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Unit Nadarajah and Haghighi distribution: Properties and applications in quality control

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Squared estimation.

Abstract. In practice, the data related to rates and proportion may have excess of ones wherein the beta distribution does not fit well. To deal with the inflation of ones, this article introduces unit Nadarajah and Haghighi distribution. Besides deriving statistical properties of the proposed distribution, several estimation methods are discussed. In particular, maximum likelihood estimation, least squares estimation, weighted least squares estimation, maximum product of spacing, minimum spacing absolute distance estimation, minimum spacing absolute log-distance estimation, Cramér-Von-Mises, Anderson-Darling method and right-tail Anderson-Darling method are considered. Using real data sets, it is shown that the new distribution outperforms some well-known existing distributions. Furthermore, the application of the proposed distribution in quality control is also discussed. A control chart using unit Nadarajah and Haghighi distribution is constructed and its performance is evaluated using the average run length.

1. Introduction

Recently many distributions have been introduced in statistics to accommodate natural phenomena arising from diverse fields. In lifetime data analysis, Weibull distribution has a special significance and considered as the benchmark model. Depending on the shape parameter, the Weibull distribution is flexible to model increasing, decreasing, and constant hazard function. In addition, its closed form cumulative distribution function also exists. However, to deal the data with range between zero and one, beta distribution is more appropriate and many absolutely continuous distributions have been used to generate flexible distributions to accommodate the data of proportion. For example, Mazucheli et al. [1] introduced the unit-Weibull distribution and showed its flexibility over the beta distribution. Similarly, the unit-gamma distribution [2], unit logistic distribution [3], unit Lindley distribution [4], unit Gompertz distribution [5], Topp-Leone

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generated distributions [6], reflected generalized Topp-Leone power series distribution [7], etc., are introduced to deal proportion data.

Nadarajah and Haghighi [8] introduced a new extension of the exponential distribution, known as the Nadarajah and Haghighi (NH) distribution, to deal with the inflation of zeros in absolutely continuous data. Motivated by the application of NH distribution, the aim of this article is to introduce Unit Nadarajah and Haghighi (UNH) distribution. A distinct feature of UNH distribution is that it is not constructed by taking into account the positive part of the real line and neither includes special functions nor additional parameters in the formulation but it is constructed in the unit interval. As a consequence, very few distributions with unit interval/finite support are available in the literature. However, while considering real life data sets concerning percentages, proportions or fractions, etc., one needs to consider values in a limited range [9]. Likewise, survival time of units/items/subjects of interest are normally greater than zero and also the lifetime of units/items/subjects of interest cannot arrive at infinite point. In such cases, it is necessary to use a bounded model [10,11]. Similarly, there are many random variables and random processes that appear in real life applications whose values are bounded both at the lower and upper ends [12–16]. Besides, in the context of reliability measurement, Genç [17] stated that to get plausible results of reliability, it is better to have models defined on the unit interval.

In the premise of the above, the UNH distribution is suitable to handle the inflation of ones in the proportion data. For example, let compare the mean proportion of days out of 30 wherein people do some physical exercises for at least 30 minutes. If people do exercise 30 out of 30 days, then data will have inflation of one and the response will be highly skewed. In such situation, beta distribution cannot be used because it does not accommodate the occurrence of one. Similarly, comparing the proportion of rain in two cities can also lead to inflation of one when both cities have the same amount of rain in a given time. Besides introducing UNH, we estimate the parameters of the UNH using nine different methods, including Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE), Weighted Least Square Estimation (WLSE), Maximum Product of Spacing (MPS), Minimum Spacing Absolute Distance Estimation (MSADE), Minimum Spacing Absolute Log-Distance Estimation (MSALDE), Cramér-Von-Mises (CVM), Anderson-Darling (AD), Percentile Estimation (PCE), and Right-Tail Anderson-Darling (RAD). In addition to estimation of the parameters of the model, we also construct control charts using UNH distribution to show its practical application for monitoring data.

The rest of the article is organized as follows. Section 2 presents the derivation of the UNH distribution while properties including quantile function, moments, entropies, order statistic are discussed in Section 3. Section 4 discusses different estimation methods to estimate the unknown parameters of the proposed distribution. The simulation study is presented in Section 5. Control charts and their performance assessment are presented in Section 6. Real data applications are presented in Section 7, whereas concluding remarks are given in Section 8.

2. UNH distribution

The main aim of the proposed model is to deal with the inflation of ones. To this end, the probability density function and cumulative distribution function of the NH distribution with two parameters α, λ are defined as:

$$f(x;\alpha,\lambda)=\alpha\lambda(1+\lambda x)^{\alpha-1}\exp\left[1-(1+\lambda x)^{\alpha}\right],\quad x>0,\alpha,\lambda>0, \tag{1}$$

$$F(x;\alpha,\lambda)=1-\exp\left[1-(1+\lambda x)^{\alpha}\right],\quad x>0,\alpha,\lambda>0. \tag{2}$$

Now, using the transformation $Y=\exp(-X)$, we obtain the following probability density function:

$$f(y;\alpha,\lambda)=\frac{\alpha\lambda}{y}(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\quad 0<y<1. \tag{3}$$

Figure 1(a) represents the shape of the UNH distribution which is decreasing and increasing for different values of the parameters. The parameters $\alpha,\lambda>0$ are non-negative where α is the shape parameter and λ is the rate parameter.

The expression of the CDF of the UNH distribution is:

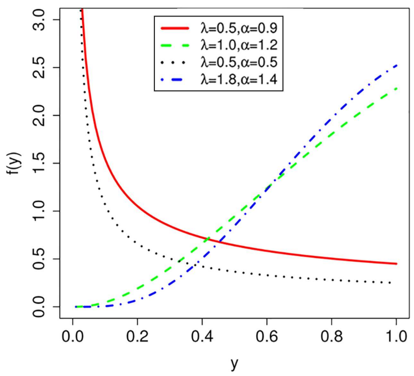
$$F(y;\alpha,\lambda)=\exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\qquad 0<y<1, \tag{4}$$

whereas the graphical depiction is given in Figure 1(b).

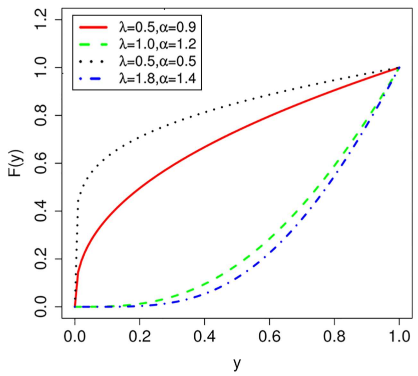
The survival function is a function that provides the probability that a particular object will survive after a specific time. The term survival function is extensively used in human mortality to show the survival time of a patient beyond a specific time. In reliability, it is used to show the performance of electric devices beyond a specific time. The survival function is given by:

$$S(y;\alpha,\lambda)=1-\exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\quad 0<y<1. \tag{5}$$

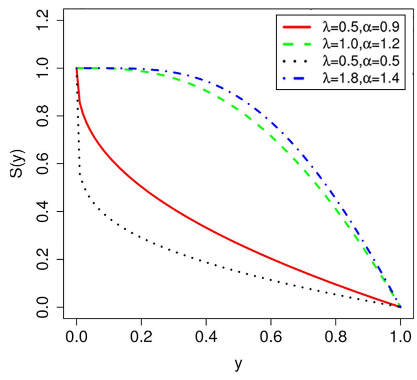
The hazard function is the ratio of probability density



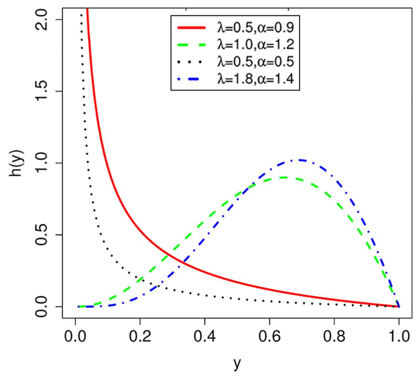
(a) PDF



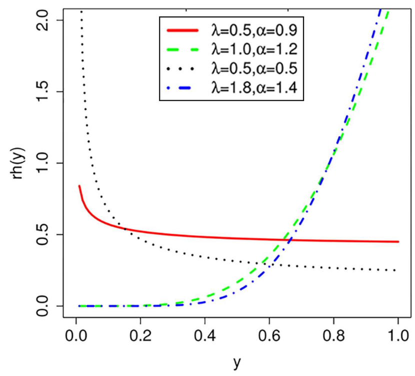
(b) CDF



(c) Survival function



(d) Hazard function



(e) Reversed hazard function

Figure 1. PDF, CDF, survival, hazard, and reverse hazard function of the UNH distribution.

function and survival function. For the UNH, it is obtained as:

$$h(y) = \frac{\alpha\lambda(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right]}{y(1-\exp\left[1-(1-\lambda\ln y)^{\alpha}\right])}. \quad (6)$$

Figure 1(d) depicts the hazard function of the UNH distribution where it can be noticed that the distribution has decreasing, increasing-decreasing hazard function for different choices of the parameters. This shows the flexibility of the UNH distribution.

