

# An accurate mathematical approach for calculating capillary pressure in drainage and imbibition from centrifuge data

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## Keywords

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Imbibition;  
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Space (RKHS) method;  
Singular integral equation.

## Abstract

The conversion of centrifuge data into capillary pressure curves is crucial for rock capillary pressure measurement in various applications. This process involves converting measured fluid productions into local saturation values, and the accuracy and efficiency of this procedure are essential. This paper addresses the challenge of achieving accuracy without sacrificing computational efficiency by introducing a new method based on the Reproducing Kernel Hilbert Space (RKHS) technique. This approach enables the conversion of capillary pressure versus average saturation data into capillary pressure versus local (outlet) saturation. The RKHS method is applied to both drainage and imbibition centrifuge data, and its efficiency and accuracy are evaluated using both artificially generated and experimental datasets. The results obtained with the RKHS method are compared and validated against other existing methods.

## 1. Introduction

Capillary pressure curves are extensively used in the petroleum industry, such as two and three-phase fluid flow simulation in porous media, making decisions about Enhanced Oil Recovery (EOR) management, determining rock wettability, and estimating pore size distribution [1-7]. The equation for centrifuge capillary pressure represents a singular integral equation, and researchers have proposed various approximate solutions to this equation [8,9]. Recently, Bursey et al. [10] even employed machine learning techniques for predicting centrifuge capillary pressure.

In centrifuge experiments, average saturation  $\bar{S}$  is measured against capillary pressure  $P_{c1}$  at the inlet face of a rock sample during rotation at various angular velocities  $\omega$  (Figure 1). Brunner [11] and Hermansen et al. [12] established the relationship between average saturation  $\bar{S}(P_{c1})$  and local saturation  $S(P_c)$ . Their assumptions include hydrostatic equilibrium in each phase and Dirichlet boundary condition of  $P_c = 0$  at the outflow. Capillary pressure can be measured at any distance ( $r$ ) from the centrifuge axis for drainage experiments using the following equation:

$$P_{c_r} = \frac{1}{2} \Delta \rho \omega^2 (r_2^2 - r^2), \quad (1)$$

where:  $P_{c_r}$  is the capillary pressure at radius  $r$ ;  $\Delta \rho$  the difference between the phase densities;  $\omega$  the rotation speed;  $r_1$  and  $r_2$  the radii at the core faces having minimum and maximum distance from the center, respectively.

Based on Eq. (1), capillary pressure at the face near the center is:

$$P_{c1} = \frac{1}{2} \Delta \rho \omega^2 (r_2^2 - r_1^2). \quad (2)$$

Using Eqs. (1) and (2), we have:

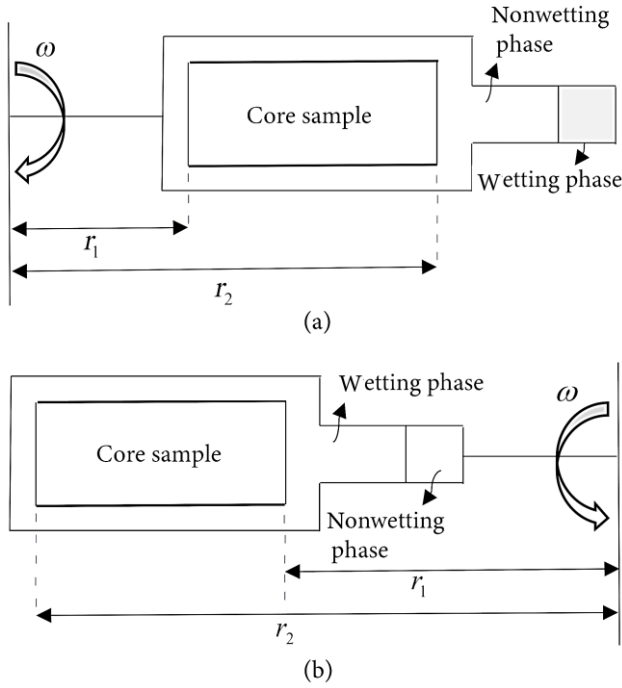
$$P_c = P_{c1} \frac{(r_2^2 - r^2)}{(r_2^2 - r_1^2)}. \quad (3)$$

The proportion between  $S$  and its average is:

$$\bar{S} = \frac{1}{(r_2 - r_1)} \int_{r_1}^{r_2} S(r) dr. \quad (4)$$

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**Figure 1.** Centrifuge experiment setup: (a) Drainage and (b) imbibition experiment setup.

Finally, the following equation can be derived after mathematical calculation:

$$\bar{S}(P_{C_1}) = \frac{r_1 + r_2}{2\sqrt{P_{C_1}}} \int_0^{P_{C_1}} \frac{S(Pc)}{\sqrt{r_2^2 P_{C_1} - (r_2^2 - r_1^2) Pc}} dPc, \quad (5)$$

or:

$$\bar{S}(P_{C_1}) = \frac{r_1 + r_2}{2r_2\sqrt{P_{C_1}}} \int_0^{P_{C_1}} \frac{S(Pc)}{\sqrt{P_{C_1} - BPc}} dPc, \quad (6)$$

where:

$$B = 1 - \left(\frac{r_1}{r_2}\right)^2, \quad 0 \leq B \leq 1. \quad (7)$$

For imbibition experiments, similar equations are obtained exchanging  $r_1$  for  $r_2$ :

$$B = 1 - \left(\frac{r_2}{r_1}\right)^2, \quad (8)$$

and  $P_{C_1}$  for  $P_{C_2}$ , where  $P_{C_2}$  is the capillary pressure at radius  $r_2$  [12,13]. Both drainage and imbibition experiments, capillary pressure ( $S(Pc)$ ) aim to obtain from centrifuge data ( $\bar{S}(P_{C_1})$ ) using Eq. (6).

Various methods have been proposed to approximate the solution of Eq. (6). Hassler and Brunner assumed that the pressure field in the core is linear and the gravity has no influence on the pressure field [14]. These assumptions are valid for narrow and short samples inside a centrifuge with a long rotational axis. So, this method gives smaller saturations compared to the exact solution [15]. The Hoffman method [16], van Domselaar method [17], Rajan

method [18], Forbes methods [19], and Nazari Moghaddam [20] address the solution of fundamental Eq. (6). It is revealed that all mentioned methods use a simplified version of Eq. (6). It is because the weakly singular Volterra integral Eq. (6) is known to be ill-conditioned [11,12]. In other words, error in local saturation increases during inverting noisy average saturation data. In this regard, various interpretation procedures were carried out [15,21, 22]. However, for better accuracy, these methods need special experimental facilities. Forbes methods have simple implementation but are sensitive to pressure step size and give fluctuated results [19]. Except for the Forbes methods, other mentioned methods such as Nazari Moghaddam and Hassler and Brunner address the solution of only drainage experiments. Considering the numerous industrial applications of the centrifuge technique, a robust method that provides accurate results with less computational time is still demanded in both drainage and imbibition experiments.

