a_{000}+mz\right)G(x,z).\end{aligned}\tag{18}$$

By calculating the terms with z^{n+1} in Eq. (18), we get $y\frac{\partial g_n(x,y)}{\partial y}=mg_n(x,y)$.

That is $g_n(x,y)=g_n(x)y^m$, where $g_n(x)$ is a polynomial in variable x.

By calculating the terms with z^n in Eq. (18), we get

$$y\frac{\partial g_n(x,y)}{\partial x}+\left(\left(\lambda+x^2-\beta x^4\right)y-\omega_0^2x\right)\frac{\partial g_n(x,y)}{\partial y}+y\frac{\partial g_{n-1}(x,y)}{\partial y}=\left(-m\beta x^4+m\,x^2+2a_{000}\right)g_n(x,y)+mg_{n-1}(x,y).$$

Solving it, we get

$$g_{n-1}(x,y)=g_n(x)(-\lambda m+2a_{000})y^m\ln(y)+\left(-\frac{d}{dx}g_n(x)y-\frac{g_n(x)m\omega^2x}{y}+g_{n-1}(x)\right)y^m,$$

where $g_{n-1}(x)$ is arbitrary function in variable x only. In order for $g_{n-1}(x,y)$ be polynomial we must have

$$g_n(x)(-\lambda m+2a_{000})=0.\tag{19}$$

Therefore, $g_n(x)=0$ or $a_{000}=\frac{\lambda m}{2}$. If $g_n(x)=0$, this implies $g_n(x,y)=0$, and consequently $G=g_0(x,y)$.

Substituting G in Eq. (18) and by calculating coefficient of z^i, we obtain

$$i=0:y\frac{\partial g_0(x.y)}{\partial x}+\left(\left(\lambda+x^2-\beta x^4\right)y-\omega_0^2x\right)\frac{\partial g_0(x,y)}{\partial y}=\left(-m\beta x^4+m\,x^2+2a_{000}\right)g_0(x,y),\tag{20}$$

$$i=1:y\frac{\partial g_0(x,y)}{\partial y}=mg_0(x,y).\tag{21}$$

Solving Eq. (21) implies $g_0(x,y)=g_0(x)y^m$, where $g_0(x)$ is arbitrary function in variable x only, while Eq. (20) implies that $g_0(x)=Ce^{-\frac{\frac{1}{2}x\left(-m\omega_0^2x+2\lambda my-4a_{000}y\right)}{y^2}}$, which leads to a contradiction, as $g_0(x)$ is function of x only. Consequently, $a_{000}=\frac{\lambda m}{2}$.

By calculating the terms with z^{n-1} in Eq. (18), we get

$$y\frac{\partial g_{n-1}(x,y)}{\partial x}+\left(\left(-\beta x^4+x^2+\lambda\right)y-\omega_0^2x\right)\frac{\partial g_{n-1}(x,y)}{y}+y\frac{g_{n-2}(x,y)}{\partial y}+ng_n(x,y)\left(\mu-x^2\right)=m\left(-\beta x^4+x^2+\lambda\right)g_{n-1}(x,y)+mg_{n-2}(x,y).$$

By solving this partial differential equation, we arrive at the following function

$$\begin{aligned}g_{n-2}(x,y)=&\left(\omega_0^2x\frac{d}{dx}+m\omega_0^2-n\left(\mu-x^2\right)\right)g_n(x)y^m ln(y)+(\frac{1}{2}y^2\frac{d^2}{dx^2}g_n(x)-\left(\frac{d}{dx}g_n(x)\right)\beta x^4y+x^2y\frac{d}{dx}g_n(x)+\lambda y\frac{d}{dx}g_n(x)\\&-y\frac{d}{dx}g_{n-1}(x)-\frac{m\omega_0^2x^5g_n(x)\beta}{y}+\frac{m\omega_0^2x^3g_n(x)}{y}+\frac{m\omega_0^2xg_n(x)\lambda}{y}-\frac{m\omega_0^2xg_{n-1}(x)}{y}+\frac{1}{2}\frac{g_n(x)m^2\omega_0^4x^2}{y^2}-\frac{1}{2}\frac{g_n(x)m\omega_0^4x^2}{y^2}+g_{n-3}(x))y^m.\end{aligned}$$

Since $g_{n-2}(x,y)$ needs to be a polynomial, we conclude that

$$\left(-\omega_0^2x\frac{d}{dx}+m\omega_0^2-n\left(\mu-x^2\right)\right)g_n(x)=0.\tag{22}$$

This is satisfied only when $\omega_0=0$ and $n=0$, as $g_n(x)\neq 0$, consequently, $G=g_0(x,y)$. From calculating z^j coefficients of $i=0,1$ in Eq. (18), we obtain

$$j=1: y\frac{\partial g_0}{\partial y}=mg_0\left(x,y\right), \tag{23}$$

$$j=0: y\frac{\partial g_0\left(x,y\right)}{\partial x}+\left(\lambda+x^2-\beta x^4\right)y\frac{\partial g_0\left(x,y\right)}{\partial y}=m\left(\lambda+x^2-\beta x^4\right)g_0\left(x,y\right). \tag{24}$$

Solving Eq. (23) implies $g_0\left(x,y\right)=F_0\left(x\right)y^m$, while Eq. (24) implies $F_0\left(x\right)=C$, where C is a constant. Consequently, $G=Cy^m$, where $m\geq 1$. So, for $\beta\neq 0$, y is the only irreducible Darboux polynomial of Eq. (1) if and only if $\omega_0=0$.

Lemma 3.7 For $\beta=0$, if g is an irreducible Darboux polynomial for Eq. (1) with cofactor

$$K=\sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{k-0}^{2-i-j}a_{i,j,k}x^i y^j z^k \ ,$$

then $G=g\cdot\phi g$ is a Darboux polynomial invariant by ϕ with a cofactor of the form

$$K=2x^2a_{200}+2xya_{110}+2y^2a_{020}+2z^2a_{002}+2za_{001}+2a_{000}.$$

Proof. The proof follows the same reasoning as in Lemma 3.3, and therefore, is omitted.

Lemma 3.8 For $\beta=0$, the coefficients $a_{200}=a_{001}=\frac{1}{2}m, a_{002}=a_{110}, a_{020}=0$ of cofactor for Darboux polynomial of Eq. (1) are zero, moreover, the coefficients $a_{400}=-\frac{\beta\cdot m}{2}$, $a_{200}=a_{001}=\frac{1}{2}m$,where $m\in \mathrm{N}\cup\{0\}$.

Proof. Consider G a Darboux polynomial of Eq. (1) has the form $G=\sum_{i=0}^{n}G_i\left(x,y,z\right)$, where each $G_i\left(x,y,z\right)$ is a homogenous polynomial of degree i, and $G_n\left(x,y,z\right)\neq 0$.

From the definition of Darboux polynomial, we obtain

$$y\frac{\partial G}{\partial x}+\left(\left(\lambda+z+x^2\right)y-\omega_0^2x\right)\frac{\partial G}{\partial y}+\left(\mu-x^2\right)\frac{\partial G}{\partial z}=\left(2x^2a_{200}+2xya_{110}+2y^2a_{020}+2z^2a_{002}+2za_{001}+2a_{000}\right)G. \tag{25}$$

First, calculate terms with degree $n+2$ in Eq. (25), we get

$$x^2y\frac{\partial}{\partial y}G_n\left(x,y,z\right)=\left(2x^2a_{200}+2xya_{110}+2y^2a_{020}+2z^2a_{002}\right)G_n\left(x,y,z\right).$$

That is $G_n\left(x,y,z\right)=g_n\left(x,z\right)y^{2a_{200}+\frac{2z^2a_{002}}{x^2}}\mathrm{e}^{\frac{y\left(2xa_{110}+ya_{020}\right)}{x^2}}$, where $g_n\left(x,z\right)$ is arbitrary function in the variables x and z.

In order $G_n\left(x,y,z\right)$ to be polynomial we must have $a_{200}=\frac{1}{2}m$, and $a_{002}=a_{110}=a_{020}=0$. Therefore $G_n\left(x,y,z\right)=g_n\left(x,z\right)y^m$.