The cumulative hazard is the sum of all the hazard values to a particular time. The cumulative hazard function of the UNH is given by:

$$H(y;\alpha,\lambda) =$$

$$\int_0^y \frac{\alpha\lambda(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right]}{y\left[1-\exp(1-(1-\lambda\ln y)^{\alpha})\right]}dy. \quad (7)$$

Similarly, the Reversed Hazard Function (RHF) is a important tool in reliability. The RHF of the UNH distribution is defined as:

$$r(y;\alpha,\lambda) = \frac{f(y;\alpha,\lambda)}{F(y;\alpha,\lambda)}, \quad (8)$$

and depicted in Figure 1(e).

3. Statistical properties

This section derives some important statistical properties of the UNH distribution.

3.1. Quantile function

The quantile function of the UNH is obtained by $F(y) = u$, where $u \sim Uniform(0,1)$, that is, $u = \exp(1-(1-\lambda\log(y))^{\alpha})$. The simplified form of the quantile function of the UNH is given by:

$$y = \exp\left(\frac{1}{\lambda}(1-(1-\ln(u))^{\frac{1}{\alpha}})\right). \quad (9)$$

The pth quantile function of UNH distribution is defined as:

$$y_p = \exp\left(\frac{1}{\lambda}(1-(1-\ln(p))^{\frac{1}{\alpha}})\right). \quad (10)$$

Using $p = 0.5$ in Eq. (10), one can obtain the median of the UNH distribution as under:

$$y_{0.5} = \exp\left(\frac{1}{\lambda}(1-(1-\log(0.5))^{\frac{1}{\alpha}})\right). \quad (11)$$

3.2. The moments

In this section, we derive the rth moment for the UNH distribution. The first fourth moments are the most important to describe the shape of the distribution. Suppose the random variable Y follows the UNH (λ, α), then the rth moment is given as:

$$u_r' = \int_0^1 y^r f(y, \alpha, \lambda) dy$$

$$= \int_0^1 \alpha\lambda y^{r-1}(1-\lambda\log y)^{\alpha-1}\exp(1-(1-\lambda\log y)^{\alpha}). \quad (12)$$

Using the binomial expansion on $[1-F(y)]^i$, i.e.,:

$$[1-F(y;\alpha,\lambda)]^i = \sum_{k=0}^{i}\binom{i}{k}(-1)^k[F(y)]^k. \quad (13)$$

We obtain:

$$(1-\lambda\log y)^{\alpha-1} = \sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(\lambda\log y)^i, \quad (14)$$

$$u_r' = \alpha\lambda\int_0^1 y^{r-1}\sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(\lambda\log y)^i\exp(1$$
$$-(1-\lambda\log y)^{\alpha})dy, \quad (15)$$

$$u_r' = \sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\alpha\lambda\int_0^1 y^{r-1}(\lambda\log y)^i\exp(1$$
$$-(1-\lambda\log y)^{\alpha})dy. \quad (16)$$

The rth moment of the UNH distribution cannot be expressed analytically further but can be solved numerically.

3.3. Rényi entropy

The Rényi entropy measures uncertainty of a random variable and defined as:

$$R_v(y) = \frac{1}{1-v}\log\left[\int_0^1 (f(y))^v dy\right], \quad (17)$$

$$R_v(y) = \frac{1}{1-v}\log\left[\int_0^1 \frac{(\alpha\lambda)^v}{(y)^v}(1-\lambda\ln y)^{v(\alpha-1)}\right.$$
$$\left.\exp\left(v(1-(1-\lambda\ln y))^{\alpha}\right)\right]dy. \quad (18)$$

Using the Taylor series:

$$\exp\left(v(1-(1-\lambda\ln y)^{\alpha})\right) =$$
$$\sum_{i=0}^{1}\frac{(v)^i(1-(1-\lambda\ln y)^{\alpha})^i}{i!}, \quad (19)$$

and

$$[1-(1-\lambda\ln y)^{\alpha}]^i = \sum_{k=0}^{i}\binom{i}{k}(-1)^k[(1-\lambda\ln y)^{\alpha}]^k.$$

We get:

$$R_v(y) = \frac{1}{1-v}\ln\left[\int_0^1 \frac{(\alpha\lambda)^v}{(y)^v}(1-\lambda\ln y)^{v(\alpha-1)}\right.$$
$$\left.\sum_{i=0}^{1}\frac{(v)^i(1-(1-\lambda\ln y)^{\alpha})^i}{i!}\right]dy$$
$$= \frac{1}{1-v}\ln\left[\sum_{i=0}^{1}\frac{(v)^i}{i!}\left(\int_0^1\frac{(\alpha\lambda)^v}{(y)^v}(1\right.\right.$$
$$\left.\left.-\lambda\ln y)^{v(\alpha-1)}(1-(1-\lambda\ln y)^{\alpha})^i\right)\right]dy$$
$$= \frac{1}{1-v}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{(v)^i}{i!}\right.$$
$$\left.\left(\int_0^1\frac{(\alpha\lambda)^v}{(y)^v}(1-\lambda\ln y)^{v(\alpha-1)}[(1-\lambda\ln y)^{\alpha}]^k\right)\right]dy$$
$$= \frac{1}{1-v}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{(v)^i}{i!}\right.$$
$$\left.\left(\int_0^1\frac{(\alpha\lambda)^v}{(y)^v}(1-\lambda\ln y)^{(v(\alpha-1)-k\alpha)}\right)\right]dy. \quad (20)$$

Again using:

$$(1-\lambda\ln y)^{(v(\alpha-1)-k\alpha)} = \sum_{s=0}^{v(\alpha-1)-k\alpha}\binom{v(\alpha-1)-k\alpha}{s}$$
$$(-1)^s[\lambda\log y]^s. \quad (21)$$

We obtain:

$$R_v(y) = \frac{1}{1-v}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\sum_{s=0}^{v(\alpha-1)-k\alpha}\binom{v(\alpha-1)-k\alpha}{s}\right.$$
$$\binom{i}{k}\frac{(v)^i}{i!}(-1)^s(-1)^k(\alpha\lambda)^v$$
$$\left.\left(\int_0^1 (y)^{-v}(\lambda\ln y)^s\right)\right]dy. \quad (22)$$

3.4. Stress and strength modeling

Suppose Y_1 and Y_2 are two independent continuous random variables, where $Y_1 \sim UNH(\alpha_1, \lambda_1)$ and $Y_2 \sim UNH(\alpha_2, \lambda_2)$. Then, the stress and strength, denoted by R, is determined as:

$$R = P(y_1 > y_2) = \int_{-\infty}^{\infty} f_{y_1}(y)F_{y_2}(y)dy. \quad (23)$$

$$R = P(y_1 > y_2) = \int_0^1 \frac{\alpha_1\lambda_1}{y}(1-\lambda_1 \ln y)^{\alpha_1-1}$$
$$\exp(1-(1-\lambda_1 \ln y)^{\alpha_1})\exp(1-(1-\lambda_2 \ln y)^{\alpha_2})dy.$$
$$= \int_0^1 \frac{\alpha_1\lambda_1}{y}(1-\lambda_1 \ln y)^{\alpha_1-1}$$
$$\exp(2-(1-\lambda_1 \ln y)^{\alpha_1}-(1-\lambda_2 \ln y)^{\alpha_2})dy. \quad (24)$$