In this paper, Reproducing Kernel Hilbert Space (RKHS) method is introduced for solving the fundamental centrifuge equation (Eq. (6)) to obtain local saturation from centrifuge data for both drainage and imbibition experiments.

## 2. RKHS method

The theory of reproducing kernel, initially introduced by Zaremba [23], has been widely used to solve boundary value problems [24]. Aronszajn [25] further developed the theory, introducing Bergman kernel functions that have found applications in solving various problems. Cui and Lin [26], after summarizing RKHS theory, have solved different problems. Karatas Akgül [27] has proposed the RKHS method for nonlinear boundary-value problems. Foroutan et al. [28] used the RKHS method for computing solutions of nonlinear third-order ordinary differential equations under multipoint boundary conditions. Recently, the RKHS method has been used to address problems across different scientific domains [29–36].

Fundamentally, the centrifuge equation (Eq. (6)) can be rearranged as follows:

$$F(P_{C_1}) = \int_0^{P_{C_1}} K(P_{C_1}, Pc) S(Pc) dPc, \quad P_{C_1} \geq 0, \quad (9)$$

where

$$F(P_{C_1}) = \frac{2r_2}{r_1 + r_2} \left[ \bar{S}(P_{C_1}) \sqrt{P_{C_1}} \right], \quad (10)$$

and

$$K(P_{C_1}, Pc) = \frac{1}{\sqrt{P_{C_1} - BPc}}. \quad (11)$$

Above equations can be rewritten as:

$$f(P_{C_1}) = \int_0^{P_{C_1}} K(P_{C_1}, Pc) s(Pc) dPc, \quad 0 \leq P_{C_1} \leq 1, \quad (12)$$

where

$$f(P_{C_1}) = \frac{1}{\sqrt{P_{C_{\max}}}} F(P_{C_{\max}} P_{C_1}), \quad (13)$$

and

$$s(P_{C_1}) = S(P_{C_{\max}} P_{C_1}). \quad (14)$$

$P_{C_{\max}}$  is defined as the maximum capillary pressure data that centrifuge experiments are conducted. The space  $W^m[0,1]$  is defined as follows:

$$W^m[0,1] = \{s(x) \mid s^{(m-1)}(x) \in AC[0,1], s^{(m)}(x) \in L^2[0,1]\}, \quad (15)$$

where  $AC[0,1]$  stands for an absolutely continuous real-valued function on  $[0,1]$ . The following concepts:

$$(s, q)_m = \sum_{i=0}^{m-1} s^{(i)}(0) q^{(i)}(0) + \int_0^1 s^{(m)}(x) q^{(m)}(x) dx, \quad (16)$$

and

$$\|s\|_m = \sqrt{(s, s)_m}, s, q \in W^m[0,1], \quad (17)$$

are the inner product and norm in  $W^m[0,1]$  respectively.  $W^m[0,1]$  is a complete Hilbert space and contains a unique reproducing kernel function [37]. Therefore,  $W^m[0,1]$  is defined as a (RKHS). Let  $R_x^m(y)$  denote the reproducing kernel of  $W^m[0,1]$ . For every  $s \in W^m[0,1]$  and fixed  $x \in [0,1]$ ,  $R_x^m(y)$  satisfies the following reproducing property:

$$s(x) = (s(y), R_x^m(y))_m. \quad (18)$$

For more detail of  $R_x^m(y)$  calculation see Appendix A.

$R_x^1(y)$  is as follow:

$$R_x^1(y) = \begin{cases} 1+y, & y \leq x, \\ 1+x, & y > x. \end{cases} \quad (19)$$

For  $\{x_j\}_{j=1}^n, x_j \in [0,1]$ , we define:

$$\psi_j(x) = R_x(x_j) + \int_0^{x_j} K(x_j, t) R_x(t) dt. \quad (20)$$

An orthonormal system of  $\{\bar{\psi}_j\}_{j=1}^n$  is derived by applying the Gram-Schmidt orthogonalization process on  $\{\psi_j\}_{j=1}^n$ . In other words:

$$\bar{\psi}_j(x) = \sum_{k=1}^j \beta_{jk} \psi_k(x), \quad (21)$$

where  $\beta_{jk}, k = 1, 2, \dots, j$ , are the Gram-Schmidt orthogonalization coefficients. The approximate solution of Eq. (12) can be written as:

$$s_n(x) = \sum_{j=1}^n \sum_{k=1}^j \beta_{jk} f(x_k) \bar{\psi}_j(x). \quad (22)$$

For detailed computational procedures and proof, readers are referred to Beyrami et al. [24] and the provided appendix.

### 3. Results and discussion

Two sets of examples are used in this study. The first step (validation part) was an accuracy assessment on an

artificially generated dataset. These examples have exact solutions reported in the literature and are used for error analysis and validation [19,20]. Different numbers of data points are used to demonstrate the convergence of the approximate solution to the theoretical function as the data points increased. The results of the different values of  $B$  are used to consider the effect of  $B$  on the approximate solution of the RKHS method. The next two sets of experimental centrifuge data were used to consider the performance of the RKHS method. The next two sets of experimental centrifuge data that contains capillary pressure versus average saturation, were used to consider the performance of the RKHS method. These examples lack exact responses and error evaluation cannot be done for them. Therefore, these examples are presented as case studies for practical application.

#### 3.1. Validation

In this section, the applicability of the proposed method is tested on artificially generated datasets for both drainage and imbibition experiments. Moreover, the accuracy is evaluated and compared with the methods from the literature for different numbers of experimental data points and the values of  $B$ . Forbes's second method is believed to be the most accurate and simple method [19]. Therefore, in the following figures, the RKHS method is compared with the Forbes method, and comparison results with other methods are presented in tables. For this purpose, the Root Mean Square Error (RMSE) and the relative error with the following definitions:

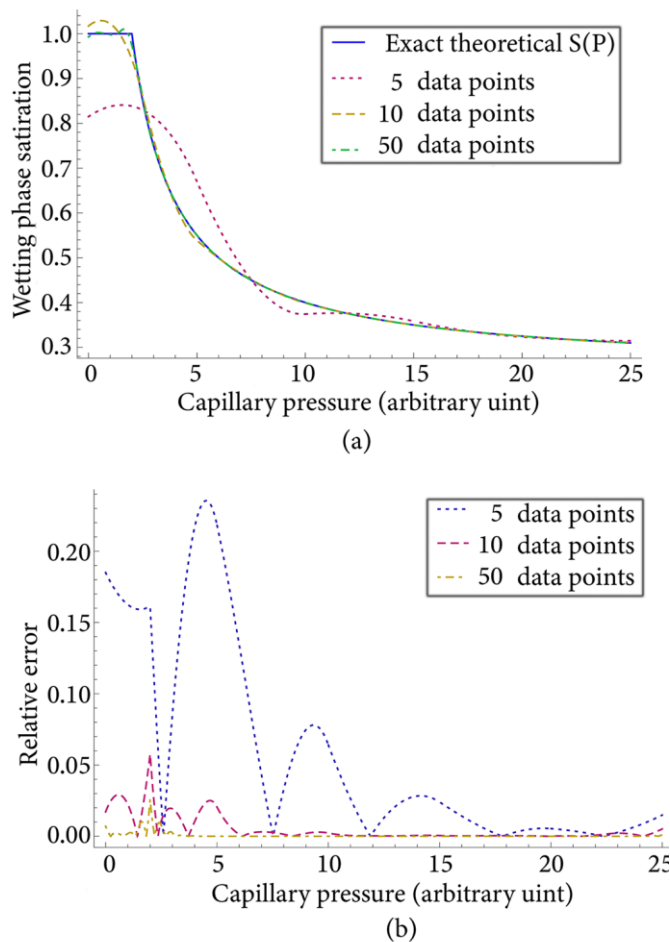
$$\begin{aligned} \text{RMSE} &= \sqrt{\frac{\sum_{i=1}^n (S_{\text{theoretical},i} - S_{\text{predicted},i})^2}{n-1}}, \\ \text{Relative Error} &= \frac{|S_{\text{theoretical}} - S_{\text{predicted}}|}{|S_{\text{theoretical}}|}, \end{aligned} \quad (23)$$

are reported in the tables and figures, respectively. Used functions are assumed to be the exact local capillary function or saturation function  $S(P_{C_1})$ . Using Eq. (6), the centrifuge data that are average saturation,  $\bar{S}(P_{C_1})$ , are artificially obtained. Predicted local saturation function  $S_{n,B}(P_{C_1})$  for different values of  $n$  and  $B$  are obtained by (22) as  $S_{n,B}(P_{C_1}) = s_{n,B}(P_{C_1} / P_{C_{\max}})$ .

**Example D1:** The following function is used for drainage experiments as the exact theoretical local saturation by Forbes [19] and Nazari Moghaddam [20]:

$$S(P_c) = \begin{cases} 1, & 0 < P_c \leq 2, \\ \frac{1.5}{P_c} + 0.25, & P_c > 2. \end{cases} \quad (24)$$

**Example D2:** The following function is used for drainage experiments as the exact theoretical local saturation by Nazari Moghaddam [20]:



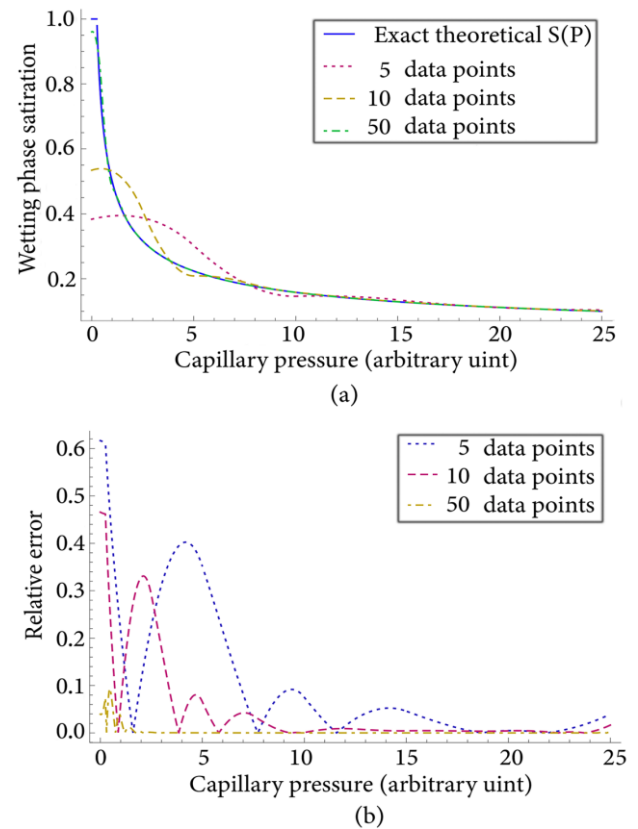
**Figure 2.** (a) Exact and local saturation obtained by RKHS method for Example D1 with  $B=0.75$  and  $n=5, 10$ , and 50 data points; (b) the relative error of obtained approximate solutions.

$$S(P_c) = \begin{cases} 1, & 0 < P_c \leq 0.25, \\ \frac{0.5}{\sqrt{P_c}}, & P_c > 0.25. \end{cases} \quad (25)$$

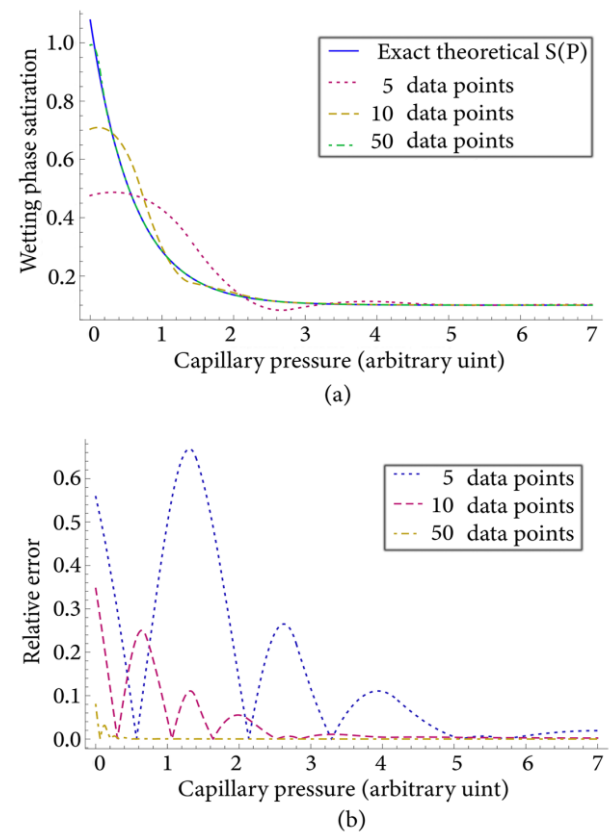
**Example D3:** The following function is used for drainage experiments as the exact theoretical local saturation by Forbes [19]:

$$S(P_c) = 0.1 - 0.9 \exp(-1.66P_c + 0.083). \quad (26)$$

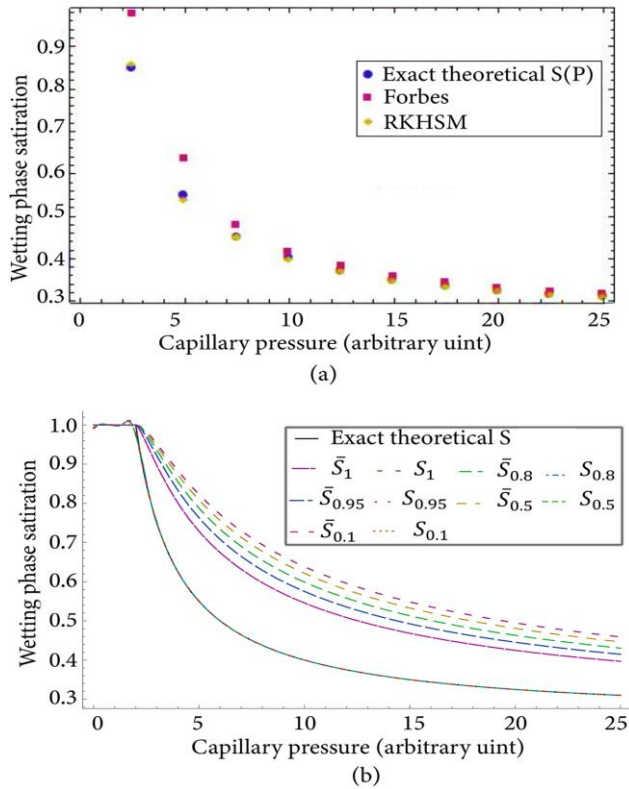
Figures 2, 3, and 4 illustrate the solution and the relative error of the RKHS method for  $B = 0.75$  and the varying numbers of data points for the drainage experiment. The derived solution converges to the precise answer as the number of data points grows. Figures 5(a), 6(a), and 7(a) show the comparison result of RKHS and the Forbes method with the precise theoretical  $S(P_c)$  for  $B = 0.75$  and  $n = 10$  data points for the drainage experiment. Figures 5(b), 6(b), and 7(b) indicate the interpretation of  $S_B(P_c)$  and  $\bar{S}_B(P_c)$  for  $n = 50$  data points and  $B = 0.1, 0.5, 0.8, 0.95$ , and 1 for the drainage experiment. As a result, the RKHS method has good accuracy for different values of  $B$ . Table 1 also presents the superior results of the RKHS method for drainage experiments versus the Hassler and Brunner solution [14]



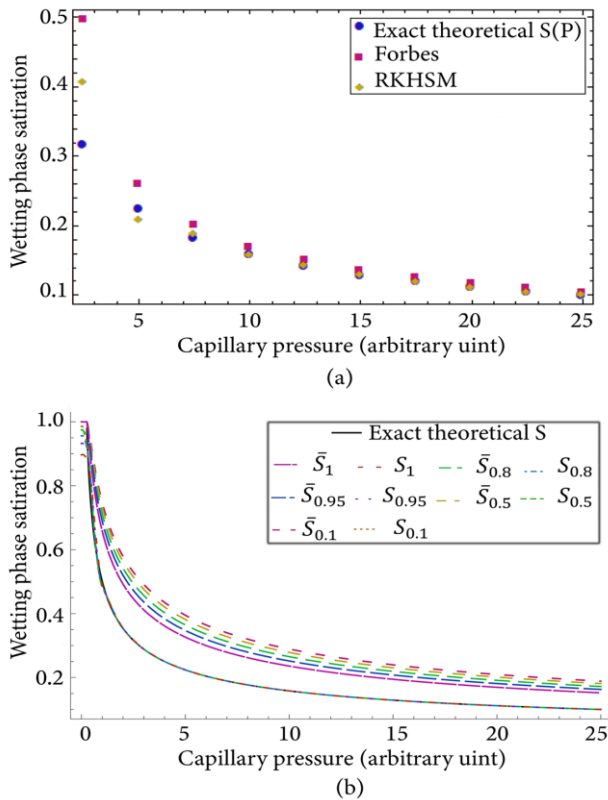
**Figure 3.** (a) Exact and local saturation obtained by RKHS method for Example D2 with  $B=0.75$  and  $n=5, 10, 50$  data points; (b) the relative error of obtained approximate solutions.



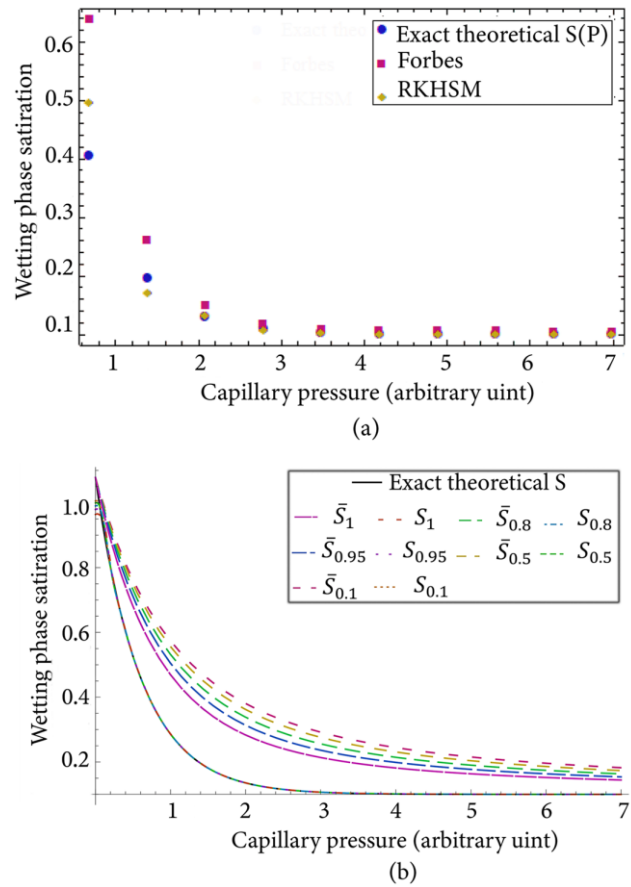
**Figure 4.** (a) Exact and local saturation obtained by RKHS method for Example D3 with  $B=0.75$  and  $n=5, 10$ , and 50 data points; (b) the relative error of obtained approximate solutions.



**Figure 5.** (a) Comparison of local saturation obtained by RKHS and Forbes methods versus the exact function for Example D1 with  $B=0.75$  and  $n=10$  data points; (b) interpretation of  $S_B(Pc)$  and  $\bar{S}_B(Pc_1)$  for Example D1 with  $n=50$  data points and  $B=0.1, 0.5, 0.8, 0.95$ , and  $1$ .