Next, proceed to calculate terms with degree $n+1$ in Eq. (25), we get

$$zy\left(\frac{\partial}{\partial y}G_n\left(x,y,z\right)\right)+x^2y\left(\frac{\partial}{\partial y}G_{n-1}\left(x,y,z\right)\right)-x^2\left(\frac{\partial}{\partial z}G_n\left(x,y,z\right)\right)=mx^2G_{n-1}\left(x,y,z\right)+2za_{0,0,1}G_{n-1}\left(x,y,z\right).$$

Hence

$$G_{n-1}\left(x,y,z\right)=y^m g_{n-1}\left(x,z\right)+\frac{ln\left(y\right)\left(x^2\frac{\partial}{\partial z}g_n\left(x,z\right)-g_n\left(x,z\right)mz+2g_n\left(x,z\right)za_{001}\right)y^m}{x^2}.$$

For $G_{n-1}\left(x,y,z\right)$ to be polynomial we must have $x^2\frac{\partial}{\partial z}g_n\left(x,z\right)-g_n\left(x,z\right)mz+2g_n\left(x,z\right)za_{001}=0.$

This equation is the same as Eq. (14), as in Lemma 3.4, we arrive that $a_{001}=\frac{1}{2}m$.

Lemma 3.9 For $\beta=0$ the only irreducible Darboux polynomial G of Eq. (1) is y with cofactor $\lambda+z+x^2$ if and only if $\omega_0=0$.

Proof. Suppose that $G=\sum_{i=1}^{n}g_i\left(x,\mathrm{y}\right)z^i$ where each $g_i\left(x,y\right)$ for $i=0,\ldots,n$ is polynomials in variables x and y only. Then we have

$$y\frac{\partial\sum_{i=1}^{n}g_i\left(x,y\right)z^i}{\partial x}+\left(\left(\lambda+z+x^2\right)y-\omega_0^2x\right)\frac{\partial\sum_{i=1}^{n}g_i\left(x,y\right)z^i}{\partial y}+\left(\mu-x^2\right)\frac{\partial\sum_{i=1}^{n}g_i\left(x,y\right)z^i}{\partial x}=\left(mx^2+2a_{000}+mz\right)\sum_{i=1}^{n}g_i\left(x,y\right)z^i. \tag{26}$$

First, calculating the coefficient of z^{n+1} in Eq. (26), we obtain $y\left(\frac{\partial}{\partial y}g_n(x,y)\right)=mg_n(x,y)$, hence $g_n(x,y)=g_n(x)y^m$, where $g_n(x)$ is an arbitrary function of variable x.

By calculating the terms with z^n in Eq. (26), we get

$$y\frac{\partial}{\partial x}g_n(x,y)+\left(\left(\lambda+x^2\right)y-\omega_0^2x\right)\frac{\partial}{\partial y}g_n(x,y)+y\frac{\partial}{\partial y}g_{n-1}(x,y)=mg_n(x,y)+\left(mx^2+2a_{000}\right)g_n(x,y).$$

Therefore, $g_{n-1}(x,y)=g_n(x)\left(-\lambda m+2a_{000}\right)y^m\ln(y)+\left(-y\frac{\mathrm{d}}{\mathrm{d}x}g_n(x)-\frac{g_n(x)m\omega_0^2x}{y}+g_{n-1}(x)\right)y^m.$

In order for $g_{n-1}(x,y)$ be polynomial we must have $g_n(x)\left(-\lambda m+2a_{000}\right)=0$.

This equation is the same to Eq. (19) as in Lemma 3.6, we arrive to $a_{000}=\frac{\lambda m}{2}$.

By calculating the terms with z^{n-1} in Eq. (26), we get

$$y\frac{\partial g_{n-1}(x,y)}{\partial x}+\left(\left(x^2+\lambda\right)y-\omega_0^2x\right)\frac{\partial g_{n-1}(x,y)}{y}+y\frac{g_{n-2}(x,y)}{\partial y}+ng_n(x,y)\left(\mu-x^2\right)=m\left(x^2+\lambda\right)g_{n-1}(x,y)+mg_{n-2}(x,y)$$

That is

$$g_{n-2}(x,y)=\left(-\omega^2x\frac{d}{dx}+m\omega_0^2-n\left(\mu-x^2\right)\right)g_n(x)y^m ln(y)+(\frac{1}{2}y^2\frac{d^2}{dx^2}g_n(x)+x^2y\frac{d}{dx}g_n(x)+\lambda y\frac{d}{dx}g_n(x)-y\frac{d}{dx}g_{n-1}(x)$$
$$+\frac{m\omega_0^2x^3g_n(x)}{y}+\frac{m\omega_0^2xg_n(x)\lambda}{y}-\frac{m\omega_0^2xg_{n-1}(x)}{y}+\frac{1}{2}\frac{g_n(x)m^2\omega_0^4x^2}{y^2}-\frac{1}{2}\frac{g_n(x)m\omega_0^4x^2}{y^2}+g_{n-2}(x))y^m.$$

Since $g_{n-2}(x,y)$ needs to be a polynomial, we conclude $\left(-\omega_0^2x\frac{d}{dx}+m\omega_0^2-n\left(\mu-x^2\right)\right)g_n(x)=0.$

Also, this equation is the same as Eq. (22). As in Lemma 3.6, we arrive that y will be the only irreducible Darboux polynomial of Eq. (1) if and only if $\omega_0=0$. Moreover, the cofactor is $\lambda+z+x^2$.

Proof of Theorem 3.1
from Lemmas 3.6 and 3.9 it follows that $y$is the only irreducible Darboux polynomial of Eq. (1) if and only if $\omega_0=0$.

4 Exponential Factor

In order to investigate the presence exponential factor of Eq. (1), we shall provide some well-known results.

Proposition 4.1 The following statements hold [15].

(i) If $E=\exp\left(g_0/g\right)$ is an exponential factor for the polynomial Eq. (1) and g is not a constant polynomial, then $g=0$ is an invariant algebraic hypersurface.

(ii) Eventually e^{g_0} can be an exponential factor, coming from the multiplicity of the infinite invariant hyperplane.

Theorem 4.2 Given an irreducible invariant algebraic surface $g_1=0$ of degree m of Eq. (1), it has algebraic multiplicity k if and only if the vector field associated to Eq. (1) has $k-1$ exponential factors of the form $\exp\left(g_{0,i}/g_1^i\right)$, where $g_{0,i}$ is a polynomial of degree at most im, and $g_{0,i}$ and g_1 are coprime for $i=1,\ldots,k-1$.[34]

Theorem 4.3 For Eq. (1) with $\omega_0=0$ the statements listed below are true:

1. For $\beta\neq 0$ all exponential factors are e^x, e^{x^2}, e^{x^3}, e^{x^4}, e^z, e^{z^2}, e^{z^3}, e^{xz}, e^{x^2z}, e^{xz^2}, $e^{\left(\frac{1}{5}\beta x^5+y\right)}$, $e^{\left(\frac{1}{6}\beta x^6+xy\right)}$, $e^{\left(\frac{1}{7}\beta x^7+x^2y-\frac{1}{5}x^5\right)}$ and $e^{\left(\frac{5}{2}y^2+\frac{1}{6}x^6+\beta x^5y+\lambda xy+xyz-\frac{1}{4}\beta x^8+\frac{1}{10}\beta^2x^{10}-x^3y\right)}$ with cofactors y, $2xy$, $3yx^2$, $4x^3y$, $\left(-x^2+\mu\right)$, $\left(-2x^2z+2\mu z\right)$, $\left(-3x^2z^2+3\mu z^2\right)$, $\left(-x^3+\mu x+yz\right)$, $\left(-x^4+\mu x^2+2xyz\right)$, $(-2x^3z+2\mu xz+yz^2)$, $\left(x^2y+\lambda y+yz\right)$, $\left(x^3y+\lambda xy+xyz+y^2\right)$, $\left(\lambda x^2y+x^2yz+2xy^2\right)$ and $\left(\lambda^2xy+2\lambda xyz-x^3y+2x^2y^2+xyz^2+6\lambda y^2+\mu xy+6y^2z\right)$ respectively.
2. For $\beta=0$ all exponential factors are e^x, e^{x^2}, e^z, and e^{3y-x^3} with cofactors y, $2xy$, $\left(\mu-x^2\right)$ and $\left(3yz+3\lambda y\right)$ respectively.