Using:

$$[1-\lambda_1 \ln y]^{\alpha_1-1} = \sum_{k=0}^{\alpha_1-1}\binom{\alpha_1-1}{k}(-1)^k(\lambda_1 \ln y)^k. \quad (25)$$

$$R = P(y_1 > y_2) = \alpha_1\lambda_1 \sum_{k=0}^{\alpha_1-1}\binom{\alpha_1-1}{k}(-1)^k$$
$$\int_0^1 y^{-1}(\lambda_1 l y)^k \exp(2-(1-\lambda_1 \ln y)^{\alpha_1}$$
$$-(1-\lambda_2 \ln y)^{\alpha_2})dy.$$

Since:

$$\exp(2-(1-\lambda_1 \ln y)^{\alpha_1}-(1-\lambda_2 \ln y)^{\alpha_2}) =$$
$$\sum_{i=0}^{1}\frac{(2-(1-\lambda_1 \ln y)^{\alpha_1}-(1-\lambda_2 \ln y)^{\alpha_2})^i}{i!}, \quad (26)$$

$$R = P(Y_1 > Y_2) = \frac{\alpha_1\lambda_1}{i!}\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k$$
$$\int_0^1 y^{-1}(\lambda_1 \ln y)^k (2^i - (1-\lambda_1 \ln y)^{i\alpha_1}$$
$$-(1-\lambda_2 \ln y)^{i\alpha_2})dy. \quad (27)$$

Again using:

$$[1-\lambda_1 \log y]^{i\alpha_1} = \sum_{j=0}^{i\alpha_1}\binom{i\alpha_1}{j}(-1)^j(\lambda_1 \ln y)^j.$$

$$R = P(y_1 > y_2) = (-1)^j(-1)^k(-1)^j\frac{\alpha_1\lambda_1}{i!}$$
$$\sum_{j=0}^{i\alpha_1}\sum_{j=0}^{i\alpha_2}\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i\alpha_1}{j}\binom{i\alpha_2}{j}\binom{i}{k}$$
$$\left(2^i{\lambda_1}^k\int_0^1 y^{-1}(\ln y)^k\,dy\right)$$
$$-\left({\lambda_2}^{k+j}\int_0^1 y^{-1}(\ln y)^{k+j}\,dy\right)$$
$$-\left({\lambda_1}^k{\lambda_2}^j\int_0^1 y^{-1}(\ln y)^{k+j}dy\right). \quad (28)$$

3.5. Order statistics

In this section, we define the probability density function of the ith order statistic of the UNH distribution. Suppose a sample of size k, $Y_{(1)},...,Y_{(k)}$, be the order statistic obtained from a random sample $Y_1,...,Y_k$ of size k from a continuous population with distribution function $F(y;\varphi)$ and probability density function $f(y;\varphi)$. Then, the probability density function of $y_{(i)}$ is given by:

$$f_{Y(i)}(y) = \frac{k!}{(i-1)!(k-i)!}f_Y(y)[F(y;\varphi)]^{i-1}$$
$$[1-F(y;\varphi)]^{k-i} \quad (29)$$

where $i = 1,2,...,k$. For the UNH distribution, we have:

$$f_{Y_{(i)}}(y) = \frac{k!}{(i-1)!(k-i)!}(\frac{\alpha\lambda}{y}(1-\lambda \ln y)^{\alpha-1}$$
$$\exp(1-(1-\lambda \ln y)^{\alpha}))\Big[\exp(1-(1-\lambda \ln y)^{\alpha})\Big]^{i-1}$$
$$[1-\exp(1-(1-\lambda \ln y)^{\alpha})]^{k-i}. \quad (30)$$

While the probability density function of the largest order statistic $y_{(k)}$ is given by:

$$f_{Y_{(k)}}(y) = \frac{\alpha\lambda k}{y}(1-\lambda \ln y)^{\alpha-1}\exp(1-(1-\lambda \ln y)^{\alpha})$$
$$[\exp(1-(1-\lambda \ln y)^{\alpha})]^{k-1}, \quad (31)$$

and the probability density function of the smallest order statistic $y_{(1)}$ is given by:

$$f_{Y(1)}(y) = \frac{\alpha\lambda k}{y}(1-\lambda \ln y)^{\alpha-1}\exp(1-(1-\lambda \ln y)^{\alpha})$$
$$[1-\exp(1-(1-\lambda \ln y)^{\alpha})]^{k-1}. \quad (32)$$

4. Estimation of parameters

In this section, we discuss the unknown parameters estimation of the UNH distribution using the MLE, LSE, PCE, MPS, MADE, MSALDE, CVM, AD, and RAD methods [18,19].

4.1. MLE method

Suppose $Y_1, Y_2, ..., Y_n$ be a simple random sample from the UNH distribution. Then, the likelihood function is given by:

$$L(\lambda,\alpha,\mathbf{y}) = \prod_{i=1}^{n} f(y_i,\lambda,\alpha) = \prod_{i=1}^{n}\frac{\alpha\lambda}{y_i}(1-\lambda \log y_i)^{\alpha-1}$$
$$\exp(1-(1-\lambda \log y_i)^{\alpha}). \quad (33)$$

The log-likelihood function is given by:

$$\ln L(\lambda,\alpha,\mathbf{y}) = n\ln(\lambda\alpha) - \sum_{i=1}^{n}\ln(y_i) + (\alpha-1)\sum_{i=1}^{n}\ln(1-\lambda\ln y_i) + n - \sum_{i=1}^{n}(1-\lambda\ln y_i)^{\alpha}. \quad (34)$$

It follows that the maximum likelihood estimators MLEs of the parameters are obtained by differentiating the log-likelihood function with respect to the parameters λ and α and then equating the resulting equations to zero.

$$\frac{\partial \ln L(\lambda,\alpha,\mathbf{y})}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-\lambda\ln y_i) - \sum_{i=1}^{n}(1-\lambda\ln y_i)^{\alpha}\ln(1-\lambda\ln y_i) = 0, \quad (35)$$

$$\frac{\partial \ln L(\lambda,\alpha,\mathbf{y})}{\partial\lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{(\ln y_i)}{(1-\lambda\ln y_i)} + \alpha\sum_{i=1}^{n}(\ln y_i)(1-\lambda\ln y_i)^{\alpha-1} = 0. \quad (36)$$

The MLEs of the UNH distribution cannot be obtained in closed forms. Thus, it needs to be solved numerically for the parameters λ and α.

4.2. Least Squares Estimators (LSE)

Let $Y_1,...,Y_n$ is a random sample of size n from the distribution function $F(.)$ and $Y_{(i)} < ... < Y_{(n)}$ denote the corresponding order sample. The ordinary least squares estimators can be obtained by minimizing:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[F(y_{(i)}) - E(F(y_{(i)}))\right]^2. \quad (37)$$

Using:

$$E(F(Y_{(i)})) = \frac{i}{n+1}. \quad (38)$$

We get:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[F(Y_{(i)}) - \frac{i}{n+1}\right]^2. \quad (39)$$

Therefore, in the case of the UNH distribution, the ordinary least squares estimators of λ and α, say λ_{LSE} and α_{LSE}, respectively, can be obtained by minimizing:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]^2. \quad (40)$$

Differentiate Eq. (40) with respect to the unknown parameters λ and α and equating the resulting equations to zero, one can get the LSE estimators.

$$\frac{\partial Z(\lambda,\alpha)}{\partial\lambda} = 2\sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right] \exp(1-(1-\lambda\ln y_{(i)})^{\alpha})\alpha (1-\lambda\ln y_{(i)})^{\alpha-1}\ln y_{(i)} = 0, \quad (41)$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial\alpha} = 2\sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right] \exp(1-(1-\lambda\ln y_{(i)})^{\alpha})(1-\lambda\ln y_{(i)})^{\alpha}\ln (1-\lambda\ln y_{(i)}) = 0. \quad (42)$$

As these equations cannot be solved analytically, the non-linear equations need to be solved numerically. The weighted least squares estimators of the unknown parameters can be obtained to minimizing:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i\left[F(Y_{(i)}) - E(F(Y_{(i)}))\right]^2. \quad (43)$$