**Figure 6.** (a) Comparison of local saturation obtained by RKHS and Forbes methods versus the exact function for Example D2 with  $B=0.75$  and  $n=10$  data points; (b) interpretation of  $S_B(Pc)$  and  $\bar{S}_B(Pc_1)$  for Example D2 with  $n=50$  data points and  $B=0.1, 0.5, 0.8, 0.95$ , and  $1$ .



**Figure 7.** (a) Comparison of local saturation obtained by RKHS and Forbes methods with the exact function for Example D3 with  $B=0.75$  and  $n=10$  data points; (b) interpretation of  $S_B(Pc)$  and  $\bar{S}_B(Pc_1)$  for Example D3 with  $n=50$  data points and  $B=0.1, 0.5, 0.8, 0.95$ , and  $1$ .

**Table 1.** The calculated RMSE of different methods for different drainage examples.

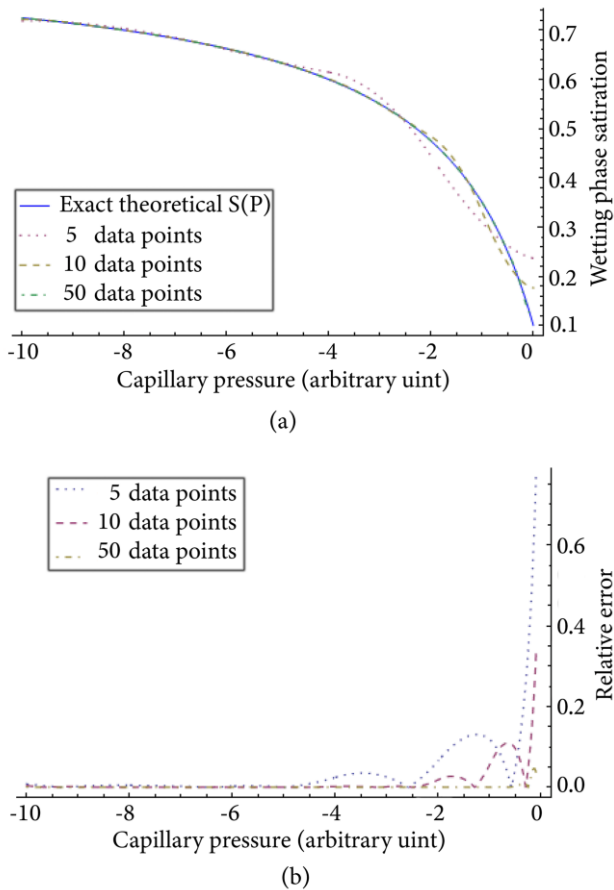
	Example D1	Example D2	Example D3
Hassler and Brunner [14]	0.184	0.118	0.158
Nazari Moghaddam [20]	0.048	0.059	0.088
Forbes [19]	0.053	0.062	0.080
RKHSM	4.91E-3	0.031	0.031

and the Nazari Moghaddam solution [20]. The RMSE values for drainage examples D1, D2, and D3 stand at  $4.19\text{E-}3$ ,  $0.031$ , and  $0.031$ , respectively, marking them as the lowest values among the compared methods. As shown, the RKHS method is more accurate than the other methods.

**Example 11:** The following function is used for imbibition experiments as the exact theoretical local saturation by Forbes [19]:

$$S(Pc) = \begin{cases} 0.1, & 0 \geq Pc \geq -0.001, \\ 0.1 + 0.1 \log(-1000 Pc), & Pc < -0.001. \end{cases} \quad (27)$$

**Example 12:** The following function is used for imbibition experiments as the exact theoretical local saturation by Forbes [19]:



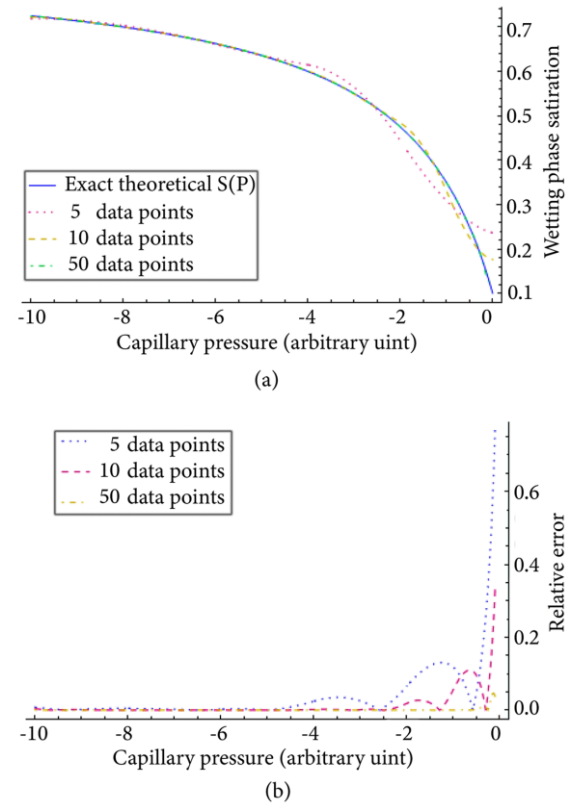
**Figure 8.** (a) Exact and local saturation obtained by RKHS method for Example I1 with  $B=-4.0$  and  $n=5, 10, 50$  data points; (b) The relative error of obtained approximate solutions.

**Table 2.** The calculated RMSE of different methods for different drainage examples.

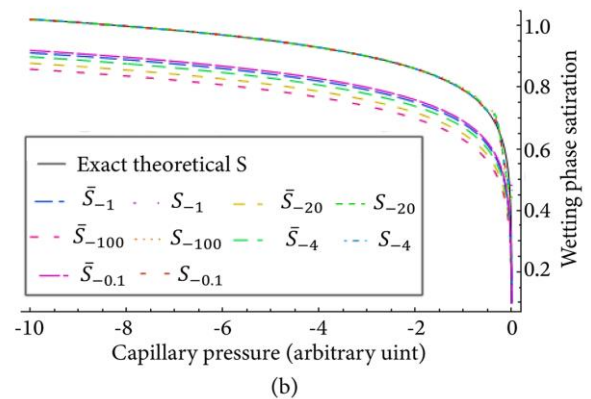
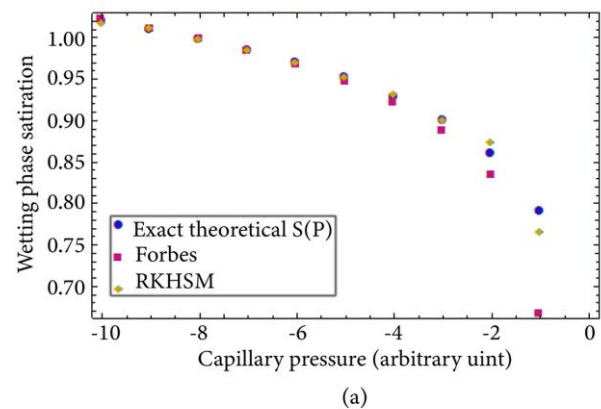
	Example I1	Example I2
Forbes [19]	0.041	0.047
RKHSM	9.41E-3	7.29E-3

