Sharif University of Technology
Scientia Iranica
Transactions on Industrial Engineering
https://scientiairanica.sharif.edu

An exact iterative algorithm to solve a linear fractional programming problem

Hale Gonce Kocken [a,*], **Beyza Ahlatcioglu Ozkok** [b], **Mehmet Ahlatcioglu** [c]

a. *Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey.*
b. *Department of Business Administration, Yildiz Technical University, Istanbul, Turkey.*
c. *Department of Mathematics, Yildiz Technical University, Istanbul, Turkey.*

Received 3 June 2021; received in revised form 31 January 2022; accepted 29 August 2022

KEYWORDS
Linear Fractional Programming (LFP);
Iterative optimization;
Asymptotic solution;
Unboundedness.

Abstract. The Linear Fractional Programming (LFP) problem that optimizes the ratio of two linear objective functions under linear constraints has a wide range of application areas. Based on the traditional definition of continuity, we developed an exact iterative algorithm that does not depend on big-M coefficients. Removing the nonlinearity in the fractional objective function by converting the objective function into a linear form, an equivalent linear-iterative problem is obtained and a computationally efficient algorithm is proposed. We also analyze the unbounded and asymptotic solution case of the LFP. To demonstrate the efficiency of the proposed method, illustrative numerical examples are provided for all solution cases. Also, we analyze the validity of our algorithm and compare our results with the existing algorithm from the literature by generating random large-scale test problems.

1. Introduction

Linear Fractional Programming (LFP) problems are of great interest because of their extensive application areas such as resource allocation, transportation, production, finance, location theory, stochastic processes, Markov renewal programs, information theory, applied linear algebra, large scale programming, game theory, etc. In many practical applications like cutting stock problems, ore blending problems, shipping schedules problems, the optimal policy for a Markovian chains, the sensitivity of linear programming problems, optimization of ratios of criteria gives more insight into the situation than the optimization of each criterion. To study relative efficiency in different fields such as education, hospital administration, court systems, air force maintenance units, Bank branches, etc., fractional programming solves more efficiently the above type of problems. In real world decision situations, decision makers, sometimes, may face up with the decision to optimize inventory/sales, actual cost/standard cost, output/employee, etc. with respect to some constraints [1]. Reviews of various fractional programming applications are given by Schaible [2–4], and Craven [5]. Further references are listed in Schaible's bibliography

*. *Corresponding author.*
E-mail addresses: hgonce@yildiz.edu.tr (H. Gonce Kocken);

To cite this article:
H. Gonce Kocken, B. Ahlatcioglu Ozkok, M. Ahlatcioglu "An exact iterative algorithm to solve a linear fractional programming problem", *Scientia Iranica*, (2025) **32**(16): 5685 https://doi.org/10.24200/sci.2022.58352.5685

of this field [2], which covers more than 550 articles. There are also comprehensive books related to LFP problems, such as that of Bajalinov [6].

A usual LFP problem is a special case of a nonlinear programming problem. Isbell and Marlow [7] presented an algorithm for the LFP problem of the maximization of a linear fractional function under linear constraints. An LFP problem can be transformed into a LP problem by using the variable transformation method of Charnes and Cooper [8], or it can be solved by adopting the updated objective function method of Bitran and Novaes [9]. Bitran and Magnanti [10] considered algorithms, duality, and sensitivity analysis for optimization problems that they, called fractional, whose objective function is the ratio of two real-valued functions. Assuming that the denominator of the objective function of the LFP problem over the feasible region is positive, Swarup [11] extended the well-known simplex method to solve LFP problems, with the object of giving an algorithm for the solution of programming problems with linear fractional functions without reducing them to LP problems. Singh [12] extended the saddlepoint and stationary point theory of optimality in nonlinear programming to nonlinear fractional programming problems. Bajalinov [13] considered a special problem in the context of linear and LFP: given an objective function on a feasible bounded set S, the optimal vertex, and a neighboring vertex $\bar{x}$, adjust the objective function to make $\bar{x}$ the new optimum. Considering the presented literature up to [13], it can be concluded that almost all of the methods developed for LFP are analytical methods.

As an iterative method, Tantawy [14] focused on LFP problems with inequality constraints and presented a method based on the conjugate gradient projection that is applied to solve nonlinear programming problems with linear constraints. Tantawy [15] proposed an iterative method for solving LFP problems and showed that this method can be used for sensitivity analysis when a scalar parameter is introduced into the objective function coefficients. Effati and Pakdaman [16] introduced an interval-valued LFP problem and proved that an interval-valued LFP problem can be converted to an optimization problem with an interval-valued objective function whose bounds are linear fractional functions.

Tantawy [17] presented a concept of duality for the LFP problem in which the objective function is a linear fractional function and the constraint functions are in the form of linear inequalities. Using the concept of duality, Simi and Talukder [18] presented a new approach for solving LFP problem. Biswas et al. [19] discussed the theory of LFP and presented some proof for the optimality and convexity of LFP. Fu et al. [20] proposed simulation-based linear fractional programming model, which integrates a runoff simulation model into a LFP framework, is developed for optimal water resource planning. Ozkok [21] presented an efficient iterative algorithm to solve LFP problem. In their paper they noticed that in order to improve their algorithm they should determine the good big-M values. Based on this idea, we improved their algorithm by removing the big-M parameter in their approach. Moreover, we demonstrated the effectiveness of our new algorithm, which we developed by eliminating big-M from Ozkok's algorithm, on large-scale examples that we have produced, by comparing the results.

As another type of LFP, Das et al. [22] focused on the LFP problem with absolute value functions. They applied the traditional unrestricted variable transformation and convert the LFP into an LP problem. As a real-life application of LFP, there are some current studies. Considering the challenge of solving large-scale LFP problems, Mohammed and Lomte [23] proposed a secure and verifiable scheme to offload the computations on the cloud side. Mohammed et al. [24] also focused on outsourcing of scientific computations and shown how to use certificate validation to obtain correctness guarantees for privacy preserving out sourcing of LFP problem. Adding some parameters obtained by solving a LFP problem, [25] presented an extended version of the Tikhonov regularization method. In [26], an LFP is converted to an LP problem by using the duality concept, and a traditional data envelopment analysis model is provided to demonstrate the applicability.

Considering the objective function is continuous at every point of the feasible region of LFP problem, a convergence condition is obtained by the traditional definition of continuity. We combine our convergence condition and the objective function of the LFP problem to create an iterative constraint. Using this constraint, we construct a new iterative LP problem to obtain the optimal solution of the LFP problem. This iterative LP problem can solve all LFP problems that have a bounded feasible region.

This paper is organized as follows: In Section 2, we present the definition of the LFP problem and some preliminaries. In Section 3, we present our methodology and give the steps of our new algorithm and its flow chart. Finally, Section 4 and Section 5 consist of our numerical examples and conclusions, respectively.