Using:

$$E(F(Y_{(i)})) = \frac{i}{n+1}. \quad (44)$$

We get:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i\left[F(Y_{(i)}) - \frac{i}{n+1}\right]^2. \quad (45)$$

The weight w_i are equal to:

$$\frac{1}{V(y_{(i)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in the case of the UNH distribution, the weighted least squares estimators of λ and α, say $\hat{\lambda}_{WLSE}$ and $\hat{\alpha}_{WLSE}$, respectively, can be obtained by minimizing:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]^2, \quad (46)$$

that is, differentiate with respect to the unknown parameters λ and α and equating to zero, we get the following equations:

$$\frac{\partial Z(\lambda,\alpha)}{\partial\lambda} = 2\sum_{i=1}^{n} w_i \left[\exp\left(1-(1-\lambda\ln y_{(i)})^{\alpha}\right) - \frac{i}{n+1}\right] \exp\left(1-(1-\lambda\ln y_{(i)})^{\alpha}\right)\alpha \left(1-\lambda\ln y_{(i)}\right)^{\alpha-1}\ln y_{(i)} = 0, \quad (47)$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial \alpha} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

$$\left[\exp(1-\left(1-\lambda \ln y_{(i)}\right)^{\alpha}) - \frac{i}{n+1}\right]$$

$$\exp\left(1-\left(1-\lambda \ln y_{(i)}\right)^{\alpha}\right)\left(1-\lambda \ln y_{(i)}\right)^{\alpha}$$

$$\ln\left(1-\lambda \ln y_{(i)}\right) = 0. \tag{48}$$

The above equations need to be solved numerically.

4.3. PCE method

If the cumulative distribution function have a closed form, then one can estimate the unknown parameter by fitting a straight line to the percentile points. In our case:

$$F(y;\alpha,\lambda) = \exp\left(1-(1-\lambda \log y)^{\alpha}\right), \tag{49}$$

therefore:

$$y = \exp\left(\frac{1}{\lambda}\left(1-(1-\log(u))^{\frac{1}{\alpha}}\right)\right). \tag{50}$$

Let $Y_1, ..., Y_n$ is a random sample of size n from the distribution function F(.) and $Y_{(i)} < ... < Y_{(n)}$ denote the corresponding ordered sample. The estimate of λ and α can be obtained by minimizing:

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[y_{(i)} - \exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)\right]^2, \tag{51}$$

that is, differentiate with respect to α and λ:

$$\frac{\partial Z(\lambda,\alpha)}{\partial \alpha} = \sum_{i=1}^{n}\left[y_{(i)} - \exp\frac{1}{\lambda}\left(1-(1-\ln(u_i))^{\frac{1}{\alpha}}\right)\right]\frac{1}{\lambda}$$

$$\exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)$$

$$(1-\ln(u_i))^{\frac{1}{\alpha}}\ln\left(1-\ln(u_i)\right) = 0, \tag{52}$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial \lambda} = \sum_{i=1}^{n}\left[y_{(i)} - \exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)\right]$$

$$\exp\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})$$

$$\frac{1}{(\lambda)^2}(1-(1-\ln(u_i))^{\frac{1}{\alpha}}) = 0, \tag{53}$$

where $u_i = \frac{i}{n+1}$.

4.4. MPS method

For the method of MPS [20,21], we define:

$$D_j(\alpha,\lambda) = F(y_{j:k}|\alpha,\lambda) - F(y_{j-1:k}|\alpha,\lambda),$$

$$j = 1,2,...,k. \tag{54}$$

Let $\hat{\alpha}_{MPS}$ and $\hat{\lambda}_{MPS}$ are the estimators obtained using the MPS for the UNH distribution parameters α and λ. The geometric mean of the spacings is defined as:

$$G(\alpha,\lambda) = \left[\prod_{j=1}^{k+1} D_j(\alpha,\lambda)\right]^{\frac{1}{k+1}}, \tag{55}$$

or maximizing the function:

$$H(\alpha,\lambda) = \frac{1}{k+1}\sum_{j=1}^{k+1}\ln D_j(\alpha,\lambda), \tag{56}$$

$$\frac{\partial H(\alpha,\lambda)}{\partial \alpha} = \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{D_j(\alpha,\lambda)}\Big[\omega_1(y_{j:k}|\alpha,\lambda)$$

$$-\omega_1(y_{j-1:k}|\alpha,\lambda)\Big] = 0, \tag{57}$$

$$\frac{\partial H(\alpha,\lambda)}{\partial \lambda} = \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{D_j(\alpha,\lambda)}\Big[\omega_2(y_{j:k}|\alpha,\lambda)$$

$$-\omega_2(y_{j-1:k}|\alpha,\lambda)\Big] = 0, \tag{58}$$

$$\omega_1(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})$$

$$(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}), \tag{59}$$

$$\omega_2(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha$$

$$(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}). \tag{60}$$

Maximizing $H(\alpha,\lambda)$ is as efficient as the MLE and the MPS estimators are consistent under more common conditions than the MLE estimators.

4.5. MSADE method

The method of MSADE [22] and the authors showed that parameters estimation by MSADE is as efficient as MLE. Furthermore, the MSADE are consistent under more flexible condition than the MLE estimators. We define:

$$D_j(\alpha,\lambda) = F(y_{j:k}|\alpha,\lambda) - F(y_{j-1:k}|\alpha,\lambda),$$

$$j = 1,2,...,k. \tag{61}$$

Then, $\hat{\alpha}_{MSADE}$ and $\hat{\lambda}_{MSADE}$, are the UNH distribution parameters α and λ are obtained by minimizing the following function with respect to α and λ.

$$T(\alpha,\lambda)=\sum_{j=1}^{k+1}\left|D_j(\alpha,\lambda)-\frac{1}{n+1}\right|, \tag{62}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\alpha}=\sum_{j=1}^{k+1}\frac{D_j(\alpha,\lambda)-\frac{1}{n+1}}{\mid D_j(\alpha,\lambda)-\frac{1}{n+1})\mid}\Big[\omega_1(y_{j:k}|\alpha,\lambda)-\omega_1(y_{j-1:k}|\alpha,\lambda)\Big]=0, \tag{63}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\lambda}=\sum_{j=1}^{k+1}\frac{D_j(\alpha,\lambda)-\frac{1}{n+1}}{\mid D_j(\alpha,\lambda)-\frac{1}{n+1})\mid}\Big[\omega_2(y_{j:k}|\alpha,\lambda)-\omega_2(y_{j-1:k}|\alpha,\lambda)\Big]=0, \tag{64}$$

where:

$$\omega_1(y_{j:k}|\alpha,\lambda)=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}), \tag{65}$$

$$\omega_2(y_{j:k}|\alpha,\lambda)=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}). \tag{66}$$

4.6. MSALDE method

The MSALDE are obtained by minimizing T(α,λ) as follows:

$$T(\alpha,\lambda)=\sum_{j=1}^{k+1}\left|\ln D_j(\alpha,\lambda)-\ln\frac{1}{n+1}\right|, \tag{67}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\alpha}=\sum_{j=1}^{k+1}\frac{\ln D_j(\alpha,\lambda)-\ln\frac{1}{n+1}}{\mid\ln D_j(\alpha,\lambda)-\ln\frac{1}{n+1})\mid}\frac{1}{D_j(\alpha,\lambda)}[\omega_1(y_{j:k}|\alpha,\lambda)-\omega_1(y_{j-1:k}|\alpha,\lambda)]=0, \tag{68}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\lambda}=\sum_{j=1}^{k+1}\frac{\ln D_j(\alpha,\lambda)-\ln\frac{1}{n+1}}{\mid\ln D_j(\alpha,\lambda)-\ln\frac{1}{n+1})\mid}\frac{1}{D_j(\alpha,\lambda)}[\omega_2(y_{j:k}|\alpha,\lambda)-\omega_2(y_{j-1:k}|\alpha,\lambda)]=0, \tag{69}$$

where

$$\omega_1(y_{j:k}|\alpha,\lambda)=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}), \tag{70}$$

$$\omega_2(y_{j:k}|\alpha,\lambda)=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}). \tag{71}$$