$$S(Pc) = 0.85 + 1.5 / (Pc - 2). \quad (28)$$

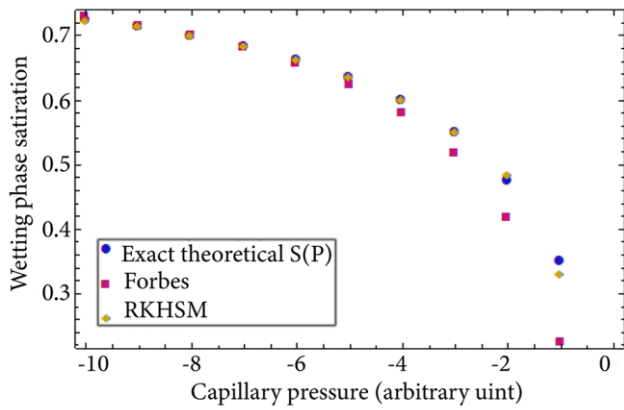
Figures 8 and 9 illustrate the solution and the absolute error of the RKHS method for  $B = -4.0$  and the different numbers of data points for the imbibition experiment. As the number of data points rises, the resultant solution converges to reliable response. Figures 10(a) and 11(a) demonstrate the comparative result of RKHS and Forbes methods with exact theoretical  $S(Pc)$  for  $B = -4.0$  and  $n = 10$  data points for the imbibition experiment. Figures 10(b) and 11(b) indicate interpretation of  $S_B(Pc)$  and  $\bar{S}_B(Pc_1)$  for  $n=50$  data points and  $B=-0.1, -1, -4, -20$ , and  $-100$  for imbibition experiment. Therefore, the RKHS method has good accuracy for different values of  $B$ . Table 2 also presents the RMSE superior results of the RKHS method for imbibition experiments versus the Forbes method solution [19]. As shown, for imbibition Examples I1 and I2, the RMSE values are  $9.41E-3$  and  $7.29E-3$ , both of which are lower than the error associated with the Forbes method.



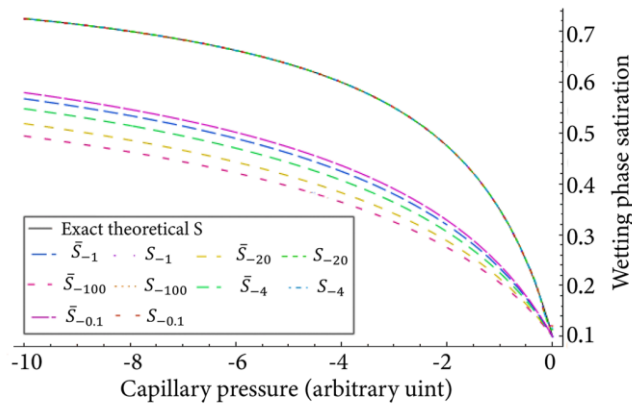
**Figure 9.** (a) Exact and local saturation obtained by RKHS method for Example I2 with  $B=-4.0$  and  $n=5, 10, 50$  data points; (b) the relative error of obtained approximate solutions.



**Figure 10.** (a) Comparison of local saturation obtained by RKHS and Forbes methods with exact function for Example I1 with  $B=-4.0$  and  $n=10$  data points; (b) interpretation of  $S_B(Pc)$  and  $\bar{S}_B(Pc_1)$  for Example I1 with  $n=50$  data points and  $B=-0.1, -1, -4, -20$ , and  $-100$ .



(a)



(b)

**Figure 11.** (a) Comparison of local saturation obtained by RKHS and Forbes methods with exact function for Example I2 with  $B=-4.0$  and  $n=10$  data points; (b) Interpretation of  $S_B(P_c)$  and  $\bar{S}_B(P_{c1})$  for Example I2 with  $n=50$  data points and  $B=-0.1, -1, -4, -20$ , and  $-100$ .

**Table 3.** Rock sample properties of Example D4.

Property	Value	Unit
Porosity	17.424	Percent
Absolute permeability	0.806	mD
Length	5.101	cm
Diameter	3.799	cm
Grain density	2.836	gr/cc

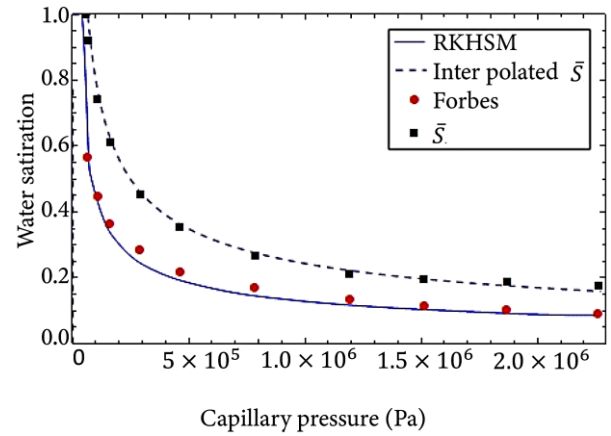
### 3.2. Case study

In this section, two sets of experimental centrifuge data for drainage and imbibition experiments are used to compare the RKHS method with the Forbes method [19]. Figure 12(a) and (b) illustrate the following fitted curve:

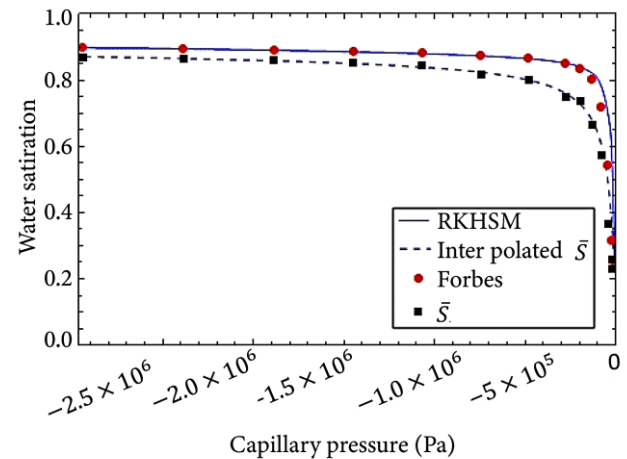
$$\bar{S}(P_c) = \frac{a P_c^b + c}{d P_c^e + f}, \quad (29)$$

with experimental centrifuge data  $\bar{S}$ , based on RKHS and Forbes methods for each case study. RKHS and Forbes methods are compared using interpolated  $\bar{S}$  data. The results of this comparison showed the robustness and accuracy of the RKHS method across different cases.