2. Problem definition and preliminaries

The mathematical model of a LFP problem can be stated as follows:

$$\max Z(\mathbf{x}) = \frac{\mathbf{N}(\mathbf{x})}{\mathbf{D}(\mathbf{x})} = \frac{\mathbf{c}^T\mathbf{x} + \alpha}{\mathbf{d}^T\mathbf{x} + \beta}, \tag{1}$$

subject to $\mathbf{x} \in S$,

where,

$$S = \{\mathbf{x}\in\mathbf{R}^n | \mathbf{A}\mathbf{x}\leq\mathbf{b}, \quad \mathbf{x}\geq 0, \quad \mathbf{b}\in\mathbf{R}^m, (\mathbf{d}^T\mathbf{x}+\beta)>0\},$$

is a convex and compact nonempty set and the fractional objective function is defined on S, that is Z: $\subset \mathbf{R}^n \rightarrow \mathbf{R}(S) \subset \mathbf{R}$. We note here that assuming that is the domain of the objective function is not a restriction, since Problem (1) aims to find an optimal solution on S.

Remark 1: Resulting from the basic assumption $\left(\mathbf{d}^T\mathbf{x} + \beta\right) > 0$, the fractional function $Z(\mathbf{x})$ is continuous on $\mathbf{R}$, and also on its domain S, that is, $Z(\mathbf{x})$ is continuous on $\forall \mathbf{x}_i \in S$ and the corresponding function value is $Z\left(\mathbf{x}_i\right) = Z_i$.

Definition 1: A neighborhood of a point $\mathbf{x}_i \in \mathbf{R}^n$ is the set:

$$B\left(\mathbf{x}_i, r\right) = \left\{\mathbf{x} \in \mathbf{R}^n \left| |\mathbf{x} - \mathbf{x}_i| < r\right.\right\},$$

where $r \in \mathbf{R}^+$ is some positive number. The neighborhood is also called a ball with radius r and centre $\mathbf{x}_i$.

Definition 2: $Z(\mathbf{x})$ is continuous on $S \subset \mathbf{R}^n$ provided that for every point $\mathbf{x}_i \in S$ and for $\varepsilon > 0$, there exists a number $\delta > 0$ such that $Z(\mathbf{x})$ satisfies $\left|Z\left(\mathbf{x}\right) - Z\left(\mathbf{x}_i\right)\right| < \varepsilon$ whenever $\mathbf{x} \in S$ and the distance between $\mathbf{x}$ and $\mathbf{x}_i$ satisfies $|\mathbf{x} - \mathbf{x}_i| < \delta$.

Using Definitions 1 and 2 can be re-expressed in terms of neighborhoods as follows:

$\forall \varepsilon \in \mathbf{R}^+$, if there exists $\delta > 0$ such that $\forall \mathbf{x} \in B\left(\mathbf{x}_i, \delta\right)$ and $Z \in B\left(Z_i, \varepsilon\right)$ where $Z_i = Z\left(\mathbf{x}_i\right)$, then $Z\left(\mathbf{x}\right)$ is continuous at the given $\mathbf{x}_i \in \mathbf{R^n}$.

3. Our methodology

Considering the algorithm in Ozkok [21], the convergence condition for the LFP (Problem (1)):

$$Z\mathbf{x} = Z_i\mathbf{x} + \mathbf{x}_iZ - Z_i\mathbf{x}_i, \tag{2}$$

is obtained. It is obvious that the fractional objective function $Z = \frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta}$, can be written as follows:

$$\mathbf{d}^T\mathbf{x}Z + \beta Z = \mathbf{c}^T\mathbf{x} + \alpha. \tag{3}$$

Combining Eqs. (2) and (3), we have the following linear equation:

$$\left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\bar{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha. \tag{4}$$

Obviously, the structures of these two functions are different from each other. Therefore, from now on, the objective function will be expressed with Z for the LFP problem and $\bar{Z}$ for the Linear Programming (LP) problem.

Thus, Problem (1) can be converted to the following iterative LP problem:

$$\max \bar{Z}, \tag{5a}$$

$$\text{s.t.,} \quad \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\bar{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha, \tag{5b}$$

$$\mathbf{x} \in S. \tag{5c}$$

Here, the subscript $i \in \{0, 1, 2, \ldots\}$ denotes the iteration counter and Problem (5b) represents our iterative constraint. Starting with an initial solution $(\mathbf{x}_0, Z_0)$, that is $i = 0$, the sub-problem (5) generates a second feasible solution $(\mathbf{x}_1^*, Z_1^*)$ by using the initial solution, then moves to a third feasible solution $(\mathbf{x}_2^*, Z_2^*)$ by using the previous feasible solution $(\mathbf{x}_1^*, Z_1^*)$, and so on. In general, if $(\mathbf{x}_i^*, Z_i^*)$ is a feasible solution obtained at iteration i, then our algorithm finds a new feasible solution $\left(\mathbf{x}_{i+1}^*, Z_{i+1}^*\right)$ at iteration $i + 1$.

Proposition 1: The gradient vectors of the fractional objective function Z and the linear objective function $\bar{Z}$ are equal at every point $\mathbf{x}_i \in S$.

Proof: We omit the proof, the interested reader can see it in Ozkok [21].

Result 1: Since the gradient direction determines the direction of the increase of a function, an increase in the linear objective function $\bar{Z}\left(\mathbf{x}_i\right) \leq \bar{Z}(\mathbf{x}^*)$ implies an increase in the fractional objective function $Z\left(\mathbf{x}_i\right) \leq Z(\mathbf{x}^*)$.

Proposition 2: Starting with an initial point $\mathbf{x}_0 \in S$, let the successive optimal solutions of Problem (5) be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \forall i \in N$. Then, the fractional objective function values generate an increasing sequence for the successive optimal solutions $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots$, that is $Z_0 \leq Z_1 \leq \ldots \leq Z_i \leq Z_{i+1} \leq \ldots$ until reaching the optimal solution of Problem (1) where $\forall i \in N$.

Proof: We omit the proof, the interested reader can see it in Ozkok [21].

Theorem 1: If an increasing sequence $\{Z_i\}_{i\in N}$ is bounded above, then it converges.

Proof: Straightforward.

Result 2: If the feasible region S is bounded, then $\{Z_i\}_{i\in N}$ always converges since it is bounded above.

This means that successive optimization of Problem (5) guarantees obtaining the optimal solution of Problem (1).

Result 3: If the feasible region S is unbounded, then $\{Z_i\}_{i\in N}$ is either bounded above (convergent) or unbounded (divergent).

Result 4: The case of unbounded solution of Problem (5):

$$\left(\lim_{\mathbf{x}\to\infty}\bar{Z}(\mathbf{x})\to\infty\right) \text{ implies } \lim_{\mathbf{x}\to\infty}Z(\mathbf{x})=\lim_{\mathbf{x}\to\infty}\frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta},$$

which is the asymptotic solution of Problem (1). This solution can be obtained by an additional analysis which we give in the next subsection.