Proof. Consider E as an exponential factor of Eq. (1). This implies that

$$y\frac{\partial E}{\partial x}+\left(\lambda+x^2+z-\beta x^4\right)y\frac{\partial E}{\partial y}+\left(\mu-x^2\right)\frac{\partial E}{\partial z}=LE\quad, \tag{27}$$

where L is a polynomial of variables x,y,z of degree at most four. From Lemmas 3.5 and 3.9 Eq. (1) has only the irreducible Darboux polynomial $y=0$ if and only if $\omega_0\neq 0$, then in view of Theorems 4.1 and 4.2 the Eq. (1) can have an exponential factor of the form either $E=e^h$ or $E=e^{\frac{h}{y^n}}$ with $h\in \mathbb{C}[x,y,z]$ and $n\geq 1$ the greatest degree of h be n and the greatest common divisor of h and y being a constant.

We first prove that Eq. (1) with $\omega_0=0$ has no exponential factor of the form $E=e^{\frac{h}{y^n}}$. Suppose that Eq. (1) with has an exponential factor of the form $E=e^{\frac{h}{y^n}}$ with $n\geq 1$ such that y is coprime with h and $h=h(x,y,z)\in \mathbb{C}[x,y,z]$. Substituting E in Eq. (27) , which implies

$$y\frac{\partial h}{\partial x}+\left(\lambda+x^2+z-\beta x^4\right)\left(y\frac{\partial h}{\partial y}-nh\right)+\left(\mu-x^2\right)\frac{\partial h}{\partial z}=Ly^n\ .$$

Let $\breve{h}$ indicate to h restriction with $y=0$. We obtain $\breve{h}\neq 0$ and $-n\breve{h}\left(\lambda+x^2+z-\beta x^4\right)+\left(\mu-x^2\right)\frac{\partial \breve{h}}{\partial z}=0$.

Solving it, we obtain $\breve{h}=f\left(x\right)\,e^{-\frac{1}{2}\frac{nz\left(2\beta x^4-2x^2-2\lambda-z\right)}{\mu-x^2}}$.

In order $\breve{h}$ be polynomial and because $n\geq 1$ it requires $f\left(x\right)=0$, which implies $\breve{h}=0$. However, this contradicts the fact that $\breve{h}\neq 0$. Therefore, if the system has an exponential factor, it must be of the form e^H.

First, let $\beta\neq 0$, consider e^H, where $.\,H=H(x,y,z)\in \mathbb{C}[x,y,z]$ .be an exponential factor of Eq. (1) with cofactor L_4, implies that

$$y\frac{\partial H}{\partial x}+\left(\lambda+x^2+z-\beta x^4\right)y\frac{\partial H}{\partial y}+\left(\mu-x^2\right)\frac{\partial H}{\partial z}=L_4, \tag{28}$$

where L_4 is a polynomial in variables x,y,z, with degree at most four.

Consider $H=\sum_{i=0}^{n}H_i z^i$, where each $H_i=H_i\left(x,y\right)$ polynomial in variables x,y, and $n\geq 6$.

First, by calculating the z^{n+1} coefficients in Eq. (28), we obtain $y\frac{\partial H_n}{\partial y}=0$ that is $H_n=\breve{h}_n\left(x\right)$, where $\breve{h}_n\left(x\right)$ is a function in variable x only.

Now, calculating the z^n coefficients in Eq. (28), we obtain

$$y\frac{\partial H_n}{\partial x}+y\frac{\partial H_{n-1}}{\partial y}=0\text{, so }H_{n-1}=-y\frac{d}{dx}\ \breve{h}_n\left(x\right)+\breve{h}_{n-1}\left(x\right).$$

Now, calculating the z^{n-1} coefficients in Eq. (28), we obtain

$$y\frac{\partial H_{n-1}}{\partial x}+\left(\lambda+x^2-\beta x^4\right)y\frac{\partial H_{n-1}}{\partial y}+y\frac{\partial H_{n-2}}{\partial y}+n\left(\mu-x^2\right)H_n=0\,,$$

implies that

$$-y^2\left(\frac{d^2}{dx^2}\breve{h}_n\left(x\right)\right)+y\frac{d}{dx}\breve{h}_{n-1}\left(x\right)-\left(\ \lambda+x^2-\beta x^4\right)y\frac{d}{dx}\breve{h}_n\left(x\right)+y\frac{\partial}{\partial y}\breve{h}_{n-2}+n\breve{h}_n\left(x\right)(\mu-x^2)=0.$$

To satisfy the previous equation required, $\breve{h}_n\left(x\right)$ divisible by y, contradicts the truth $\breve{h}_n\left(x\right)$ is a function in x only. This implies $\breve{h}_n\left(x\right)=0$, implies $H_n=0,$ for $n\geq 6$ consequently suppose that

$$H=H_0\left(x,y\right)+H_1\left(x,y\right)z+H_2\left(x,y\right)z^2+H_3\left(x,y\right)z^3+H_4\left(x,y\right)z^4+H_5\left(x,y\right)z^5. \tag{29}$$

Substituting H and L_4 where $L_4=\sum_{i=0}^{4}\sum_{j=0}^{4-i}\sum_{k-0}^{4-i-j}b_{i,j,k}x^i y^j z^k$ in Eq. (28) and by equating in the terms with z^{i} for $i=0,\ldots,6$, we obtain

$$y\frac{\partial}{\partial x}H_0\left(x,y\right)+\left(\ \lambda+x^2-\beta x^4\right)y\frac{\partial}{\partial y}H_0\left(x,y\right)+\left(-x^2+\mu\right)H_1\left(x,y\right)-b_{400}x^4-b_{300}x^3-b_{200}x^2-b_{310}x^3y-b_{210}x^2y-b_{220}x^2y^2-$$

$$b_{110}xy-b_{120}xy^2-b_{130}xy^3-b_{000}-b_{100}x-b_{010}y-b_{020}y^2-b_{030}y^3-b_{040}y^4=0.$$

(30)

$$(y\frac{\partial}{\partial x}H_1(x,y)+(\lambda+x^2-\beta x^4)y\frac{\partial}{\partial y}H_1(x,y)+y\frac{\partial}{\partial y}H_0(x,y)+2(-x^2+\mu)H_2(x,y)-b_{301}x^3-b_{201}x^2-b_{211}x^2y-b_{111}xy$$
$$-b_{121}xy^2-b_{001}-b_{101}x-b_{011}y-b_{021}y^2-b_{031}y^3=0.$$
(31)

$$y\frac{\partial}{\partial x}H_2(x,y)+(\lambda+x^2-\beta x^4)y\frac{\partial}{\partial y}H_2(x,y)+y\frac{\partial}{\partial y}H_1(x,y)+3(-x^2+\mu)H_3(x,y)-b_{202}x^2-b_{112}xy-b_{002}$$
$$-b_{102}x-b_{012}y-b_{022}y^2=0. \quad (32)$$

$$y\frac{\partial}{\partial x}H_3(x,y)+(\lambda+x^2-\beta x^4)y\frac{\partial}{\partial y}H_3(x,y)+y\frac{\partial}{\partial y}H_2(x,y)+4(-x^2+\mu)H_4(x,y)-b_{003}-b_{103}x-b_{013}y=0. \quad (33)$$

$$y\frac{\partial}{\partial x}H_4(x,y)+(\lambda+x^2-\beta x^4)y\frac{\partial}{\partial y}H_4(x,y)+y\frac{\partial}{\partial y}H_3(x,y)+5(-x^2+\mu)H_5(x,y)-b_{004}=0. \quad (34)$$

$$y\frac{\partial}{\partial x}H_5(x,y)+(\lambda+x^2-\beta x^4)y\frac{\partial}{\partial y}H_5(x,y)+y\frac{\partial}{\partial y}H_4(x,y)=0. \quad (35)$$

$$y\frac{\partial}{\partial y}H_5(x,y)=0. \quad (36)$$

First, from Eq. (36), by direct computation we get $H_5(x,y)=h_5(x)$, where $h_5(x)$ is a polynomial in variable x.