**Example D4:** A set of experimental centrifuge data  $\bar{S}$  for drainage experiment is used to compare the RKHS method with the Forbes method [19]. The rock sample properties are in Table 3.



(a)



(b)

**Figure 12.** (a) Interpretation of measured saturation for plug Example D4 where  $B=0.683148$ , based on RKHS and Forbes method for drainage experiment; (b) interpretation of measured saturation for plug Example I3 where  $B=-1.06773$ , based on RKHS and the Forbes method for imbibition experiment.

**Table 4.** Rock sample properties of Example I3.

Property	Value	Unit
Porosity	20.827	Percent
Absolute permeability	3.362	mD
Length	5.110	cm
Diameter	3.799	cm
Grain density	2.851	gr/cc

**Example I3:** A set of experimental centrifuge data  $\bar{S}$  for the imbibition experiment is used to compare the RKHS method with the Forbes method [19]. Table 4 contains the rock sample properties.

## 4. Conclusions

In this paper, the solution of the fundamental centrifuge equation is derived using the Reproducing Kernel Hilbert Space (RKHS) method. The proposed method provides an accurate and efficient solution for calculating capillary pressure from centrifuge data. The accuracy of the RKHS method is validated and compared with other methods using both artificially generated and experimental data. The method demonstrated its versatility and computational efficiency across a range of  $B$  values, offering a promising

solution for analyzing experimental results in both drainage and imbibition studies. The results obtained with the RKHS method exhibited significantly higher precision compared to the other methods.

## Nomenclature

### Latin

$r$	Radial distance from the centrifuge axis to an arbitrary point in the centrifuge core
$r_1$	$r$ at the inner core face
$r_2$	$r$ at the outer core face
$P_c$	Capillary pressure
$P_{c_1}$	Capillary pressure at $r_1$
$P_{c_2}$	Capillary pressure at $r_2$
$P_{c_{\max}}$	Maximum capillary pressure
$S$	Wetting phase saturation defined on $[0, P_{c_{\max}}]$ , i.e., $S(P_{c_1})$ where $0 \leq P_{c_1} \leq 1$
$\bar{S}$	Average wetting phase saturation defined on $[0, P_{c_{\max}}]$
$B$	Dimensionless factor, $B = 1 - (r_2 / r_1)^2$ for drainage and $B = 1 - (r_1 / r_2)^2$ for imbibition
$F$	Known side of the Volterra integral equation defined on $[0, P_{c_{\max}}]$
$K$	Kernel of the Volterra integral operator
$f$	Known side of the Volterra integral equation defined on $[0, 1]$
$s$	Wetting phase saturation defined on $[0, 1]$ , i.e., $s(P_{c_1})$ where $0 \leq P_{c_1} \leq 1$
$s_n$	Approximate value of wetting phase saturation defined on $[0, 1]$

### Greek

$\rho$	Phase mass density
$\Delta\rho$	Difference between the phase densities
$\omega$	Centrifuge angular velocity
$\beta_{jk}$	Gram–Schmidt orthogonalization coefficients
$\psi_j$	Base functions of $W^m$
$\bar{\psi}_j$	Orthonormal base functions of $W^m$
$S$	Wetting phase saturation defined on $[0, P_{c_{\max}}]$ , i.e., $s(P_{c_1})$ where $0 \leq P_{c_1} \leq P_{c_{\max}}$
$S_n$	Approximate value of wetting phase saturation defined on $[0, P_{c_{\max}}]$

### Metric units

$P_c$	Pascal
$r$	Meter (m)
$\rho$	Kgr/m <sup>3</sup>
$\omega$	Radian per second

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## Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Authors contribution statement

All authors contributed to the reported work for conceptualization, methodology, validation, writing-review and editing, and supervision.

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## Appendix A.

This section includes computational details based on the RKHS theory. The provided appendix presents the nomenclature and mathematical procedures used in the calculations. Suppose that  $R_y$  denote the reproducing kernel function of  $W^m[0,1]$ . By the definition of reproducing property in Eq. (18), for every  $y \in [0,1]$  and  $s \in W^m[0,1]$ , we have:

$$s(y) = (s(x), R_y^m(x))_m. \quad (\text{A.1})$$

By Eq. (18), the definition of inner product in  $W^m[0,1]$ , we have:

$$(s, R_y)_m = \sum_{i=0}^{m-1} s^{(i)}(0) R_y^{(i)}(0) + \int_0^1 s^{(m)}(x) R_y^{(m)}(x) dx \quad (\text{A.2})$$

Using integration by parts  $m$  times, we can get:

$$\begin{aligned} \int_0^1 s^{(m)}(x) \frac{\partial^m R_y(x)}{\partial x^m} dx = \\ \sum_{i=0}^{m-1} (-1)^i s^{(m-i-1)}(x) \frac{\partial^{m+i} R_y(x)}{\partial x^{m+i}} \Big|_{x=0} \\ + (-1)^m \int_0^1 s(x) \frac{\partial^{2m} R_y(x)}{\partial x^{2m}} dx, \end{aligned} \quad (\text{A.3})$$

and after a change of variable, we obtain:

$$\begin{aligned} \sum_{i=0}^{m-1} (-1)^i s^{(m-i-1)}(x) \frac{\partial^{m+i} R_y(x)}{\partial x^{m+i}} = \\ \sum_{i=0}^{m-1} (-1)^i s^{(i)}(x) \frac{\partial^{2m-i-1} R_y(x)}{\partial x^{2m-i-1}}. \end{aligned} \quad (\text{A.4})$$

Therefore,

$$\begin{aligned} (s(x), R_y(x))_m = \sum_{i=0}^{m-1} s^{(i)}(0) \left[ \frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{(m-i-1)} \frac{\partial^{2m-i-1} R_y(0)}{\partial x^{2m-i-1}} \right] \\ + \sum_{i=0}^{m-1} (-1)^{(m-i-1)} s^{(i)}(1) \frac{\partial^{2m-i-1} R_y(1)}{\partial x^{2m-i-1}} \\ + (-1)^m \int_0^1 s(x) \frac{\partial^{2m} R_y(x)}{\partial x^{2m}} dx. \end{aligned} \quad (\text{A.5})$$

By Eq. (A.1),  $R_y$  satisfies:

$$\begin{cases} (-1)^m \frac{\partial^{2m} R_y(x)}{\partial x^{2m}} = \delta(x-y), \\ \frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{(m-i-1)} \frac{\partial^{2m-i-1} R_y(0)}{\partial x^{2m-i-1}} = 0, \\ \frac{\partial^{2m-i-1} R_y(1)}{\partial x^{2m-i-1}} = 0, \quad i = 0, 1, \dots, m-1. \end{cases} \quad (\text{A.6})$$