3.1. Unbounded feasible region case

Assume that the feasible region:

$$S=\{\mathbf{x}\in\mathbf{R}^n|\mathbf{A}\mathbf{x}\le\mathbf{b},\ \ \mathbf{x}\ge 0, \mathbf{b}\in\mathbf{R}^m, \left(\mathbf{d}^T\mathbf{x}+\beta\right)>0\},$$

is unbounded and Problem (5) has an unbounded solution, that is $\lim_{\mathbf{x}\to\infty}\bar{Z}(\mathbf{x})\to\infty$. By adding the constraint $x_1+x_2+\cdots+x_n\le M$ to the feasible region S, we get:

$$\overline{S}=\{\mathbf{x}\in\mathbf{R}^n|\mathbf{A}\mathbf{x}\le\mathbf{b},\ \ \mathbf{x}\ge 0,\ \ \ \mathbf{b}\in\mathbf{R}^m,$$
$$\left(\mathbf{d}^T\mathbf{x}+\beta\right)>0,\ \ \ \mathbf{1}\cdot\mathbf{x}\le M\},$$

where M is a very big number. As seen, $\overline{S}$ is the restricted form of with the hyperplane $x_1+x_2+\cdots+x_n=M$. The optimal solution in $\overline{S}$ occurs at an extreme point of the intersection of the original constraints $\mathbf{A}\mathbf{x}\le\mathbf{b}$ and the hyperplane $x_1+x_2+\cdots+x_n=M$. Let reduce the feasible region $\overline{S}$ by the following transformation for each decision variable:

$$\frac{x_j}{M}=v_j\Rightarrow x_j=M\,v_j\Rightarrow\mathbf{x}=M\cdot\mathbf{V}.$$

By this transformation, the constraint $\mathbf{1}\cdot\mathbf{x}\le M$ can be converted to $v_1+v_2+\ldots+v_n\le 1$ or $\mathbf{1}\cdot\mathbf{V}\le 1$. The original constraints $\mathbf{A}\mathbf{x}\le\mathbf{b}$ can be converted to $\mathbf{A}\mathbf{V}\le\mathbf{0}$ since the limit of $\frac{\mathbf{b}}{M}$ is zero as M approaches to infinity. The nonnegativity constraints $\mathbf{x}\ge 0$ and the assumption $\left(\mathbf{d}^T\mathbf{x}+\beta\right)>0$ can be written as $\mathbf{V}\ge 0$ and $\mathbf{d}^T\mathbf{V}>0$ respectively.

Thus, the reduced feasible region $\widehat{S}$:

$$\widehat{S}=\left\{\mathbf{V}\in\mathbf{R^n}\left|\mathbf{A}\mathbf{V}\le\mathbf{0},\,\mathbf{V}\ge 0,\,\mathbf{d}^T\mathbf{V}>0,\,1\cdot\mathbf{V}\le 1\right.\right\},$$

and the objective function of Problem (1) is given by:

$$Z\left(\mathbf{x}\right)=\frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta}=\frac{M\mathbf{c}^T\mathbf{V}+\alpha}{M\mathbf{d}^T\mathbf{V}+\beta}=\frac{M\left(\mathbf{c}^T\mathbf{V}+\frac{\alpha}{M}\right)}{M\left(\mathbf{d}^T\mathbf{V}+\frac{\beta}{M}\right)}.$$

The limit of the objective function as M approaches to infinity can be evaluated as:

$$\lim_{M\to\infty}Z(\mathbf{x})=\frac{\mathbf{c}^T\mathbf{V}}{\mathbf{d}^T\mathbf{V}}=\bar{Z}(\mathbf{V}),$$

thus the problem to find the maximum of asymptotic values of the fractional objective function $Z(\mathbf{x})$ can be written as follows:

$$\max\left(\lim_{M\to\infty}Z(\mathbf{x})\right)=\max\bar{Z}(\mathbf{V})=\frac{\mathbf{c}^T\mathbf{V}}{\mathbf{d}^T\mathbf{V}},\tag{6a}$$

$$\text{s.t.,}\quad \mathbf{A}\mathbf{V}\le\mathbf{0},\tag{6b}$$

$$\mathbf{1}\cdot\mathbf{V}=1,\tag{6c}$$

$$\mathbf{V}\ge 0.\tag{6d}$$

Note that $\mathbf{1}\cdot\mathbf{V}\le 1$ is handled in equality form as Eq. (6c), since the optimal solution is on the hyperplane $x_1+x_2+\cdots+x_n=M$.

Assume that $\bar{Z}$ is continuous at $\mathbf{V}_i\in\widehat{S}$ and $\bar{Z}_i=\frac{\mathbf{c}^T\mathbf{V}_i}{\mathbf{d}^T\mathbf{V}_i}$.

By Definition 2, $\forall\varepsilon\in R^+$ correspond to $\delta>0$ such that $\left|\bar{Z}-\bar{Z}_i\right|<\varepsilon$ for each $\mathbf{V}$ satisfies $\left|\mathbf{V}-\mathbf{V}_i\right|<\delta$. Then, $\left|\bar{Z}-\bar{Z}_i\right|\left|\mathbf{V}-\mathbf{V}_i\right|<\varepsilon.\delta$ and $\left(\bar{Z}-\bar{Z}_i\right)\left(\mathbf{V}-\mathbf{V}_i\right)=0$ imply the following:

$$\bar{Z}\mathbf{V}=\bar{Z}_i\mathbf{V}+\mathbf{V}_i\bar{Z}-\bar{Z}_i\mathbf{V}_i.\tag{7}$$

Combining Problem (7) and $\bar{Z}=\frac{\mathbf{c}^T\mathbf{V}}{\mathbf{d}^T\mathbf{V}}$, the following linear approximation equation can be written as:

$$\left(\bar{Z}_i\mathbf{d}^T-\mathbf{c}^T\right)\mathbf{V}+\bar{Z}\mathbf{d}^T\mathbf{V}_i=\bar{Z}_i\mathbf{d}^T\mathbf{V}_i.\tag{8}$$

Obviously, the objective function of Problem (6) is nonlinear whereas Eq. (8) is linear. Therefore, from now on, the objective function will expressed with $\bar{Z}$ for the LFP problem while $\widehat{Z}$ for LP problem.

Theorem 2: The fractional objective function $\bar{Z}$ and the linear objective function $\widehat{Z}$ take the same value at the point $\mathbf{V}_i$ and also their gradient vectors are the same at that point.

Proof: Straightforward.

Then, the transformed problem can be written as:

$$\max\,\widehat{Z},\tag{9a}$$

$$\text{s.t.,}\quad \mathbf{A}\mathbf{V}\le\mathbf{0},\tag{9b}$$

$$\mathbf{1}\cdot\mathbf{V}=1,\tag{9c}$$

$$\left(\bar{Z}_i\mathbf{d}^T-\mathbf{c}^T\right)\mathbf{V}+\bar{Z}\mathbf{d}^T\mathbf{V}_i=\bar{Z}_i\mathbf{d}^T\mathbf{V}_i,\tag{9d}$$

$$\mathbf{V}\ge\mathbf{0}.\tag{9e}$$

Let the optimal solution of Problem (9) and the corresponding optimal objective function value are denoted by $\mathbf{V}^* = \mathbf{V}_{i+1}$ and $\bar{Z}(\mathbf{V}^*) = \bar{Z}(\mathbf{V}_{i+1}) = \bar{Z}_{i+1}$, respectively.

Remark 2: If $\bar{Z}_{i+1} \to \infty$, then the unbounded solution occurs.