From Eq. (35), gives $H_4(x,y)=-y\frac{d}{dx}h_5(x)+h_4(x)$, where $h_4(x)$ is a polynomial in variable x.

Also, from Eq. (34), we obtain

$H_3(x,y)=(5h_5(x)x^2-5h_5(x)\mu+b_{004})\ln(y)+\frac{1}{2}\frac{d^2}{dx^2}h_5(x)y^2+(\lambda+x^2-\beta x^4)y\frac{d}{dx}h_5(x)-\frac{d}{dx}h_4(x)y+h_3(x)$, where $h_3(x)$ is a polynomial in variable x.

In order $H_3(x,y)$ be polynomial, it follows that $h_5(x)=\frac{1}{5}\frac{b_{004}}{-x^2+\mu}$, and since $h_5(x)$ must be polynomial implies $b_{004}=0$.

Consequently, $h_5(x)=0$, and $H_3(x,y)=-\frac{d}{dx}h_4(x)y+h_3(x)$.

From Eq. (33), it follows that

$H_2(x,y)=(4h_4(x)x^2-4h_4(x)\mu+b_{103}x+b_{003})\ln(y)+\frac{1}{2}y^2\frac{d^2}{dx^2}h_4(x)+(\lambda+x^2-\beta x^4)y\frac{d}{dx}h_4(x)-\mathrm{y}\frac{d}{dx}h_3(x)+b_{013}y+h_2(x)$.

Since $H_2(x,y)$ must be polynomial, we have $h_4(x)=\frac{1}{4}\frac{xb_{103}+b_{003}}{-x^2+\mu}$. Additionally, since $h_4(x)$ is a polynomial, we obtain $b_{103}=0$, $b_{003}=0$, and thus $h_4(x)=0$. Therefore, $H_2(x,y)$ becomes $H_2(x,y)=-y\frac{d}{dx}h_3(x)+b_{013}y+h_2(x)$.

From Eq. (32), we derive

$$H_1(x,y)=(3h_3(x)x^2+b_{202}x^2-3h_3(x)\mu+b_{102}x+b_{002})\ln(y)+\frac{1}{2}y^2\frac{d^2}{dx^2}h_3(x)+\frac{1}{2}b_{022}y^2+(\lambda+x^2-\beta x^4)y\frac{d}{dx}h_3(x)$$
$$+\beta x^4yb_{013}-x^2yb_{013}-\lambda yb_{013}+b_{112}xy-y\frac{d}{dx}h_2(x)+b_{012}y+h_1(x).$$

Since $H_1(x,y)$ must be polynomial, we have $h_3(x)=\frac{1}{3}\frac{x^2b_{202}+xb_{102}+b_{002}}{-x^2+\mu}$. Moreover, since $h_3(x)$ is a polynomial, we obtain $b_{102}=0$, $b_{002}=3C_0\mu$, $b_{202}=-3C_0$ and thus $h_3(x)=C_0$, where C_0 is constant. Therefore,

$$H_1(x,y)=-y\frac{d}{dx}h_2(x)+h_1(x)+\frac{1}{2}b_{022}y^2+\left(-(\lambda+x^2-\beta x^4)b_{013}+b_{112}x+b_{012}\right)y.$$

From Eq. (31), we derive $H_0(x,y)$; its detailed expression is provided in Eq. (52) of the Appendix

Since $H_0(x,y)$ must be polynomial, we obtain $b_{001}=2a_0\mu$, $b_{101}=2a_1\mu$, $b_{201}=-2a_0$, $b_{301}=-2a_1$, and thus $h_2(x)=a_0+a_1x$, where a_0, a_1 are constants.

Finally, from Eq. (30), we derive Eq. (53); which is given in Appendix.

Equating the terms y^j, for $j=0,...,4$ in Eq. (29). Thus, from the coefficient of y^4 in Eq. (53), we obtain $b_{040}=0$.

From the coefficient of y^2 in Eq. (53) and solving for $h_1(x)$, we get

$$h_1(x)=\frac{1}{2}x^2b_{111}-\frac{1}{2}x^2b_{020}+\frac{1}{3}x^3a_1+\frac{3}{10}x^5b_{013}-\frac{1}{3}x^4b_{112}-\frac{1}{3}x^3b_{012}-\frac{1}{30}x^6b_{022}+\frac{1}{20}x^5b_{121}+\frac{1}{12}x^4b_{021}-\frac{1}{6}b_{120}x^3-\frac{1}{12}b_{220}x^4+\frac{1}{3}b_{211}x^3$$
$$+\frac{2}{3}x^3b_{013}-\frac{1}{24}x^4b_{022}+\frac{1}{15}\beta\lambda x^6b_{022}-\frac{3}{5}\beta\lambda x^5b_{013}+\frac{1}{6}\beta^2x^9b_{013}-\frac{3}{7}\beta x^7b_{013}+\frac{1}{5}\beta x^6b_{112}+\frac{1}{5}\beta x^5b_{012}-\lambda x^2b_{112}-\frac{1}{2}\lambda^2b_{022}x^2+\frac{1}{2}\lambda b_{021}x^2$$
$$+\frac{1}{4}\mu b_{022}x^2-\frac{1}{30}\beta x^6b_{021}+\frac{1}{28}\beta x^8b_{022}-\frac{1}{5}\beta x^5a_1-\frac{1}{6}\lambda x^4b_{022}-\frac{1}{42}\beta x^7b_{121}-\frac{1}{90}\beta^2x^{10}b_{022}+\frac{1}{6}\lambda x^3b_{121}++\lambda x^3b_{013}+C_1x+C_2,$$

where C_1 and C_2 are constants.

From the coefficient of y in Eq. (53) and solving for $h_0(x)$, we obtain $h_0(x)$ (see Appendix Eq. (54)).

where C_3 is constant.

From the coefficient of y^3 in Eq. (53), we find that $b_{013}=b_{022}=b_{031}=b_{130}=0$, and $b_{121}=2b_{030}$.

From the coefficient of y^0 in Eq. (53), we find that $C_1=-b_{300}$, $C_2=-\mu b_{400}-b_{200}$, $a_1=b_{012}$, $b_{000}=-\mu(\mu b_{400}+b_{200})$, $b_{020}=4\lambda b_{112}+b_{111}+2b_{400}$, $b_{021}=6b_{112}$, $b_{030}=0$, $b_{100}=-b_{300}\mu$, $b_{120}=2b_{211}$, and $b_{220}=2b_{112}$.

From all the previous information and after some easy computations, we obtain

$$H=\left(\left(-x^2-\mu\right)z+2xy-\frac{1}{2}x^4+\frac{1}{3}\beta x^6-\lambda x^2\right)b_{400}+\left(-xz+y-\frac{1}{3}x^3+\frac{1}{5}\beta x^5-\lambda x\right)b_{300}+\left(x^2y-\frac{1}{3}\lambda x^3+\frac{1}{7}\beta x^7-\frac{1}{5}x^5\right)b_{211}-zb_{200}$$
$$+\left(\frac{1}{4}x^4+xyz+\left(\beta x^5-x^3-\lambda x\right)y+\frac{1}{2}x^4\lambda-\frac{1}{3}\beta x^6\lambda+\frac{1}{2}\lambda^2x^2+\frac{1}{6}x^6+\frac{5}{2}y^2-\frac{1}{2}\mu x^2-\frac{1}{4}\beta x^8+\frac{1}{10}\beta^2x^{10}\right)b_{112}$$
$$+\left(\frac{1}{6}\beta x^6-\frac{1}{2}\lambda x^2+xy-\frac{1}{4}x^4\right)b_{111}+a_0z^2+xz^2b_{012}+\left(-\lambda x-\frac{1}{3}x^3+y+\frac{1}{5}\beta x^5\right)b_{011}+C_3+\frac{1}{3}b_{210}x^3+\frac{1}{2}b_{110}x^2+\frac{1}{4}b_{310}x^4+C_0z^3+b_{010}x,$$

and

$$L_4=\left(x^4-\mu^2+2y^2\right)b_{400}+\left(x^3-\mu x\right)b_{300}+\left(x^2yz+2xy^2\right)b_{211}+\left(x^2-\mu\right)b_{200}+\left(2x^2y^2+xyz^2+4\lambda y^2+6y^2z\right)b_{112}+\left(xyz+y^2\right)b_{111}$$
$$+(-2x^3z+2\mu xz+yz^2)b_{012}+\left(-2x^2z+2\mu z\right)a_0+\left(-3x^2z^2+3\mu z^2\right)C_0+b_{310}x^3y+b_{210}x^2y+b_{110}xy+b_{010}y+b_{011}yz.$$