In case of  $x \neq y$ ,  $R_y$  satisfies:

$$(-1)^m \frac{\partial^{2m} R_y(x)}{\partial x^{2m}} = 0, \quad (\text{A.7})$$

with the boundary conditions:

$$\begin{cases} \frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{(m-i-1)} \frac{\partial^{2m-i-1} R_y(0)}{\partial x^{2m-i-1}} = 0, \\ \frac{\partial^{2m-i-1} R_y(1)}{\partial x^{2m-i-1}} = 0, \quad i = 0, 1, \dots, m-1. \end{cases} \quad (\text{A.8})$$

Equation  $\lambda^{2m} = 0$ , is known as the characteristic equation of Eq. (A.7) with  $2m$  eigenvalue multiplicity for  $\lambda = 0$ . Hence, Eq. (A.6) has the following general solution:

$$R_y(x) = \begin{cases} lR_y(x) = \sum_{i=1}^{2m} c_i(y)x^{i-1}, & x < y, \\ rR_y(x) = \sum_{i=1}^{2m} d_i(y)x^{i-1}, & x \geq y. \end{cases} \quad (\text{A.9})$$

Since,

$$(-1)^m \frac{\partial^{2m} R_y(x)}{\partial x^{2m}} = \delta(x-y), \quad (\text{A.10})$$

we conclude

$$\frac{\partial^{2m-1} rR_y(y^+)}{\partial x^{2m-1}} = \frac{\partial^{2m-1} lR_y(y^-)}{\partial x^{2m-1}}, \quad (\text{A.11})$$

$$i = 0, 1, \dots, 2m-2,$$

and

$$(-1)^m \left( \frac{\partial^{2m-1} rR_y(y^+)}{\partial x^{2m-1}} - \frac{\partial^{2m-1} lR_y(y^-)}{\partial x^{2m-1}} \right) = 1. \quad (\text{A.12})$$

Eqs. (A.11), (A.12), and  $2m$  equations from (A.8) are coupled for computing  $4m$  unknowns  $c_i(y)$  and  $d_i(y)$ ,  $i=1,2,\dots,2m$ , in (A.9). Beyrami et al. [37] presented the Mathematica source code of the above equations.

Spouse that  $\psi_i$ ,  $j=1,\dots,N$ , have been determined by Eq. (20). Throughout what follows, calculation of the approximate solution  $s_n$  in Eq. (22) is described. Let  $\beta_i^* = (\beta_{i,1}^*, \beta_{i,2}^*, \dots, \beta_{i,i}^*)^T$  denote the solution of the following linear system,

$$\begin{bmatrix} (\psi_1, \psi_1)_m & (\psi_1, \psi_2)_m & \dots & (\psi_1, \psi_{i-1})_m \\ (\psi_2, \psi_1)_m & (\psi_2, \psi_2)_m & \dots & (\psi_2, \psi_{i-1})_m \\ \vdots & \vdots & \ddots & \vdots \\ (\psi_{i-1}, \psi_1)_m & (\psi_{i-1}, \psi_2)_m & \dots & (\psi_{i-1}, \psi_{i-1})_m \end{bmatrix} \begin{bmatrix} \beta_{i,1}^* \\ \beta_{i,2}^* \\ \vdots \\ \beta_{i,i-1}^* \end{bmatrix} = - \begin{bmatrix} (\psi_1, \psi_i)_m \\ (\psi_2, \psi_i)_m \\ \vdots \\ (\psi_{i-1}, \psi_i)_m \end{bmatrix}, \quad (\text{A.13})$$

and  $\beta_{i,i}^* = 1$ . Let  $\mathbf{A}$  denote the coefficient matrix of the above linear system. In order to get the entries of  $\mathbf{A}$ , using the integral operator in Eq. (12), we define  $L$  as:

$$(Ls)(x) = \int_0^x K(x,t)s(t)dt, \quad 0 \leq x \leq 1, \quad (\text{A.14})$$

$\mathbf{A}$  is Hermitian matrix and its entries can be obtained by:

$$\begin{aligned} (\psi_i(x), \psi_j(x))_m &= (\psi_i(x), [L_y R_x(y)](x_j))_m \\ &= [\bar{L}_y(\psi_i(x), R_x(y))_m](x_j) \\ &= [\bar{L}_y \psi_i(y)](x_j). \end{aligned} \quad (\text{A.15})$$

We define  $\beta_i^* = \frac{\beta_i^*}{\|\beta_i^*\|_m}$ . In order to get the solution  $s_N$ , we proceed as follow. Let  $\gamma_{i,k} = (\psi_i, \bar{\psi}_k)_m$  denote the coefficients of the following Fourier series:

$$\psi_i = \sum_{k=1}^{i-1} \gamma_{i,k} \bar{\psi}_k + \gamma_{i,i} \bar{\psi}_i. \quad (\text{A.16})$$

Hence, we have:

$$(\psi_i, \psi_i)_m = \sum_{k=1}^{i-1} \gamma_{i,k}^2 + \gamma_{i,i}^2,$$

and

$$\gamma_{i,i}^2 = \sum_{k=1}^{i-1} \gamma_{i,k}^2 - (\psi_i, \psi_i)_m,$$

We denote the matrix form of Eq. (A.16) as  $\xi = \Gamma \bar{\xi}$ , where  $\xi = [\psi_1, \psi_2, \dots, \psi_N]^T$ ,  $\bar{\xi} = [\bar{\psi}_1, \bar{\psi}_2, \dots, \bar{\psi}_N]^T$ , and  $\Gamma = [\gamma_{i,j}]_{i,j=1}^N$  is a lower triangular matrix. For lower triangular Matrix  $\mathbf{B}$ , whose  $i$ th row is  $\beta_i^T$ , we have  $\Gamma \mathbf{B} = \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix. In what follows, c source code of the above procedure is presented.

```
void CalcBeta()
{
    double sum;
    for (int i = 0; i <= N - 1; i++)
    {
        for (int k = 0; k < i; k++)
        {
            BETA[i][k] = 0.0;
            for (int q = 0; q <= k; q++)
                BETA[i][k] += Beta[k][q] * A[i][q];
        }
        sum = 0.0;
        for (int q = 0; q < i; q++)
            sum += pow(BETA[i][q], 2.0);

        Beta[i][i] = 1.0 / (sqrt(A[i][i] - sum));

        for (int j = 0; j < i; j++)
        {
            sum = 0.0;
            for (int q = j; q <= i - 1; q++)
                sum += BETA[i][q] * Beta[q][j];
            Beta[i][j] = Beta[i][i] * (-sum);
        }
    }
}

double yn(double x)
{
    double result = 0.0;
    for (int i = 0; i < N; i++)
        for (int k = 0; k <= i; k++)
            result += Beta[i][k] * f(k) * SaiBar(i, x);

    return result;
}
```

## Biographies

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