Remark 3: If $\bar{Z}_{i+1} = \bar{Z}_i$, then the maximum of asymptotic values $\max\left(\lim_{M\to\infty} Z(\mathbf{x})\right) = \max \bar{Z}(\mathbf{V}) = \bar{Z}_i$ is found by the optimal solution of Problem (9).

Remark 4: If $\bar{Z}_{i+1} > \bar{Z}_i$, then the iterative process is continued by setting $i = i + 1$.

Remark 5: Assume that $\bar{Z}_i = \max\left(\lim_{M\to\infty} Z(\mathbf{x})\right)$.

Case 1: If $Z(\mathbf{x}_{i+1}) < \bar{Z}_i$ for all $\mathbf{x}_{i+1} \in S$, $\bar{Z}_i$ is the supremum of the objective function at the feasible region S, that is, $\sup_S Z(\mathbf{x}) = \bar{Z}_i$.

Case 2: If $Z(\mathbf{x}_{i+1}) > \bar{Z}_i$ for $\exists\ \mathbf{x}_{i+1} \in S$, by setting $Z_i = \bar{Z}_i$, the problem.

$$\max \widehat{Z}, \tag{10a}$$

$$\text{s.t.,}\ \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\widehat{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha, \tag{10b}$$

$$\mathbf{x} \in S, \tag{10c}$$

is solved iteratively until either $Z_{i+1} \to \infty$ (unbounded solution) or $Z_{i+1} = Z_i$ (optimal solution) is obtained.

Result 5: Problem (10) and the following problem:

$$\max\ \left(\mathbf{c}^T - Z_i\mathbf{d}^T\right)\mathbf{x}, \tag{11a}$$

$$\text{s.t.} \qquad \mathbf{x} \in S, \tag{11b}$$

have the same optimal solution.

Proof: From Problem (10b), $\widehat{Z}$ can be written as:

$$\widehat{Z} = \frac{Z_i\mathbf{d}^T\mathbf{x}_i + \alpha - \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x}}{\left(\mathbf{d}^T\mathbf{x}_i + \beta\right)}.$$

Since $Z_i\mathbf{d}^T\mathbf{x}_i + \alpha$ and $\mathbf{d}^T\mathbf{x}_i + \beta$ are constants, and $\mathbf{d}^T\mathbf{x}_i + \beta > 0$ by the assumption, handling the object function $\widehat{Z}$ as $\left(\mathbf{c}^T - Z_i\mathbf{d}^T\right)\mathbf{x}$ does not change the optimal solution.

Similar conclusion can be made for Problem (9) and the following problem:

$$\max\ \left(\mathbf{c}^T - Z_i\mathbf{d}^T\right)\mathbf{V}, \tag{12a}$$

$$\text{s.t.,}\quad \mathbf{AV} \leq \mathbf{0}, \tag{12b}$$

$$\mathbf{1}\cdot\mathbf{V} = \mathbf{1}, \tag{12c}$$

$$\mathbf{V} \geq \mathbf{0}. \tag{12d}$$

3.2. Finding an initial objective function value

An initial objective function value Z_i can be found by solving one of the following problems over the feasible region S or selecting an arbitrary value.

$$\max_{\mathbf{x}\in S}\ 0, \tag{13a}$$

$$\max_{\mathbf{x}\in S}\ \mathbf{c}^T\mathbf{x}, \tag{13b}$$

$$\max_{\mathbf{x}\in S}\ -\mathbf{d}^T\mathbf{x}, \tag{13c}$$

$$\max_{\mathbf{x}\in S}\ \left(\mathbf{c}^T - \mathbf{d}^T\right)\mathbf{x}. \tag{13d}$$

We will choose the initial value by maximizing $\left(\mathbf{c}^T - \mathbf{d}^T\right)\mathbf{x}$ over S.

Remark 6: If an unbounded solution is encountered in Problem 13(b),(c) or (d); Problem (13a) can be used to determine an initial objective function value.

Remark 7: Problem (13a)-(13d) has no notable advantage over each other in terms of the number of iterations. However, it can be interpreted that Problem (13d) may require the least number of iterations since it considers both the numerator and the denominator of the fractional objective function, while Problem (13a) may require the greatest number of iterations.

3.3. A stopping criterion

As stated in Proposition 2, Problem (5) generates an increasing sequence for the fractional objective function. If Problem (1) has an optimal solution $(\mathbf{x}^*, Z^*)$, then Z^* will be a finite value, so the corresponding sequence will be bounded above. In each iteration, the solution becomes closer to $(\mathbf{x}^*, Z^*)$. Thus, the stopping criterion can be defined as to obtain the same objective function value in two successive iterations, $Z^*_{i+1} = Z^*_i$.

3.4. Statement of our algorithm

Based on the previous sections, we are now ready to give our algorithm:

Step 0 Load the LFP Problem (1).

Step 1 Select an initial objective function value Z_0 and set $i = 0$.