4.7. CVM method

To encourage our decision of CVM estimators, MacDonald [23] presented an empirical proof that the bias of these estimators is smaller than the other small distance type estimators. The CVM estimators $\hat{\alpha}_{CVM}$ and $\hat{\lambda}_{CVM}$ of the UNH distribution parameters α and λ are obtained by minimizing the following function:

$$C(\alpha,\lambda)=\frac{1}{12n}+\sum_{j=1}^{n}\left(F(y_{j:n|\alpha,\lambda})-\frac{2j-1}{2n}\right)^2. \tag{72}$$

These estimators can also be obtained by solving the following non-linear equations:

$$\frac{\partial C(\alpha,\lambda)}{\partial\alpha}=\sum_{i=1}^{n}\left(\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})-\frac{2j-1}{2n}\right)(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k})=0, \tag{73}$$

$$\frac{\partial C(\alpha,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\left(\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})-\frac{2j-1}{2n}\right)\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k})=0. \tag{74}$$

4.8. AD and RTADE methods

In this section, we define the method of AD estimation for the UNH distribution as:

$$A(\alpha,\lambda)=-k-\frac{1}{k}\sum_{j=1}^{k}(2j-1)\{\ln F(y_{j:k}|\alpha,\lambda)+\ln\bar{F}(y_{k+1-j:k}|\alpha,\lambda)\}. \tag{75}$$

These estimators can also be obtained by solving non-linear Eqs. (76)–(79) are shown in Box I. Similarly, the RTADE estimators $\hat{\alpha}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the UNH parameters α and λ are obtained by minimizing:

$$R(\alpha,\lambda)=\frac{k}{2}-2\sum_{j=1}^{k}\ln F(y_{j:k}|\alpha,\lambda)-\frac{1}{k}\sum_{j=1}^{k}(2j-1)\ln\bar{F}(y_{k+1-j:k}|\alpha,\lambda). \tag{80}$$

These estimators can also be obtained by solving non-linear Eqs. (81)–(82) are shown in Box II.

5. Simulation study

The performance of ten different estimation methods is compared using a comprehensive simulation study. For all methods, we computed biases, mean squared errors, average absolute difference between the theoretical

$$\frac{\partial A(\alpha,\lambda)}{\partial \alpha}=\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})}$$
$$-\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{(1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha}))}=0, \tag{76}$$

$$\frac{\partial A(\alpha,\lambda)}{\partial \lambda}=\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}\ln(y_{j:k})}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$
$$-\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y))^{\alpha-1}\ln(y_{j:k})}{(1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha}))}=0, \tag{77}$$

$$\frac{\partial F(\alpha,\lambda)}{\partial \alpha}=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}), \tag{78}$$

$$\frac{\partial F(\alpha,\lambda)}{\partial \lambda}=\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}). \tag{79}$$

Box I

$$\frac{\partial R(\alpha,\lambda)}{\partial \alpha}=-2\sum_{j=1}^{k}\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$
$$+\frac{1}{k}\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}=0, \tag{81}$$

$$\frac{\partial R(\alpha,\lambda)}{\partial \lambda}=-2\sum_{j=1}^{k}\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y_{j:k}))^{\alpha-1}\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$
$$+\frac{1}{k}\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y_{j:k}))^{\alpha-1}\ln(y_{j:k}))}{(1-\exp(1-(1-\lambda ln(y_{j:k}))^{\alpha}))}=0. \tag{82}$$

Box II

and empirical estimate of the distribution functions (Dabs), and the maximum absolute difference between the theoretical and empirical distribution functions (Dmax). The experiments were repeated N=10000 times by taking samples of sizes n = 20,40,60,80 and 100, with (α,λ) = (0.5,0.5), (0.5,2.0), (1.5,2.0), (1.5,0.5), (3.5,2.0), (3.0,0.5).

It is noticed from Tables 1–3 that the biases and Root Mean Square Error (RMSE) of α and λ decrease when sample size increased for all methods of estimation. The average absolute difference between the theoretical and empirical estimate of the distribution functions (Dabs) is smaller than the maximum absolute difference between the theoretical and empirical distribution functions (Dmax) for all methods of estimation. The simulation results suggest that the WLS perform better in terms of biases and RMSEs. The second better performing estimators is the MPS estimators.

Table 1. Simulation results for α=0.5 and λ=0.5.

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias (α)	0.766^{7}	0.400^{4}	0.094^{2}	-0.385^{3}	-0.085^{1}	1.787^{8}	7.415^{10}	0.513^{5}	0.647^{6}	3.388^{9}
	$RMSE$ (α)	2.409^{6}	0.400^{3}	0.219^{1}	0.712^{4}	0.221^{2}	3.050^{7}	8.825^{9}	1.017^{5}	3.919^{8}	11.528^{10}
	Bias (λ)	0.395^{2}	167.377^{10}	0.813^{7}	-0.400^{3}	-0.043^{1}	1.3234^{8}	7.892^{9}	0.546^{4}	0.666^{5}	0.792^{6}
	$RMSE$ (λ)	0.957^{2}	177.911^{10}	1.520^{5}	0.400^{1}	1.943^{6}	2.706^{7}	9.206^{9}	1.302^{3}	1.406^{4}	2.760^{8}
	Dabs	0.166^{1}	0.201^{7}	0.168^{4}	0.310^{8}	0.167^{2}	0.448^{9}	0.657^{10}	0.168^{5}	0.168^{3}	0.169^{6}
	$D\max$	0.266^{4}	0.311^{7}	0.249^{1}	0.719^{8}	0.250^{2}	0.784^{9}	0.958^{10}	0.270^{5}	0.260^{3}	0.277^{6}
	Total	22^{3}	41^{7}	20^{2}	$27^{4.5}$	14^{1}	48^{9}	57^{10}	$27^{4.5}$	29^{6}	45^{8}
40	Bias (α)	0.246^{4}	$-0.400^{6.5}$	0.115^{2}	$-0.400^{6.5}$	-0.0964^{1}	2.009^{9}	8.752^{10}	0.295^{5}	0.190^{3}	0.808^{8}
	RMSE (α)	0.637^{7}	$0.400^{3.5}$	0.195^{2}	$0.400^{3.5}$	0.140^{1}	3.639^{8}	10.085^{10}	0.585^{6}	0.572^{5}	3.915^{9}
	Bias (λ)	0.378^{1}	162.444^{10}	0.541^{7}	-0.400^{2}	-0.506^{6}	1.629^{8}	9.381^{9}	0.426^{3}	0.489^{4}	0.502^{5}
	RMSE (λ)	0.654^{2}	167.240^{10}	0.837^{5}	0.400^{1}	1.056^{7}	3.181^{8}	10.531^{9}	0.8014^{3}	0.812^{4}	1.032^{6}
	Dabs	0.166^{2}	0.199^{7}	0.168^{5}	0.310^{8}	0.165^{1}	0.460^{9}	0.659^{10}	0.168^{4}	0.167^{3}	0.168^{6}
	$D\max$	0.256^{4}	$0.312^{7.0}$	0.250^{2}	0.761^{8}	0.247^{1}	0.835^{9}	0.973^{10}	0.262^{5}	0.254^{3}	0.266^{6}
	Total	20^{2}	44^{8}	23^{4}	29^{6}	17^{1}	51^{9}	$58^{1}0$	26^{5}	22^{3}	40^{7}
60	Bias (α)	0.169^{4}	$-0.400^{7.5}$	0.120^{2}	$-0.400^{7.5}$	-0.096^{1}	2.258^{9}	$9.535^{1}0$	0.217^{5}	0.149^{3}	0.335^{6}
	RMSE (α)	0.289^{4}	$0.400^{6.5}$	0.182^{2}	$0.400^{6.5}$	0.122^{1}	3.995^{9}	10.811^{10}	0.398^{5}	0.250^{3}	1.391^{8}
	Bias (λ)	0.374^{1}	161.036^{10}	0.462^{6}	-0.400^{3}	-0.638^{7}	1.875^{8}	10.258^{9}	0.395^{2}	0.439^{4}	0.439^{5}
	RMSE (λ)	0.555^{2}	164.130^{10}	0.660^{5}	0.400^{1}	0.910^{7}	3.528^{8}	11.294^{9}	0.654^{4}	0.652^{3}	0.785^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.165^{1}	0.473^{9}	0.660^{10}	0.167^{5}	0.167^{3}	0.168^{6}
	$D\max$	0.253^{4}	0.312^{7}	0.250^{2}	0.777^{8}	0.246^{1}	0.859^{9}	0.979^{10}	0.258^{5}	0.252^{3}	0.261^{6}
	Total	17^{1}	48^{8}	21^{4}	34^{6}	18^{2}	52^{9}	58^{10}	26^{5}	19^{3}	37^{7}
80	Bias (α)	0.145^{4}	$-0.400^{7.5}$	0.123^{2}	$-0.400^{7.5}$	-0.094^{1}	2.915^{9}	9.744^{10}	0.188^{5}	0.138^{3}	0.255^{6}
	RMSE (α)	0.208^{3}	$0.400^{6.5}$	0.175^{2}	$0.400^{6.5}$	0.113^{1}	4.437^{9}	10.955^{10}	0.316^{5}	0.209^{4}	1.029^{8}
	Bias (λ)	0.371^{1}	159.932^{10}	0.421^{6}	-0.400^{4}	-0.707^{7}	2.468^{8}	10.459^{9}	0.372^{2}	0.409^{5}	0.395^{3}
	RMSE (λ)	0.507^{2}	162.206^{10}	0.571^{5}	0.400^{1}	0.875^{7}	4.043^{8}	11.430^{9}	0.569^{4}	0.568^{3}	0.650^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.166^{1}	0.522^{9}	0.661^{10}	0.167^{5}	0.167^{3}	0.167^{6}
	$D\max$	0.251^{4}	0.312^{7}	0.251^{2}	0.786^{8}	0.246^{1}	0.890^{9}	0.982^{10}	0.256^{5}	0.256^{3}	0.259^{6}
	Total	17^{1}	48^{8}	$21^{3.5}$	$35^{6.5}$	18^{2}	52^{9}	58^{10}	26^{5}	$21^{3.5}$	$35^{6.5}$
100	Bias (α)	0.137^{4}	$-0.400^{7.5}$	0.124^{2}	$-0.400^{7.5}$	-0.092^{1}	3.545^{9}	9.836^{10}	0.172^{5}	0.133^{3}	0.207^{6}
	RMSE (α)	0.186^{3}	$0.400^{6.5}$	0.169^{2}	$0.400^{6.5}$	0.108^{1}	4.761^{9}	11.036^{10}	0.272^{5}	0.191^{4}	0.479^{8}
	Bias (λ)	0.369^{2}	159.557^{10}	0.401^{6}	-0.400^{5}	-0.746^{7}	2.986^{8}	10.484^{9}	0.362^{1}	0.395^{4}	0.377^{3}
	RMSE (λ)	0.482^{2}	161.409^{10}	0.528^{5}	0.400^{1}	0.872^{7}	4.324^{8}	11.452^{9}	0.525^{3}	0.527^{4}	0.590^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.166^{1}	0.571^{9}	0.660^{10}	0.167^{5}	0.167^{3}	0.167^{6}
	$D\max$	0.251^{3}	0.312^{7}	0.250^{2}	0.791^{8}	0.246^{1}	0.916^{9}	0.982^{10}	0.255^{5}	0.251^{4}	0.257^{6}
	Total	16^{1}	48^{8}	21^{3}	36^{7}	18^{2}	52^{9}	58^{10}	24^{5}	22^{4}	35^{6}