Second, let $\beta=0$, consider e^H be an exponential factor of Eq. (1), write H as a polynomial in the variable z in the form $H=\sum_{i=0}^{n}H_i(x,y)z^i$, where each H_i is a polynomial in the variables x,y and $n\geq 4$. Thus

$$y\frac{\partial H}{\partial x}+\left(\lambda+x^2+z\right)y\frac{\partial H}{\partial y}+\left(\mu-x^2\right)\frac{\partial H}{\partial z}=L_2, \tag{37}$$

where L_2 is a polynomial of degree at most two in the variables x,y,z has the form

$$L_2=\sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{k-0}^{2-i-j}b_{i,j,k}x^iy^jz^k .$$

In the same way as in the proof of previous case, we obtain $H_n(x,y)=0$ for $n\geq 4$. Consequently,

$$H=H_0(x,y)+H_2(x,y)z+H_2(x,y)z^2+H_3(x,y)z^3. \tag{38}$$

Substitute H and L_2 i n Eq. (37) equating the terms with z^i for $i=0,1,2,3,4$, we obtain

$$y\frac{\partial}{\partial y}H_3(x,y)=0 . \tag{39}$$

$$y\frac{\partial}{\partial x}H_3(x,y)+\left(x^2+\lambda\right)y\frac{\partial}{\partial y}H_3(x,y)+y\frac{\partial}{\partial y}H_2(x,y)=0. \tag{40}$$

$$y\frac{\partial}{\partial x}H_2(x,y)+\left(x^2+\lambda\right)y\frac{\partial}{\partial y}H_2(x,y)+y\frac{\partial}{\partial y}H_1(x,y)+3\left(-x^2+\mu\right)H_3(x,y)-b_{002}=0. \tag{41}$$

$$y\frac{\partial}{\partial x}H_1(x,y)+\left(x^2+\lambda\right)y\frac{\partial}{\partial y}H_1(x,y)+y\frac{\partial}{\partial y}H_0(x,y)+2\left(-x^2+\mu\right)H_2(x,y)-b_{101}x-b_{011}y-b_{001}=0. \tag{42}$$

$$y\frac{\partial}{\partial x}H_0(x,y)+\left(x^2+\lambda\right)y\frac{\partial}{\partial y}H_0(x,y)+\left(-x^2+\mu\right)H_1(x,y)-b_{200}x^2-b_{110}xy-b_{020}y^2-b_{100}x-b_{010}y-b_{000}=0 . \tag{43}$$

Solving (39), we obtain $H_3(x,y)=h_3(x)$, where $h_3(x)$ is a polynomial in the variable x.

Substituting $H_3(x,y)$ into Eq. (40) and solving it, we obtain $H_2(x,y)=-y\frac{d}{dx}h_3(x)+h_2(x)$, where $h_2(x)$ is a polynomial in the variable x.

Solving Eq. (41), we obtain

$$H_1(x,y)=\left(-3h_3(x)\left(\mu-x^2\right)+b_{002}\right)\ln(y)+\frac{1}{2}y^2\frac{d^2}{dx^2}h_3(x)+x^2y\frac{d}{dx}h_3(x)+\lambda y\frac{d}{dx}h_3(x)-y\frac{d}{dx}h_2(x)+h_1(x),$$

where $h_1(x)$ is a polynomial of variable x.

In order for $H_1(x,y)$ be polynomial, while $h_3(x)$ is also a polynomial, it follows that $b_{002}=h_3(x)=0$, and thus $H_3(x,y)=0$. Thus, $H_1(x,y)$ becomes $H_1(x,y)=-\frac{d}{dx}h_2(x)y+h_1(x)$.

Using the expression of $H_1(x,y)$ and $H_2(x,y)$ and solving Eq. (42), we obtain

$$H_0(x,y)=\left(2h_2(x)\left(\mu-x^2\right)+b_{101}x+b_{001}\right)ln(y)+\frac{1}{2}y^2\frac{d^2}{d^2}h_2(x)+x^2y\frac{d}{dx}h_2(x)+\lambda y\frac{d}{dx}h_2(x)-y\frac{d}{dx}h_1(x)+b_{011}y+h_0(x),$$

where $h_0(x)$ is a polynomial in variable x.

In order for $H_0(x,y)$ be polynomial, and since $h_2(x)$ is a polynomial, we obtain $b_{101}=b_{001}=0$ and $h_2(x)=0$, which implies that $H_0(x,y)=-\frac{d}{dx}h_1(x)+b_{011}y+h_0(x)$.

Finally, from Eq. (43), we derive

$$\left(-\frac{d^2}{dx^2}h_1(x)-b_{020}\right)y^2+\left(\frac{d}{dx}h_0(x)+\left(x^2+\lambda\right)\left(-\frac{d}{dx}h_1(x)+b_{011}\right)-b_{110}x-b_{010}\right)y+\left(-x^2+\mu\right)h_1(x)-b_{200}x^2-b_{100}x-b_{000}=0. \quad (44)$$

Equating the terms of y^i for $i=2,1,0$ in Eq. (44), we obtain the following results.

$$h_1(x)=-\frac{1}{2}b_{020}x^2+e_1x+e_2,$$

$$h_0(x)=-\frac{1}{4}x^4b_{020}+\frac{1}{3}e_1x^3-\frac{1}{2}\lambda x^2b_{020}-\frac{1}{3}x^3b_{011}+e_1\lambda x-\lambda b_{011}x+\frac{1}{2}b_{110}x^2+b_{010}x+e_3,$$

and

$$\frac{1}{2}x^4b_{020}-e_1x^3+\left(-\frac{1}{2}\mu b_{020}-e_2-b_{200}\right)x^2+\left(+\mu e_1-b_{100}\right)x+\mu e_2-\mathrm{b}_{000}=0, \quad (45)$$

where . e_1,e_2,e_3 . are constants of integration.

Solving Eq. (45), we obtain $e_1=b_{020}=b_{100}=0$, $e_2=-b_{200}$, $b_{000}=-\mu b_{200}$.

Thus, $h_0(x)=-\frac{1}{3}x^3b_{011}-\lambda b_{011}x+\frac{1}{2}b_{110}x^2+b_{010}x+e_3$. From Eq. (38) we conclude that

$$H=\left(y-\frac{1}{3}x^3-\lambda x\right)b_{011}+\frac{1}{2}b_{110}x^2+b_{010}x+e_3-b_{200}z, \text{ and}$$

$$L_2=yzb_{011}+\left(x^2-\mu\right)b_{200}+b_{110}xy+b_{010}y.$$

This concludes the proof of the theorem.

Remark The exponential factor is not considered for $\omega_0\neq 0,$, as the absence of Darboux polynomials in the system implies that this analysis would not yield a first integral.

5 Darboux and Rational First Integral

Theorem 5.1 The Eq. (1) cannot have Darboux first integral for any value of parameters.

Proof. Consider H is a Darboux first integral of Eq. (1).