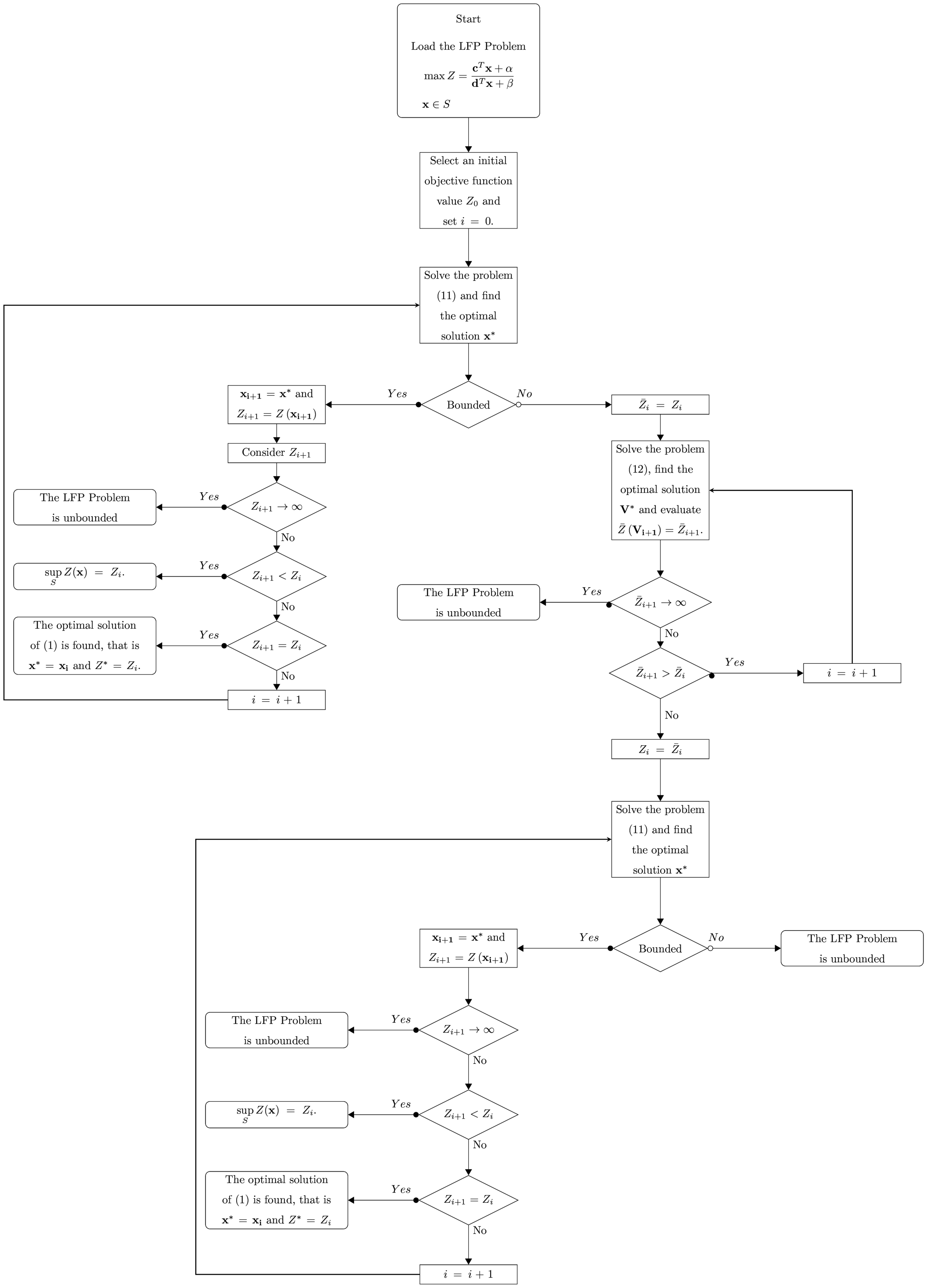

Figure 1. Flow chart of our algorithm.

Step 2 Solve Problem (11) and find the optimal solution $\mathbf{x}^*$.

- ***Step 2a.*** If Problem (11) is bounded, set $\mathbf{x_{i+1}} = \mathbf{x}^*$ and $Z_{i+1} = Z(\mathbf{x_{i+1}})$, and go to Step 3.
- ***Step 2b.*** If Problem (11) is unbounded, set $\bar{Z}_i = Z_i$ and go to Step 4.

Step 3 Consider the objective function value Z_{i+1}.

- ***Step 3a.*** If $Z_{i+1} \rightarrow \infty$, then the LFP Problem (1) is unbounded. STOP.
- ***Step 3b.*** If $Z_{i+1} < Z_i$, then $\sup_S Z(\mathbf{x}) = Z_i$. STOP.
- ***Step 3c.*** If $Z_{i+1} = Z_i$, then the optimal Solution of (1) is found, that is $\mathbf{x}^* = \mathbf{x_i}$ and $Z^* = Z_i$. STOP.
- ***Step 3d.*** Otherwise, set $i = i + 1$ and go to Step 2.

Step 4 Solve Problem (12), find the optimal solution $\mathbf{V}^*$ and evaluate $\bar{Z}(\mathbf{V_{i+1}}) = \bar{Z}_{i+1}$.

- ***Step 4a.*** If $\bar{Z}_{i+1} \rightarrow \infty$, then the LFP problem (1) is unbounded. STOP.
- ***Step 4b.*** If $\bar{Z}_{i+1} > \bar{Z}_i$, set $i = i + 1$ and go to Step 4.
- ***Step 4c.*** Otherwise, set $Z_i = \bar{Z}_i$ and go to Step 5.

Step 5 Solve Problem (11) and find the optimal solution $\mathbf{x}^*$.

- ***Step 5a.*** If Problem (11) is bounded, set $\mathbf{x_{i+1}} = \mathbf{x}^*$ and $Z_{i+1} = Z(\mathbf{x_{i+1}})$, and go to Step 6.
- ***Step 5b.*** If Problem (11) is unbounded, then the LFP, Problem (1) is unbounded. STOP.

Step 6 Consider the objective function value Z_{i+1}.

- ***Step 6a.*** If $Z_{i+1} \rightarrow \infty$, then the LFP Problem (1) is unbounded. STOP.
- ***Step 6b.*** If $Z_{i+1} < Z_i$, then $\sup_S Z(\mathbf{x}) = Z_i$. STOP.
- ***Step 6c.*** If $Z_{i+1} = Z_i$, then the optimal solution of Problem (1) is found, that is $\mathbf{x}^* = \mathbf{x_i}$ and $Z^* = Z_i$. STOP.
- ***Step 6d.*** Otherwise, set $i = i + 1$ and go to Step 5.

The flow chart of our algorithm is given in Figure 1.

4. Numerical experiments

In the following illustrative examples, the same step-numbering scheme as that used in the proposed algorithm is retained to enable the reader to directly follow and match each computational stage with its corresponding algorithmic step.

4.1. Illustrative examples

Example 1: Optimal solution case, consider the following example:

$$\max Z(\mathbf{x}) = \frac{2x_1 + 3x_2 - x_3 + 5}{x_1 + 2x_2 + 3x_3 + 2},$$
$$\text{s.t.,} \; -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{14}$$

Step 1. The initial point $\mathbf{x}_0$=(0, 0, 0.2) and the objective function value Z_0=1.8462 are determined by solving the following LP problem:

$$\max \; 0,$$
$$\text{s.t.,} \; -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{15}$$

Step 2. With these initial values, the LP problem corresponding to Problem (11) is formed as:

$$\max \; 0.1538x_1 - 0.6924x_2 - 6.5386x_3$$
$$\text{s.t.,} \; -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{16}$$

The optimal solution of Problem (16) is: $\mathbf{x_1} = (0, 1, 0)$ and the objective function value is $Z_1 = 2$.

Step 2a. Since the Problem (16) is bounded, then $\mathbf{x}_2$=$\mathbf{x}_1{}^*$=(0, 1, 0) and Z_2=$Z(\mathbf{x_2})$=2, and go to Step 3.

Step 3d. Since $Z_1 \neq Z_0$, then set $i = 1$, and go to Step 2.

Step 2. With these values, the LP problem corresponding to Problem (11) is formed as:

$$\max\ -x_2-7x_3,$$
$$\text{s.t.,}\ -2x_1+x_2+3x_3\le 2,$$
$$-x_1+x_2+5x_3\ge 1,$$
$$x_1\ge 0,\quad x_2\ge 0,\quad x_3\ge 0. \tag{17}$$

The optimal solution of Problem (17) is $\mathbf{x}_2=(0,1,0)$ and the objective function value is $Z_2=2$.

Step 3c. Since $Z_2=Z_1$, then the optimal solution of Example 1 and the optimal objective value are $\mathbf{x}^*=\mathbf{x}_2^*=(0,1,0)$ and $Z^*=Z_2^*=2$, respectively.