Table 2. Simulation results for α=3.5 and λ=0.5

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias (α)	3.759^{5}	-3.400^{3}	-0.912^{1}	5.960^{7}	1.008^{4}	-3.400^{2}	-3.400^{9}	4.157^{6}	9.003^{8}	12.338^{10}
	$RMSE$ (α)	8.115^{6}	3.400^{2}	2.282^{1}	6.074^{7}	6.680^{4}	3.400^{3}	3.400^{8}	9.325^{5}	25.405^{9}	27.828^{10}
	Bias (λ)	1.482^{6}	1509.383^{10}	3.258^{2}	0.288^{7}	3.292^{3}	-0.400^{8}	-0.400^{9}	2.514^{5}	2.922^{4}	4.143^{1}
	$RMSE$ (λ)	3.207^{1}	1598.423^{10}	5.411^{5}	0.342^{2}	5.238^{4}	0.400^{3}	0.400^{9}	4.814^{6}	5.990^{7}	14.414^{8}
	Dabs	0.172^{2}	0.210^{8}	0.168^{4}	0.408^{1}	0.170^{3}	0.331^{10}	0.331^{9}	0.168^{6}	0.169^{5}	0.166^{7}
	$D\max$	0.267^{6}	0.338^{8}	0.252^{2}	0.605^{1}	0.253^{3}	0.798^{10}	0.798^{9}	0.261^{5}	0.259^{4}	0.259^{7}
	Total	33^{6}	$39^{8.5}$	19^{1}	33^{6}	29^{2}	$31^{3.5}$	$31^{3.5}$	33^{6}	$39^{8.5}$	43^{10}
40	Bias (α)	1.774^{4}	-3.400^{6}	-0.954^{3}	-0.707^{2}	-0.182^{1}	$-3.400^{7.5}$	$-3.400^{7.5}$	2.500^{5}	3.512^{9}	6.515^{10}
	RMSE (α)	6.390^{7}	3.400^{2}	2.064^{1}	4.403^{5}	4.936^{6}	$3.400^{3.5}$	$3.400^{3.5}$	7.747^{8}	15.072^{9}	18.595^{10}
	Bias (λ)	1.311^{3}	1479.768^{10}	2.200^{6}	4.754^{9}	2.323^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.973^{4}	2.061^{5}	2.545^{8}
	RMSE (λ)	2.192^{3}	1529.969^{10}	3.353^{6}	5.745^{9}	3.295^{4}	$0.400^{1.5}$	$0.400^{1.5}$	3.357^{7}	3.346^{5}	4.810^{8}
	Dabs	0.167^{3}	0.209^{7}	0.167^{4}	NaN^{10}	0.168^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{2}	0.168^{5}	0.166^{1}
	$D\max$	0.260^{6}	0.339^{7}	0.254^{2}	NaN^{10}	0.252^{1}	$0.854^{8.5}$	$0.854^{8.5}$	0.258^{5}	0.256^{3}	0.258^{4}
	Total	26^{3}	42^{9}	22^{1}	45^{10}	25^{2}	31^{5}	31^{5}	31^{5}	36^{7}	41^{8}
60	Bias (α)	0.627^{2}	-3.400^{7}	-1.049^{4}	-0.358^{1}	-0.756^{3}	$-3.400^{8.5}$	$-3.400^{8.5}$	1.416^{6}	1.308^{5}	3.815^{10}
	RMSE (α)	4.889^{7}	3.400^{3}	1.951^{2}	1.529^{1}	3.867^{6}	$3.400^{4.5}$	$3.400^{4.5}$	6.509^{8}	9.887^{9}	13.866^{10}
	Bias (λ)	1.266^{3}	1470.299^{10}	1.878^{6}	11.397^{9}	2.002^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.778^{7}	1.819^{6}	2.167^{8}
	RMSE (λ)	1.861^{3}	1509.415^{10}	2.647^{5}	11.710^{9}	2.617^{4}	$0.400^{1.5}$	$0.400^{1.5}$	2.788^{7}	2.662^{6}	3.612^{8}
	Dabs	0.166^{1}	0.209^{7}	0.167^{4}	0.585^{10}	0.168^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	$D\max$	0.257^{4}	0.340^{7}	0.254^{2}	0.842^{8}	0.251^{1}	$0.878^{9.5}$	$0.878^{9.5}$	0.257^{6}	0.255^{3}	0.257^{5}
	Total	20^{1}	44^{10}	23^{2}	38^{8}	27^{3}	34^{6}	34^{6}	34^{6}	33^{4}	43^{9}
80	Bias (α)	-0.057^{1}	-3.400^{9}	-1.109^{5}	1.133^{6}	-1.061^{4}	-3.400^{9}	-3.400^{9}	0.743^{3}	0.239^{2}	2.438^{7}
	RMSE (α)	4.084^{7}	3.400^{5}	1.875^{2}	1.411^{1}	3.183^{3}	3.400^{5}	3.400^{5}	5.619^{8}	7.174^{9}	11.097^{10}
	Bias (λ)	1.247^{3}	1466.142^{10}	1.713^{6}	16.729^{9}	1.846^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.657^{4}	1.684^{5}	1.924^{8}
	RMSE (λ)	1.710^{3}	1497.992^{10}	2.295^{4}	16.729^{9}	2.299^{5}	$0.400^{1.5}$	$0.400^{1.5}$	2.435^{7}	2.310^{6}	2.982^{8}
	Dabs	0.165^{1}	0.209^{7}	0.167^{4}	0.637^{10}	0.167^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	$D\max$	0.255^{4}	0.339^{7}	0.253^{2}	0.938^{10}	0.251^{1}	$0.894^{8.5}$	$0.894^{8.5}$	0.256^{6}	0.255^{3}	0.256^{5}
	$Total$	19^{1}	48^{10}	23^{2}	45^{9}	26^{3}	$34^{6.5}$	$34^{6.5}$	31^{5}	30^{4}	40^{8}
100	Bias (α)	-0.296^{3}	-3.400^{9}	-1.156^{4}	2.411^{7}	-1.197^{5}	-3.400^{9}	-3.400^{9}	0.283^{2}	-0.243^{1}	1.580^{6}
	RMSE (α)	3.559^{7}	3.400^{5}	1.826^{1}	2.465^{2}	2.766^{3}	3.400^{5}	3.400^{5}	4.972^{8}	5.848^{9}	9.347^{10}
	Bias (λ)	1.246^{3}	1460.600^{10}	1.630^{6}	21.344^{9}	1.7441^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.599^{4}	1.618^{5}	1.823^{8}
	RMSE (λ)	1.631^{3}	1491.261^{10}	2.116^{4}	21.372^{9}	2.117^{5}	$0.400^{1.5}$	$0.400^{1.5}$	2.250^{7}	2.138^{6}	2.708^{8}
	Dabs	0.166^{1}	0.209^{7}	0.167^{4}	0.650^{10}	0.167^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	$D\max$	0.255^{4}	0.340^{7}	0.253^{2}	0.964^{10}	0.252^{1}	$0.903^{8.5}$	$0.903^{8.5}$	0.256^{5}	0.254^{3}	0.256^{6}
	$Total$	$21^{1.5}$	48^{10}	$21^{1.5}$	$47^{9.0}$	$27^{3.0}$	$34^{6.5}$	$34^{6.5}$	$29^{4.5}$	$29^{4.5}$	40^{8}