First if $\omega_0=0$, from Theorems 3.1 and 4.3, if Eq. (1) has a Darboux first integral H, and in view of its definition in Eq. (6) then $H=y^{\lambda_1}\exp(A)$

$$A=\mu_1x+\mu_2x^2+\mu_3x^3+\mu_4x^4+\mu_5z+\mu_6z^2+\mu_7z^3+\mu_8xz+\mu_9x^2z+\mu_{10}xz^2+\mu_{11}\left(\frac{1}{5}\beta x^5+y\right)+\mu_{12}\left(\frac{1}{6}\beta x^6+xy\right)+\mu_{13}\left(\frac{1}{7}\beta x^7+x^2y-\frac{1}{5}x^5\right)$$
$$+\mu_{14}\left(\frac{5}{2}y^2+\frac{1}{6}x^6+\beta x^5y+\lambda xy+xyz-\frac{1}{4}\beta x^8+\frac{1}{10}\beta^2x^{10}-x^3y\right)$$

where $\lambda_1,\mu_i\in\mathrm{C}$ for $i=1,2,\ldots,14$. So H satisfies Eq. (3). Thus

$$\lambda_1\left(\lambda+z+x^2-\beta^4 y\right)+\mu_1 y+2\mu_2 xy+3\mu_3 yx^2+4\mu_4 x^3 y+\mu_5\left(-x^2+\mu\right)+\mu_6\left(-2x^2z+2\mu z\right)+\mu_7\left(-3x^2z^2+3\mu z^2\right)$$
$$+\mu_8\left(-x^3+\mu x+yz\right)+\mu_9\left(-x^4+\mu x^2+2xyz\right)+\mu_{10}(-2x^3z+2\mu xz+yz^2)+\mu_{11}\left(x^2y+\lambda y+yz\right)+\mu_{12}\left(x^3y+\lambda xy+xyz+y^2\right)$$
$$\mu_{13}\left(\lambda x^2y+x^2yz+2xy^2\right)+\mu_{14}\left(\lambda^2xy+2\lambda xyz-x^3y+2x^2y^2+xyz^2+6\lambda y^2+\mu xy+6y^2z\right)=0.$$

By some calculation, we obtain that this equation is satisfied if and only if $\lambda_1=\mu_i=0$ for $i=0,\ldots,14$. Hence H is not a Darboux first integral because it is a constant function.
Second if $\omega_0\neq 0$, then in view of its definition in Eq. (7) and by Theorem 3.1, we conclude that a Darboux first integral H must take the form $H=e^G$, where $G=\sum_{i=1}^{m}\mu_i\, g_i$ and e^{g_i} be exponential factor with cofactor μ_i for $i=1,\ldots,m$. since e^G is a first integral of Eq. (1), it satisfies Eq. (3). After simplification, we get

$$y\frac{\partial G}{\partial x}+\left(\left(\lambda+z+x^2-\beta x^4\right)y-\omega_0^2x\right)\frac{\partial G}{\partial y}+\left(\mu-x^2\right)\frac{\partial G}{\partial z}=0\,.$$

Thus, G is a polynomial first integral of Eq. (1). However, according to Theorem 2.1**,** this is impossible. Therefore, Eq. (1) does not possess Darboux first integral.

Theorem 5.2 Eq. (1) does not possess a rational first integral.
To prove Theorem 5.2 we will recall the following auxiliary lemma.

Lemma 5.3 If Eq. (1) has a rational first integral then either it has a polynomial first integral or two Darboux polynomials with the same nonzero cofactor [35].

Proof Theorem 5.2
According to Theorems 2.1 and 3.1, Eq. (1) has no polynomial first integrals and does not possess two distinct invariant algebraic surfaces with the same cofactor. Thus, in view of Lemma 3.2 and Lemma 5.3 it follows that the system does not admit any rational first integrals**.**

6 Zero-Hopf bifurcation Analysis

In this section, in order to prove Theorem 6.4, we first compute the equilibrium points of Eq. (1). It is easy to verify that the following statements hold:

1. Eq. (1) admits no equilibrium points when $\mu\neq 0$ and $\omega_0\neq 0$.
2. Eq. (1) admits infinitely many non-hyperbolic equilibrium points $E_{z_0}=(0,0,z_0)$ for $z_0\in\mathrm{R}$ when $\mu=0$, and the following holds:
    1. For $\lambda+z_0>0$, $\omega_0\neq 0$ then the system has at least one eigenvalue with a positive real part then E_{z_0} is an unstable equilibrium point.
    2. For $\lambda=-z_0$, $\omega_0\neq 0$ then E_{z_0} is zero-Hopf equilibrium point of the system.
3. Eq. (1) has a non-isolated set of non-hyperbolic equilibrium points $\left(\pm\sqrt{\mu},0,\ a\right)$ for $a\in\mathrm{R}$ when $\omega_0=0$.

Secondly, we will recall the following supporting theorems:

Theorem 6.1 (First-Order Averaging Theory) [22]
Let us consider the following two initial value problem

$$\dot{x}=\varepsilon F\left(t,x\right)+\varepsilon^2 G\left(t,x,\varepsilon\right),\ x\left(0\right)=x_0, \tag{46}$$

and the corresponding averaged differential equation

$$\dot{y}=\varepsilon f\left(y\right),\qquad y\left(0\right)=x_0\,, \tag{47}$$

where each of $x,y,x_0\in\varphi$, $\varphi\subset\mathrm{R}^n$ where φ is an open subset, ε is small positive parameters, both F and G are period of period T in the variable t, $t\geq 0$, g is averaged function of $F\left(t,x\right)$, and $g:\varphi\to\mathrm{R}^n$, where

$$g\left(y\right)=\frac{1}{T}\int_0^T F\left(t,y\right).$$

Under suitable conditions, equilibrium points of the averaged system are associated with T-periodic solutions of Eq. (46).

Theorem 6.2 [25]
Consider the two initial value problems Eq. (46) and Eq. (47). Suppose the following condition holds:

I. The functions F, $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x^2}$, G, $\frac{\partial G}{\partial x}$ are all defined, continuous, and bounded by a constant that is cot contain ε, for all $\left(t,x\right)\in\left[0,\infty\right)\times\varphi$ and $\varepsilon\in\left(0,\varepsilon_0\right]$.

II. F and g are T-periodic in t (T independent of ε). Then the following holds:

i. Suppose $det\left(\frac{\partial g}{\partial y}\right)|_{y=p} \neq 0$, where p is an equilibrium point of the averaged system (47), then, for sufficiently small ε, system (46) admits a T-periodic solution $\varnothing(t,\varepsilon)$ (a limit cycle) close to p, with $\varnothing(t,\varepsilon) \to p$.

ii. The stability of the limit cycle $\varnothing(t,\varepsilon)$ is determined by the stability of the equilibrium point p of the averaged system (47). Specifically, the point p exhibits the same stability properties as the Poincaré map associated with the limit cycle $\varnothing(t,\varepsilon)$.

Proposition 6.3 There is only a one-parameter family of Eq. (1) in which the equilibrium point is localized at $(0,0,z_0)$ of coordinates is a zero-Hopf equilibrium point. That is $\lambda = -z_0$ and $\omega_0 \neq 0$. Furthermore, the eigenvalues at $(0,0,z_0)$ are zero and $\mp i\omega_0$.

Proof. Since Eq. (1) has equilibrium point only if $\mu = 0$. The characteristic equation of the Jacobian matrix of Eq. (1) at $(0,0,z_0)$ for $\mu = 0$ is given by $L^3 - (\lambda + z_0)L^2 + \omega_0{}^2 L = 0$, so the equilibrium point $(0,0,z_0)$ is a zero Hopf if and only if $\lambda = -z_0$ and $\omega_0 \neq 0$.

In the next theorem, we utilize the first-order averaging theory to study the limit cycles bifurcating from the origin of Eq. (1).

Theorem 6.4 Consider Eq. (1). Suppose $(\lambda,\beta,\omega_0) = (-z_0 + \varepsilon\,\lambda_2, \beta_1 + \varepsilon\,\beta_2, \omega_1 + \varepsilon\,\omega_2)$ where ε is a small positive parameter. We observe that detecting any periodic solution of the system that bifurcates from the zero-Hopf point on the z-axis cannot be done by applying the first-order averaging theory.