Example 2: An asymptotic solution to demonstrate an asymptotic solution for the LFP problem, let us consider the following example:

$$\max Z(\mathbf{x})=\frac{2x_1+3x_2-x_3}{x_1+2x_2+3x_3},$$
$$\text{s.t.,}\ -2x_1+x_2+3x_3\le 2,$$
$$-x_1+x_2+5x_3\ge 1,$$
$$x_1\ge 0,\quad x_2\ge 0,\quad x_3\ge 0. \tag{18}$$

Step 1. The initial point $\mathbf{x}_0=(0,0,0.2)$ and the objective function value $Z_0=-0.3333$ are determined by solving the following LP problem:

$$\max\ 0$$
$$\text{s.t.,}\ -2x_1+x_2+3x_3\le 2,$$
$$-x_1+x_2+5x_3\ge 1,$$
$$x_1\ge 0,\quad x_2\ge 0,\quad x_3\ge 0. \tag{19}$$

Step 2. With these initial values, the LP problem corresponding to Problem (11) is formed as:

$$\max\ 2.3333x_1+3.6667x_2$$
$$\text{s.t.,}\ -2x_1+x_2+3x_3\le 2,$$
$$-x_1+x_2+5x_3\ge 1,$$
$$x_1\ge 0,\quad x_2\ge 0,\quad x_3\ge 0. \tag{20}$$

The Problem (20) is unbounded, then go to Step 2b.

Step 2b. Since the Problem (20) is unbounded, then set $\bar{Z}_0=Z_0=-0,3333$, and go to Step 4.

Step 4. Consider the following problem:

$$\max\ 2.3333V_1+3.6667V_2$$
$$\text{s.t.,}\ -2V_1+V_2+3V_3\le 0,$$
$$-V_1+V_2+5V_3\ge 0,$$
$$V_1+V_2+V_3=1,$$
$$V_1\ge 0,\quad V_2\ge 0,\quad V_3\ge 0. \tag{21}$$

The optimal solution of Problem (21) is $\mathbf{V}_1=(0.3333,0.6667,0)$ and the objective function value is $\bar{Z}(\mathbf{V_1})=\bar{Z}_1=1.6$.

Step 4b. Since $\bar{Z}_1>\bar{Z}_0$, set $i=1$ and go to Step 4.

Step 4. Consider the following problem:

$$\max\ 0.4V_1-0.2V_2-5.8V_3$$
$$\text{s.t.,}\ -2V_1+V_2+3V_3\le 0,$$
$$-V_1+V_2+5V_3\ge 0,$$
$$V_1+V_2+V_3=1,$$
$$V_1\ge 0,\quad V_2\ge 0,\quad V_3\ge 0. \tag{22}$$

The optimal solution of Problem (22) is: $\mathbf{V}_2=(0.5,0.5,0)$ and the objective function value is $\bar{Z}_2=1.6667$.

Step 4b. Since $\bar{Z}_2>\bar{Z}_1$, set $i=2$ and go to Step 4.

Step 4. Consider the following problem:

$$\max\ 0.3333V_1-0.3334V_2-6.0001V_3$$
$$\text{s.t.,}\ -2V_1+V_2+3V_3\le 0,$$
$$-V_1+V_2+5V_3\ge 0,$$
$$V_1+V_2+V_3=1,$$
$$V_1\ge 0,\quad V_2\ge 0,\quad V_3\ge 0. \tag{23}$$

The optimal solution of Problem (23) is $\mathbf{V}_3$=(0.5, 0.5, 0) and the objective function value is $\bar{Z}_3=1.6667$.

Step 4c. Since $\bar{Z}_2=\bar{Z}_3$, set $Z_3=\bar{Z}_3$, and go to Step 5.

Step 5. Consider the following problem:

$$\max \ 0.3333x_1 - 0.3334x_2 - 6.0001x_3$$
$$\text{s.t.,} \ -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

The optimal solution of Problem (24) is: $\mathbf{x}_4 = (0,1,0)$ and the objective function value is $Z_4 = 1.5$.

Step 5a. The Problem (24) is bounded. Then, set $\mathbf{x}_4 = \mathbf{x}^*$ and $Z_4 = Z(\mathbf{x_4})$ go to Step 6b.

Step 6b. Since $Z_3 > Z_4$, then $\sup_S Z(\mathbf{x}) = Z_3 = 1.6667$.

Example 3: An unbounded solution to demonstrate an unbounded solution for the LFP problem, let us consider the following example:

$$\max Z(\mathbf{x}) = \frac{2x_1 + 3x_2 - x_3}{x_1 + 2x_2 + 3x_3 - 1}$$
$$\text{s.t.,} \ -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{25}$$

Step 1. The initial point $\mathbf{x}_0 = (0,0,0.2)$ and the objective function value $Z_0 = 0.5$ are determined by solving the following LP problem:

$$\max \ 0$$
$$\text{s.t.,} \ -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{26}$$

Step 2. With these initial values, the LP problem corresponding to Problem (11) formed as:

$$\max \ 1.5x_1 + 2x_2 - 2.5x_3$$
$$\text{s.t.,} \ -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{27}$$

Problem (27) is unbounded, then go to Step 2b.

Step 2b. Since Problem (27) is unbounded. Then, set $\bar{Z}_0 = Z_0 = 0.5$, and go to Step 4.

Step 4. Consider the following problem:

$$\max \ 0.3333V_1 + 0.6667V_2$$
$$\text{s.t.,} \ -2V_1 + V_2 + 3V_3 \leq 0,$$
$$-V_1 + V_2 + 5V_3 \geq 0,$$
$$V_1 + V_2 + V_3 = 1,$$
$$V_1 \geq 0, \quad V_2 \geq 0, \quad V_3 \geq 0. \tag{28}$$

The optimal solution of Problem (28) is: $\mathbf{V}_1 = (0.3333, 0.6667, 0)$ and the objective function value is $\bar{Z}(\mathbf{V_1}) = \bar{Z}_1 = 4$.

Step 4b. Since $\bar{Z}_1 > \bar{Z}_0$, set $i = 1$ and go to Step 4.

Step 4. Consider the following problem:

$$\max \ -2V_1 - 5V_2 - 13V_3$$
$$\text{s.t.,} \ -2V_1 + V_2 + 3V_3 \leq 0,$$
$$-V_1 + V_2 + 5V_3 \geq 0,$$
$$V_1 + V_2 + V_3 = 1,$$
$$V_1 \geq 0, \quad V_2 \geq 0, \quad V_3 \geq 0. \tag{29}$$

The optimal solution of Problem (28) is: $\mathbf{V}_2 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_2 = 5$.

Step 4b. Since $\bar{Z}_2 > \bar{Z}_1$, set $i = 2$ and go to Step 4.

Step 4. Consider the following problem:

$$\max \ -3V_1 - 7V_2 - 16V_3$$
$$\text{s.t.,} \ -2V_1 + V_2 + 3V_3 \leq 0,$$
$$-V_1 + V_2 + 5V_3 \geq 0,$$
$$V_1 + V_2 + V_3 = 1,$$
$$V_1 \geq 0, \quad V_2 \geq 0, \quad V_3 \geq 0. \tag{30}$$

The optimal solution of Problem (29) is $\mathbf{V}_3 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_3 = 5$.

Step 4c. Since $\bar{Z}_2 = \bar{Z}_3$, set $Z_3 = \bar{Z}_3$, and go to Step 5.

Step 5. Consider the following problem:

$$\max \ -3x_1 - 7x_2 - 16x_3$$
$$\text{s.t.,} \ -2x_1 + x_2 + 3x_3 \leq 2,$$
$$-x_1 + x_2 + 5x_3 \geq 1,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \tag{31}$$

Step 5b. The Problem (31) is unbounded, then the LFP problem is unbounded.