Table 3. Simulation results for α=1.5 and λ=0.5

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias (α)	3.219^{7}	-1.400^{5}	-0.116^{1}	10.128^{9}	0.632^{2}	-1.368^{4}	-1.292^{3}	2.291^{6}	5.582^{8}	11.301^{10}
	$RMSE$ (α)	6.600^{7}	1.400^{2}	0.785^{1}	10.337^{8}	2.971^{5}	1.479^{3}	1.890^{4}	4.850^{6}	18.081^{9}	25.595^{10}
	Bias (λ)	0.698^{4}	588.098^{10}	1.648^{7}	-0.243^{1}	1.694^{8}	-0.369^{3}	-0.283^{2}	1.202^{5}	1.374^{6}	1.863^{9}
	$RMSE$ (λ)	1.646^{4}	635.295^{10}	2.920^{8}	0.270^{1}	2.898^{7}	0.670^{2}	1.466^{3}	2.606^{5}	2.866^{6}	6.559^{9}
	Dabs	0.169^{5}	0.209^{7}	0.168^{2}	0.385^{10}	0.169^{6}	0.327^{8}	0.328^{9}	0.168^{3}	0.168^{4}	0.167^{1}
	$D\max$	0.267^{6}	0.342^{7}	0.251^{1}	0.584^{8}	0.252^{2}	0.786^{9}	0.786^{10}	0.265^{4}	0.260^{3}	0.266^{5}
	Total	33^{6}	41^{9}	20^{1}	37^{8}	30^{4}	$29^{2.5}$	31^{5}	$29^{2.5}$	36^{7}	44^{10}
40	Bias (α)	1.735^{4}	-1.400^{6}	-0.084^{1}	12.951^{10}	0.117^{2}	1.969^{8}	-1.396^{5}	1.312^{3}	1.734^{7}	5.270^{9}
	RMSE (α)	4.231^{6}	1.400^{2}	0.666^{1}	13.138^{9}	1.801^{4}	6.633^{7}	1.424^{3}	3.527^{5}	8.483^{8}	15.045^{10}
	Bias (λ)	0.649^{3}	564.159^{10}	1.076^{6}	-0.292^{1}	1.171^{8}	2.493^{9}	-0.396^{2}	0.914^{4}	0.985^{5}	1.148^{7}
	RMSE (λ)	1.134^{3}	600.154^{10}	1.651^{5}	0.300^{1}	1.681^{7}	6.308^{9}	0.486^{2}	1.651^{6}	1.627^{4}	2.271^{8}
	Dabs	0.166^{1}	0.208^{7}	0.167^{4}	0.399^{9}	0.168^{6}	0.431^{10}	0.328^{8}	0.167^{5}	0.167^{3}	0.167^{2}
	$D\max$	0.259^{4}	0.350^{7}	0.253^{2}	0.609^{8}	0.251^{1}	0.860^{10}	0.841^{9}	0.261^{5}	0.257^{3}	0.262^{6}
	Total	21^{2}	$42^{8.5}$	19^{1}	38^{7}	$28^{3.5}$	53^{10}	29^{5}	$28^{3.5}$	30^{6}	$42^{8.5}$
60	Bias (α)	0.589^{3}	$-1.400^{7.5}$	-0.091^{2}	11.197^{10}	-0.077^{1}	-1.400^{6}	$-1.400^{7.5}$	0.770^{5}	0.604^{4}	2.985^{9}
	RMSE (α)	2.536^{6}	$1.400^{4.5}$	0.597^{1}	11.408^{10}	1.191^{2}	1.400^{3}	$1.400^{4.5}$	2.657^{7}	4.494^{8}	10.405^{9}
	Bias (λ)	0.641^{3}	553.105^{10}	0.918^{6}	3.993^{9}	1.014^{8}	-0.399^{1}	-0.400^{2}	0.832^{4}	0.878^{5}	0.986^{7}
	RMSE(λ)	0.963^{3}	587.759^{10}	1.297^{4}	5.085^{9}	1.333^{6}	0.412^{2}	0.400^{1}	1.347^{7}	1.298^{5}	1.713^{8}
	Dabs	0.166^{1}	0.208^{7}	0.167^{4}	0.581^{10}	0.168^{6}	0.327^{9}	0.327^{8}	0.167^{3}	0.167^{5}	0.167^{2}
	$D\max$	0.256^{4}	0.355^{7}	0.253^{2}	0.852^{8}	0.250^{1}	0.864^{9}	0.864^{10}	0.258^{5}	0.255^{3}	0.260^{6}
	Total	20^{2}	46^{9}	19^{1}	56^{10}	24^{3}	$30^{4.5}$	33^{7}	31^{6}	$30^{4.5}$	41^{8}
80	Bias (α)	0.257^{4}	-1.400^{7}	0.096^{1}	8.634^{10}	0.161^{2}	4.728^{9}	-1.398^{6}	0.499^{5}	0.214^{3}	1.884^{8}
	RMSE (α)	1.641^{5}	1.400^{3}	0.550^{1}	8.794^{10}	0.854^{2}	7.634^{8}	1.410^{4}	$2.144^{8}6$	2.624^{7}	7.712^{9}
	Bias (λ)	0.642^{2}	542.228^{10}	0.837^{5}	19.553^{9}	0.938^{7}	4.598^{8}	-0.398^{1}	0.781^{3}	0.817^{4}	0.881^{6}
	RMSE (λ)	0.884^{2}	577.891^{10}	1.125^{3}	19.771^{9}	1.171^{5}	8.041^{8}	0.437^{1}	1.174^{6}	1.128^{4}	1.416^{7}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.661^{10}	0.167^{6}	0.568^{9}	0.328^{8}	0.167^{3}	0.167^{5}	0.167^{4}
	$D\max$	0.254^{4}	0.360^{7}	0.253^{2}	0.983^{10}	0.250^{1}	0.912^{9}	0.879^{8}	0.257^{5}	0.254^{3}	0.258^{6}
	Total	18^{2}	44^{8}	14^{1}	58^{10}	23^{3}	51^{9}	$28^{5.5}$	$28^{5.5}$	26^{4}	40^{7}
100	Bias (α)	0.128^{3}	$-1.400^{7.5}$	-0.102^{2}	11.454^{10}	-0.185^{4}	5.831^{9}	$-1.400^{7.5}$	0.334^{5}	0.101^{1}	1.270^{6}
	RMSE (α)	1.193^{3}	$1.400^{4.5}$	0.518^{1}	11.541^{10}	0.676^{2}	7.480^{9}	$1.400^{4.5}$	1.799^{6}	2.129^{7}	6.010^{8}
	Bias (λ)	0.640^{2}	542.885^{10}	0.798^{5}	24.793^{9}	0.887^{7}	5.375^{8}	-0.400^{1}	0.758^{3}	0.786^{4}	0.839^{6}
	RMSE (λ)	0.840^{2}	576.600^{10}	1.038^{3}	24.969^{9}	1.079^{5}	6.822^{8}	0.400^{1}	1.085^{6}	1.045^{4}	1.286^{7}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.664^{10}	0.167^{6}	0.609^{9}	0.327^{8}	0.167^{3}	0.167^{4}	0.167^{5}
	$D\max$	0.254^{4}	0.359^{7}	0.253^{2}	0.990^{10}	0.250^{1}	0.928^{9}	0.888^{8}	0.256^{5}	0.254^{3}	0.257^{6}
	Total	$15^{1.5}$	46^{8}	$15^{1.5}$	58^{10}	$25^{4.0}$	52^{9}	30^{6}	28^{5}	23^{3}	38^{7}