Proof. We apply the first-order averaging theory of dynamical systems, as outlined in previous Theorems 6.1 and 6.2, to analyze the existence of equilibrium points of zero-Hopf type at the $(0,0,z_0)$. To shift $(0,0,z_0)$ into the origin use the transformation $(x,y,z) = (x_1, y_1, z_1 + z_0)$; applying it to Eq. (1) gives

$$
\begin{aligned}
\dot{x}_1 &= y_1,\\
\dot{y}_1 &= \left(\lambda + z_0 + z_1 + x_1^2 - \beta x_1^4\right) y_1 - \omega_0^2\ x_1,\\
\dot{z}_1 &= -x_1^2.
\end{aligned}
\tag{48}
$$

By applying the parameter transformation $(\lambda,\beta,\omega_0) = (-z_0 + \varepsilon\lambda_2, \beta_1 + \varepsilon\beta_2, \omega_1 + \omega_2\varepsilon)$ and applying the variable transformation $(x_1, y_1, z_1) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$ to the Eq. (48), and for small positive $\varepsilon \ll 1$, we derive the following equation.

$$
\begin{aligned}
\dot{X} &= Y,\\
\dot{Y} &= -X^4 Y \varepsilon^5 \beta_2 - X^4 Y \varepsilon^4 \beta_1 + \left(X^2 Y - X\omega_2^2\right)\varepsilon^2 + \left(-2X\omega_1\omega_2 + Y\left(Z + \lambda_2\right)\right)\varepsilon - X\omega_1^2,\\
\dot{Z} &= -\varepsilon X^2.
\end{aligned}
\tag{49}
$$

To determine the variable transformation required to express Eq. (49) in normal form for the application of averaging theory, we first transform the linear part of the system at the origin (when $\varepsilon = 0$) into its real Jordan normal form, i.e., into the form

$$
\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

To achieve this, we introduce the linear transformation $X = \frac{1}{\omega_1} v, Y = u, z = w$ .With the new variables (u,v,w) and $\omega_1 = \omega$, system (49) can be written as

$$
\begin{aligned}
\dot{u} &= -\omega v - \frac{v^4 u \varepsilon^5 \beta_2}{\omega^4} - \frac{v^4 u \varepsilon^4 \beta_1}{\omega^4} + \left(\frac{v^2 u}{\omega^2} - \frac{v\omega_2^2}{\omega}\right)\varepsilon^2 + \left(-2v\omega_2 + u\left(w + \lambda_2\right)\right)\varepsilon,\\
\dot{v} &= \omega u,\\
\dot{w} &= -\frac{\varepsilon v^2}{\omega^2}.
\end{aligned}
$$

We proceed to rewrite the previous differential system in cylindrical coordinates (r,θ,w), where the transformation is given by $u = r cos\theta$ and $v = r sin\theta$. Substituting these expressions into the system yields:

$$\dot{r} = -\frac{r\left(-\omega^4\left(w+\lambda_2\right)\cos\left(\theta\right)+2\sin\left(\theta\right)\omega^4\omega_2\right)\cos\left(\theta\right)\varepsilon}{\omega^4}+O\left(\varepsilon^2\right),$$
$$\dot{\theta} = \omega+\frac{\left(-2\omega^4\omega_2\cos(\theta)^2-\omega^4\left(w+\lambda_2\right)\sin\left(\theta\right)\cos\left(\theta\right)+2\omega^4\omega_2\right)\varepsilon}{\omega^4}+O\left(\varepsilon^2\right),$$
$$\dot{w} = -\frac{\varepsilon r^2\sin(\theta)^2}{\omega^2}.$$

Therefore, the solutions to system (48) in the region $\theta > 0$ reduces to studying the solutions of the following system

$$\frac{dr}{d\theta} = F_1\left(r,\theta,w\right)\varepsilon+O\left(\varepsilon^2\right),\quad \frac{dw}{d\theta} = F_2\left(r,\theta,w\right)\varepsilon+O\left(\varepsilon^2\right),$$

where $t=\theta$, $x=\left(r,w\right)^T$ and $F\left(r,\theta,w\right)=\left(F_1\left(r,\theta,w\right),F_2\left(r,\theta,w\right)\right)^T$ such that

$$F_1\left(r,\theta,w\right) = -\frac{\left(-\omega^4\left(w+\lambda_2\right)cos\left(\theta\right)+2\sin\left(\theta\right)\omega^4\omega_2\right)rcos\left(\theta\right)}{\omega^5},$$
$$F_2\left(r,\theta,w\right) = -\frac{r^2 sin(\theta)^2}{\omega^3}.$$

By applying the averaging theorem of the first order in the interval $\left[0,2\pi\right]$,we will obtain

$$g_1\left(r,w\right) = \frac{1}{2\pi}\int_0^{2\pi}F_1 d\theta = \frac{1}{2}\frac{r\left(w+\lambda_2\right)}{\omega},$$
$$g_2\left(r,w\right) = \frac{1}{2\pi}\int_0^{2\pi}F_2 d\theta = -\frac{1}{2}\frac{r^2}{\omega^3}.$$

Hence the averaged system is

$$\frac{dr}{d\theta} = g_1\left(r,w\right)\varepsilon,$$
$$\frac{dw}{d\theta} = g_2\left(r,w\right)\varepsilon. \tag{50}$$

The unique equilibrium point of system (50) is $\left(r,w\right)=\left(0,w\right)$ and $\det\left(\frac{\partial\left(g_1\left(r,w\right),g_2\left(r,w\right)\right)}{\partial\left(r,w\right)}\right)|_{(0,w)}=0$ where $\frac{\partial\left(g_1\left(r,w\right),g_2\left(r,w\right)\right)}{\partial\left(r,w\right)}$ is Jacobian matrix of the system (50). It means the averaging method of first order does not guarantee the existence of a periodic solution in the given system. In order to pass to the second-order averaging method, we must have both $g_1\left(r,w\right)\equiv 0$ and $g_2\left(r,w\right)\equiv 0$, but this is impossible, so the second-order cannot apply. Therefore, the averaging method is unsuitable for detecting periodic solutions that bifurcate from the zero-Hopf equilibrium.

7 Conclusion

In this paper, we examined the radio physical oscillator system, a three-dimensional polynomial system of differential equations of degree 5 that depends on four parameters λ, β, ω_0, μ. We investigated into whether polynomial first integrals, Darboux polynomials, exponential factors, Darboux first integral, and rational first integral exist. We established that the system admits a Darboux polynomial y only if $\omega_0=0$. Furthermore, we have shown that the system has no first integral of Darboux, polynomial, or rational form. Finally, we applied the averaging method to detect periodic solutions that bifurcate from the zero-Hopf equilibrium in Eq. (1), but unfortunately, we observe that it is an ill-suitable method for this system.

Appendix

$$G_{n-3}\left(x,y,z\right) = -\frac{\left(x^2\frac{\partial}{\partial z}g_n\left(x,z\right)-zmg_n\left(x,z\right)+2g_n\left(x,z\right)a_{001}z\right)x^8\beta^2y^m\ln\left(y\right)}{\beta^3x^{12}} \tag{51}$$
$$+y^m g_{n-3}\left(x,z\right)-\frac{1}{6}\frac{1}{\beta^3x^{12}}(((12a_{111}x^9yz\beta^2+6a_{021}y^2z\beta^2x^8)g_{n-2}\left(x,z\right)+g_{n-1}\left(x,z\right)(12a_{110}x^9y\beta^2+6a_{020}y^2\beta^2x^8$$
$$-12y^2z^2a_{111}^2\beta x^6-12y^3z^2a_{021}a_{111}\beta x^5-3y^4z^2a_{021}^2\beta x^4-24y^2za_{111}a_{110}\beta x^6-12y^3za_{111}a_{020}\beta x^5-12y^3za_{021}a_{110}\beta x^5)g_{n-1}\left(x,z\right)$$
$$-\left(6y^4za_{021}a_{020}\beta x^4+8y^3z^3a_{111}^3x^3+12y^4z^3a_{021}a_{111}^2x^2+6y^5z^3a_{021}^2a_{111}x+y^6z^3a_{021}^3+12yza_{111}\beta x^7+6y^2za_{021}\beta x^6\right)g_n\left(x,z\right))y^m)$$