4.2. Generation of test problems

We randomly generate a number of test instances to validate the proposed solution method. The size of an instance is given by the Number of Variables (NoV) and Number of Constraints (NoC). For each problem instance, we generate the coefficient matrix $\mathbf{A}$, the right-hand side vector $\mathbf{b}$, and the objective coefficients $\mathbf{c}$, $\mathbf{d}$ and constants α and β that are unrestricted in sign for a specified (NoV, NoC) to meet the requirement $\left(\mathbf{d}^T\mathbf{x}+\beta\right) > 0$. All these parameters are generated as random integer numbers chosen from the uniform distribution.

By the generation of test problems, we aim to assess the effect of NoV and NoC on the total execution time and the iteration number of our algorithm. The sizes of the generated test problems, the maximum and minimum of the number of iterations and the CPU times (in seconds) of the algorithm are given in Table 1. Here, we note that one iteration refers to a new point generated by our algorithm. Also, the results in Table 1 do not contain unbounded, infeasible or asymptotic solution cases. For each class, ten problem instances are generated and solved using the software GAMS 24.1.3. All the computational experiments were carried out on a computer with a 2.7 GHz processor and 8GB RAM running Windows 10 Pro.

Table 1 also presents the comparison results of our algorithm with Ozkok [21]. The same starting point is used in the executions to make a reasonable comparison. In reporting CPU times, the time to obtain the starting point has been excluded.

Table 1 shows that our algorithm can reach the optimal solution for different sizes of the problem within a reasonable number of iterations, even with large problem instances. As the size of the problem increases, the CPU time is increasing as well for both algorithm. We can clearly say that our algorithm outperforms Ozkok [21] when both iteration numbers and CPU times are taken into account.

Additionally, Figures 2 and 3 represent the average number of iterations and average CPU time (in seconds) of our algorithm for the specified (NoV, NoC), respectively.

5. Conclusion

In this paper, an exact iterative algorithm by providing an extension and improvement to Ozkok [21] based on the traditional continuity definition is developed for the LFP problem. With this new extended algorithm, we aim at removing the model dependency on the big-M coefficients. Our proposed algorithm converts the LFP problem to a Linear Programming (LP) problem and constructs an iterative procedure. Whereas most of the existing methods in the literature find the optimal solution over a bounded convex region, our algorithm is able to solve all LFP problems that have a bounded or unbounded feasible region. Moreover, our algorithm is easily applicable and is capable of generating all types

Table 1. Comparison results of the proposed algorithm and [21].

Class		Number of iterations				CPU Time (in seconds)			
	(NoV, NoC)	Min Our method	Min [21]	Max Our method	Max [21]	Min Our method	Min [21]	Max Our method	Max [21]
1	(5, 5)	1	2	4	4	0.171	0.195	0.689	0.874
2	(10, 10)	2	2	4	10	0.176	0.256	0.779	0.895
3	(20, 20)	2	5	6	10	0.365	0.369	0.826	0.917
4	(30, 30)	3	6	5	10	0.604	0.71	1.078	1.264
5	(40, 40)	3	3	6	15	0.545	0.556	1.148	1.209
6	(50, 50)	3	7	10	11	0.42	0.499	0.733	0.767
7	(60, 60)	5	6	8	14	0.457	0.494	0.895	0.967
8	(70, 70)	4	6	8	14	0.566	0.581	0.765	0.797
9	(80, 80)	5	6	8	11	0.483	0.493	0.912	0.987
10	(90, 90)	5	5	9	18	0.497	0.499	0.958	1.071
11	(100, 100)	4	6	9	12	0.697	0.882	1.271	1.382
12	(200, 200)	6	6	9	8	1.031	1.106	1.476	1.645
13	(350, 350)	6	6	8	7	1.445	1.514	2.226	2.666
14	(500, 500)	6	6	7	9	3.604	3.72	6.176	6.876
15	(750, 750)	6	7	8	11	11.965	12.217	17.825	18.684
16	(1000, 1000)	6	7	7	10	32.642	34.01	39.135	39.372

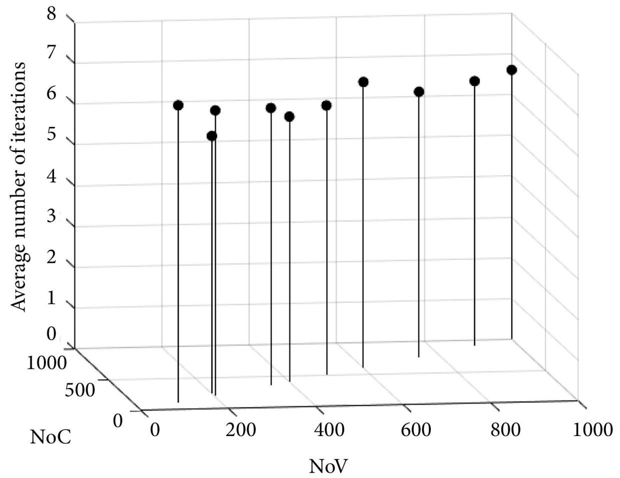

Figure 2. (NoV, NoC) versus average number of iterations.

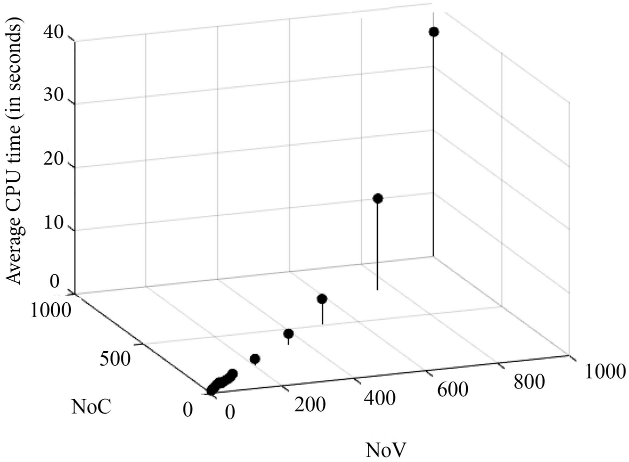

Figure 3. (NoV, NoC) versus average CPU time (in seconds).

of solutions of LFP such as optimal, unbounded or asymptotic. As a result, considering the performance of our algorithm on large-scale examples, it can be said that it is more effective than Ozkok's algorithm. A limitation of our algorithm is that it cannot solve LFP problems involving frequently encountered uncertainty situations in real life. In future studies, our algorithm will be modified to solve fuzzy LFP problems. In addition, we aim to apply our algorithm to real life problems by investigating specific application areas such as resource allocation, transportation, production, finance, location theory, etc.

Acknowledgments

This work has been supported by Yildiz Technical University Scientific Research Projects Coordination Unit under project number FBA-2020-3689.

Authors contribution statement

First author: Methodology; Software; Formal analysis; Investigation; Writing original draft; Writing review and editing; Visualization; Validation; Data curation, Funding acquisition.

Second author: Methodology; Formal analysis; Investigation; Writing original draft; Writing review and editing; Visualization; Validation; Data curation.

Third author:
Conceptualization; Methodology; Formal analysis; Validation; Supervision.

References

1. Chakraborty, M. and Gupta, S. "Fuzzy mathematical programming for multi objective linear fractional programming problem", *Fuzzy Sets and Systems*, **125**(3), pp. 335–342 (2002).
https://doi.org/10.1016/S0165-0114(01)00060-4

2. Schaible, S. "Bibliography in fractional programming", *Zeitschrift fur Operations Research*, **26**, pp. 211–241 (1982).
https://doi.org/10.1007/BF01917115

3. Schaible, S. "Fractional programming", *Zeitschrift fur Operations-Research*, **27**, pp. 39–54 (1983).
https://doi.org/10.1007/BF01916898

4. Schaible, S. "Fractional programming: Applications and algorithms", *European Journal of Operational Research*, **7**(2), pp. 111–120 (1981).
https://doi.org/10.1016/0377-2217(81)90272-1

5. Craven, B. "Fractional programming", *Sigma Series in Applied Mathematics*, Heldermann, Berlin, Germany, **4** (1988).

6. Bajalinov, E.B. "Linear-fractional programming theory, methods, applications and software", *Springer Science* and *Business Media*, **84** (2013).
https://doi.org/10.1007/978-1-4419-9174-4

7. Isbell, J. and Marlow, W. "Attrition games", *Naval Research Logistics Quarterly*, **3**(1-2), pp. 71–94 (1956).
https://doi.org/10.1002/nav.3800030108

8. Charnes, A. and Cooper, W.W. "Programming with linear fractional functionals", *Naval Research Logistics Quarterly*, **9**(3-4), pp. 181–186 (1962).
https://doi.org/10.1002/nav.3800090303

9. Bitran, G.R. and Novaes, A. "Linear programming with a fractional objective function", *Operations Research*, **21**(1), pp. 22–29 (1973).

https://doi.org/10.1287/opre.21.1.22

10. Bitran, G.R. and Magnanti, T.L. "Duality and sensitivity analysis for fractional programs", *Operations Research*, **24**(4), pp. 675–699 (1976).
https://doi.org/10.1287/opre.24.4.675

11. Swarup, K. "Letter to the editor_Linear fractional functionals programming, *Operations Research*, **13**(6), pp. 1029–1036 (1965).
https://doi.org/10.1287/opre.13.6.1029

12. Singh, C. "Optimality conditions in fractional programming", *Journal of Optimization Theory and Applications*, **33**(2), pp. 287–294 (1981).
https://doi.org/10.1007/BF00935552

13. Bajalinov, E. "Adjusting an objective function to a given optimal solution in linear and linear-fractional programming", In: *Constructing and Applying Objective Functions*, pp. 233–244 (2002).
https://doi.org/10.1007/978-3-642-56038-5

14. Tantawy, S. "A new procedure for solving linear fractional programming problems", *Mathematical and Computer Modelling*, **48**(5-6), pp. 969–973 (2008).
https://doi.org/10.1016/j.mcm.2007.12.007

15. Tantawy, S. "An iterative method for solving linear fraction programming problem with sensitivity analysis", *Mathematical and Computational applications*, **13**(3), pp. 147–151 (2008).
https://doi.org/10.3390/mca13030147

16. Effati, S. and Pakdaman, M. "Solving the interval-valued linear fractional programming problem", *American Journal of Computational Mathematics*, **2**(01), 17965 (2012).
https://doi.org/10.4236/ajcm.2012.21006

17. Tantawy, S. "A new concept of duality for linear fractional programming problems", *International Journal of Engineering and Innovative Technology*, **3**, pp. 147–149 (2014).

18. Simi, F.A. and Talukder, M.S. "A new approach for solving linear fractional programming problems with duality concept", *Open Journal of Optimization*, **6**(1), 74095 (2017).
https://doi.org/10.4236/ojop.2017.61001

19. Biswas, A., Verma, S., and Ojha, D. "Optimality and convexity theorems for linear fractional programming problem", *International Journal of Computational and Applied Mathematics*, **12**(3), pp. 911–916 (2017).

20. Fu, Q., Li, L., Li, M., et al. "A simulation-based linear fractional programming model for adaptable water allocation planning in the main stream of the Songhua River Basin, China", *Water*, **10**(5), 627 (2018).
https://doi.org/10.3390/w10050627

21. Ozkok, B.A. "An iterative algorithm to solve a linear fractional programming problem", *Computers and Industrial Engineering*, **140**, 106234 (2020).
https://doi.org/10.1016/j.cie.2019.10623

22. Das, S.K., Edalatpanah, S., and Mandal, T. "A new method for solving linear fractional programming problem with absolute value functions", *International Journal of Operational Research*, **36**(4), pp. 455–466 (2019).
https://doi.org/10.1504/IJOR.2019.104051

23. Mohammed, N.M. and Lomte, S.S. "Secure and efficient outsourcing of large scale linear fractional programming", In *Computing in Engineering and Technology*, Iyer, B., Deshpande, P.S., Sharma, S.C., and Shiurkar U. (Eds.), pp. 277–286, Springer, Singapore (2020).
https://doi.org/10.1007/978-981-32-9515-5_26

24. Mohammed, N.M., Sultan, L.R., Hamoud, A.A., and Lomte, S.S. "Verifiable secure computation of linear fractional programming using certificate validation", *International Journal of Power Electronics and Drive Systems*, **11**(1): 284, pp. 284–290 (2020).
https://doi.org/10.11591/ijpeds.v11.i1.pp284-290

25. Mohammady, S. and Eslahchi, M. "Extension of Tikhonov regularization method using linear fractional programming", *Journal of Computational and Applied Mathematics*, **371**, 112677 (2020).
https://doi.org/10.1016/j.cam.2019.112677

26. Toloo, M. "An equivalent linear programming form of general linear fractional programming: A duality approach", *Mathematics*, **9**(14), 1586 (2021).
https://doi.org/10.3390/math9141586

Biographies

Hale Gonce Kocken is an Associate Professor in the Department of Mathematical Engineering at the Yildiz Technical University (YTU), Istanbul, Turkey. She has been a faculty member of YTU since 2004. She completed her PhD entitled Fuzzy approaches to network analysis in Applied Mathematics (2011) from the same department. Her current area of research is mathematical Programming, supply chain management and some related Operational Research subjects in multi criteria and fuzzy environments. Additionally, she has doing research about multi criteria decision making methods with non-traditional fuzzy numbers and also fractional programming.

Beyza Ahlatcioglu Ozkok has been working as a Professor in the Department of Business Administration at Yildiz Technical University since 2019. She received her BS in Mathematics from Istanbul University in 2002, and an MS from Istanbul University and Marmara University in 2004. She received her PhD in Quantitative Methods from Istanbul University in 2009. During 2010-2011 she was a Visiting Researcher at Center for Applied Optimization of Industrial and Systems Engineering at the University of Florida. Her research interests are in fuzzy mathematical programming, multi criteria decision making, game theory and supply chain modeling.

Mehmet Ahlatcioglu is a Professor in the Department of Mathematics at Yildiz Technical University, Turkey. He received his PhD in Quantitative Methods from Istanbul University and in Mathematics from Yildiz Technical University in 1984 and 1989, respectively. His research interests are fuzzy mathematical programming, fractional programming, multi-objective programming, and game theory.