$$
\begin{aligned}
H_0(x,y)&=h_0(x)+\frac{1}{2}b_{121}xy^2+b_{111}xy+b_{211}x^2y+\left(\lambda+x^2-\beta x^4\right)y\frac{d}{dx}h_2(x)-2\beta x^3y^2b_{013}+\beta^2x^8yb_{013}-2\beta x^6yb_{013}+\beta x^5yb_{112}\\
&+\frac{1}{2}\beta x^4y^2b_{022}+\beta x^4yb_{012}+2\lambda x^2yb_{013}-\lambda xyb_{112}+2x^2yb_{013}-2\beta\lambda x^4yb_{013}++xy^2b_{01,3}-\frac{1}{2}x^2y^2b_{022}+\lambda^2yb_{013}-\frac{1}{2}\lambda y^2b_{022}-x^2yb_{012}\\
&-\lambda yb_{012}-2\mu yb_{013}-x^3yb_{112}+x^4yb_{013}+\frac{1}{2}b_{021}y^2+\frac{1}{3}b_{031}y^3+b_{011}y-y\frac{d}{dx}h_1(x)-\frac{1}{2}y^2b_{112}+(b_{301}x^3+2h_2(x)x^2+b_{201}x^2\\
&-2h_2(x)\mu+b_{101}x+b_{001})ln(y)+\frac{1}{2}y^2\frac{d^2}{dx^2}h_2(x).
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&-b_{040}y^4+\left(2\beta x^3b_{022}-6\beta x^2b_{013}-xb_{022}+b_{013}+\frac{1}{2}b_{121}+\left(\lambda+x^2-\beta x^4\right)b_{031}-b_{130}x-b_{030}\right)y^3+(-\frac{d^2}{dx^2}h_1(x)+8\beta^2x^7b_{013}\\
&-12\beta x^5b_{013}+5\beta x^4b_{112}+\frac{2}{3}\left(\left(-12\beta\lambda+6\right)b_{013}+\left(6b_{012}-6a_1\right)\beta\right)x^3-3x^2b_{112}+\frac{1}{3}((12\lambda+12)b_{013}-6b_{012}+6b_{211}+6a_1)x-\lambda b_{112}+b_{111}\\
&+\left(\lambda+x^2-\beta x^4\right)(\beta x^4b_{022}-4\beta x^3b_{013}-x^2b_{022}+\frac{1}{3}\left(6b_{013}+3b_{121}\right)x-\lambda b_{022}+b_{021}-b_{112})+\frac{1}{2}\left(-x^2+\mu\right)b_{022}-b_{220}x^2-b_{120}x-b_{020})y^2\\
&+(\frac{d}{dx}h_0(x)+\left(\lambda+x^2-\beta x^4\right)(-\frac{d}{dx}h_1(x)+\beta^2x^8b_{013}-2\beta x^6b_{013}+\beta x^5b_{112}+\frac{1}{6}\left(\left(-12\beta\lambda+6\right)b_{013}+\left(6b_{012}-6a_1\right)\beta\right)x^4-x^3b_{112}\\
&+\frac{1}{6}((12\lambda+12)b_{013}-6b_{012}+6b_{211}+6a_1)x^2+\frac{1}{6}\left(-6\lambda b_{112}+6b_{111}\right)x+\frac{1}{6}\left(6\lambda^2-12\mu\right)b_{013}+\frac{1}{6}\left(-6b_{012}+6a_1\right)\lambda+b_{011})+\left(-x^2+\mu\right)\\
&(-a_1+\frac{1}{2}\left(2\beta x^4-2x^2-2\lambda\right)b_{013}+b_{112}x+b_{012})-b_{310}x^3-b_{210}x^2-b_{110}x-b_{010})y+\left(-x^2+\mu\right)h_1(x)-b_{400}x^4-b_{300}x^3-b_{200}x^2-b_{000}-b_{100}x=0
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
h_0(x)&=\frac{1}{5}\beta\lambda x^5a_1-\frac{1}{5}\beta\lambda x^5b_{012}-\frac{3}{5}\beta\mu x^5b_{013}+\frac{7}{30}\beta\lambda^2x^6b_{022}-\frac{1}{5}\beta\lambda x^6b_{021}-\frac{1}{12}\beta\mu x^6b_{022}+\frac{1}{4}\lambda\mu x^2b_{022}-\frac{2}{21}\beta\lambda x^7b_{121}\\
&+\frac{71}{420}\beta\lambda x^8b_{022}-\frac{23}{450}\beta^2\lambda x^{10}b_{022}-\frac{3}{7}\beta\lambda x^7b_{013}+\frac{1}{5}\beta\lambda x^6b_{112}+\frac{1}{6}\beta^2\lambda x^9b_{013}-\lambda^2a_1x+C_1\lambda x-\lambda b_{011}x-\lambda^3b_{013}x\\
&+\lambda^2b_{012}x-\mu b_{012}x+a_1\mu x-\frac{1}{26}\beta^3x^{13}b_{013}+\frac{3}{22}\beta^2x^{11}b_{013}-\frac{1}{50}\beta^2x^{10}b_{112}-\frac{1}{2}\mu x^2b_{112}-\frac{1}{3}\lambda x^3a_1+\frac{1}{6}\beta x^6b_{020}-\frac{1}{2}\lambda x^2b_{020}\\
&+\frac{17}{350}\beta x^{10}b_{022}-\frac{13}{90}\lambda x^6b_{022}-\frac{5}{108}\beta x^9b_{121}+\frac{3}{20}\lambda x^5b_{121}-\frac{1}{15}\beta x^8b_{021}+\frac{1}{3}\lambda x^4b_{021}+\frac{1}{24}\beta x^8b_{220}-\frac{1}{12}\lambda x^4b_{220}+\frac{1}{14}\beta x^7b_{120}\\
&-\frac{1}{6}\lambda x^3b_{120}-\frac{1}{5}d_1\beta x^5+\frac{1}{5}\beta x^5b_{011}-\frac{1}{6}\beta x^9b_{013}+\frac{3}{10}\lambda x^5b_{013}+\frac{1}{3}\lambda x^3b_{012}+\frac{1}{15}\beta x^8b_{112}-\frac{1}{3}\lambda x^4b_{112}-\frac{5}{12}\lambda^2x^4b_{022}-\frac{1}{2}\lambda^3x^2b_{022}\\
&+\frac{1}{2}\lambda^2x^2b_{021}+\frac{1}{8}\mu x^4b_{022}+\frac{1}{6}\lambda^2x^3b_{121}+\frac{1}{126}\beta^3x^{14}b_{022}-\frac{25}{756}\beta^2x^{12}b_{022}+\frac{1}{66}\beta^2x^{11}b_{121}+\frac{1}{50}\beta^2x^{10}b_{021}-\frac{1}{2}\lambda^2x^2b_{112}\\
&+b_{010}x-\frac{1}{36}x^6b_{022}-\frac{1}{5}x^5b_{013}+\frac{1}{3}x^3b_{012}+\frac{1}{4}x^4b_{112}-\frac{1}{4}x^4b_{020}-\frac{1}{40}x^8b_{022}+\frac{1}{28}x^7b_{121}+\frac{1}{18}x^6b_{021}-\frac{1}{18}x^6b_{220}-\frac{1}{10}x^5b_{120}\\
&+\frac{1}{3}C_1x^3-\frac{1}{3}x^3b_{011}+\frac{1}{14}x^7b_{013}-\frac{1}{18}x^6b_{112}+\frac{1}{2}b_{110}x^2+\frac{1}{4}b_{310}x^4+\frac{1}{3}b_{210}x^3+3\lambda\mu b_{013}x+\mu x^3b_{013}-\frac{1}{24}\lambda x^4b_{022}+\frac{1}{48}\beta x^8b_{022}\\
&+\frac{1}{7}\beta x^7b_{013}-\frac{1}{3}a_1x^3-\frac{1}{3}\lambda x^3b_{013}+C_3.
\end{aligned}
\tag{54}
$$

(54)

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