Moreover, the WLS, MPS, MLE, AD, CVM, PCE estimators are among the good estimators for the UNH distribution. The LSE does not perform well. It is also confirmed that the performance of the MLE and PCE estimators are the same, as expected, and the performance of the CVM and AD estimators is the same. The additional Tables S1-S3 are given in the supplementary data file.

6. TBE control chart and performance assessment

Time-Between-Events (TBE) control charts are frequently used in reliability and other system related applications. A TBE chart monitors the inter-arrival times so it does not require sampling intervals [24]. The defects or nonconforming items from a manufacturing system are generally modeled by a Poisson process and Poisson Cumulative Sum (CUSUM) and Shewhart c charts are the examples of such control charts. Alternatively, we could use control charts that are based on inter-arrival times. These inter-arrival times are assumed to be independent and identically distributed exponential random variables. The exponential CUSUM chart and exponential chart are the two examples of these type of charts [25]. The exponential chart is preferred because one does not have to wait for the fixed time period as it plots the information immediately as soon it is obtained. A comprehensive overview of these charts is provided by Ali et al. [26].

The aim of this section is to introduce control charts to monitor the TBE data measured between zero and one scale. Moreover, as the UNH provides better fit in the case of inflation of ones in the data, the proposed TBE chart is also suitable to monitor such data. The recent contributions to monitor data of rates and proportion can be seen in [27–31] and the references cited therein.

Let β denotes the false alarm probability. To derive the control limits of the proposed chart, we equate $F(x)=\beta/2$ and $1-\beta/2$ to obtain the two-sided control chart. Similarly, equate $F(x)=\beta$ or $1-\beta$ to obtain the lower or upper-sided control limit of the one-sided chart. The simplified expressions of the ARL and control limits for the one-sided charts are given as:

$$LCL=\exp\left((1/\lambda_0)(1-(1-\log\beta)^{(1/\alpha_0)})\right),$$

$$ARL_L=1/\exp\left(1-(1-\lambda\log(LCL))^{\alpha}\right),$$

$$UCL=\exp\left((1/\lambda_0)(1-(1-\log(1-\beta))^{(1/\alpha_0)})\right),$$

$$ARL_U=1/\left(1-\exp(1-(1-\lambda\log(UCL))^{\alpha})\right). \tag{83}$$

Similarly, the control limits and ARL expressions for the two-sided control charts are given as:

$$LCL=\exp\left((1/\lambda_0)(1-(1-\log(\beta/2))^{(1/\alpha_0)})\right),$$

$$UCL=\exp\left((1/\lambda_0)(1-(1-\log(1-(\beta/2)))^{(1/\alpha_0)})\right),$$

$$ARL_{L\cup U}=1/\Bigg(\exp\left(1-(1-\lambda\log(LCL))^{\alpha}\right)$$

$$+1-\exp(1-(1-\lambda\log(UCL))^{\alpha})\Bigg). \tag{84}$$

The most common measure to access the performance of a control chart is the Average Run Length (ARL). It is defined to be the average number of points (samples) plotted until we observe a signal indicating that the process is out-of-control. The in-control ARL (ARL_0) and the out-of-control ARL (ARL_1) are the two types of ARL. Ideally, we should have a large value of (ARL_0) so that we do not have to make unnecessary adjustments to the process while a small value of (ARL_1) so that a shift in the process may be detected quickly. Further, for the Shewhart structure, the ARL is known to have geometric distribution and thus $ARL=1/p$, where "p" is the parameter of geometric distribution which represents the probability of shift detection.

Although the ARL is widely used for performance evaluation, it is to be noted that the variance of the ARL distribution is large and in some cases, nearly equal to the mean. This implies that there would be large fluctuations in the frequencies of false alarms. To overcome this drawback, the Coefficient of Variation (CV) of the run length distribution can be utilized because of the fact that the CV values do not fluctuate drastically with the increasing/decreasing magnitude of shifts. In addition, the CV values can directly be compared especially when the ARL values do not differ greatly from each other.

We conducted the ARL analysis of UNH distribution for different values of shape and scale parameters along with some additional quantities including CV, first, second, and third quartile (Q1, Q2 and Q3). It is worth mentioning that the ARL_0 value for all combinations of in-control rate (λ_0) and shape (α_0) parameters, assuming level of significance to be 0.0027, is 370.370. Furthermore, we computed the ARL values of upper, lower and two-sided control charts for all the considered combination of in-control values of the parameters. To be more specific, in our study, we used λ_0=2.5 in combination with three different values of α_0, i.e., $\alpha_0 \in (0.75, 1, 1.50)$. Thus, we have three combinations of in-control parameters $(\lambda_0, \alpha_0) = \{(2.50, 0.75), (2.50, 1.00), (2.50, 1.50)$.

For these in-control, three cases we assumed $\lambda_1 \in (0.1, 0.4, 0.5, 0.6, 0.9, 1, 1.3, 1.5, 2, 2.5, 2.7, 3)$ and $\alpha_1 \in (0.1, 0.4, 0.5, 0.6, 0.75, 0.9, 1, 1.3, 1.5)$ to represent the out-of-control situation.

6.1. Performance analysis assuming $\lambda_0 = 2.5, \alpha_0 = 0.75$

From Table 4 and Tables S4–S10, given in the Supplementary data file, it is quite clear that when we fix the value of the shape parameter α, the two-sided control chart is the quickest to detect the downward shift in the rate parameter λ. Furthermore, for fixed α, the ARL has an increasing pattern in the lower-sided chart but an opposite pattern for the upper-sided chart. The same pattern is observed for lower and upper sided charts when we fix the value of λ. The two-sided control chart, however, behaves differently; for fixed α, its ARL values increase till the nominal value of α and when $\alpha > 0.75$, the ARL has increasing trend till $\lambda < 2$ and beyond that the ARL decreases. It can also be seen that the lower-sided control chart performs poorly for $\alpha > 0.75$ (upward shift in the shape parameter) as compared to $\alpha < 0.75$. The performance of two-sided control chart also deteriorates for $\alpha > 0.75$ but not as much as it does for the lower-sided chart. On the other hand, the upper-sided control chart performs better for $\alpha > 0.75$ than the lower-sided chart. It is also noticed that the behavior of ARL for some combination of parameters is biased, i.e., $ARL_1 > ARL_0$, and we left those cells blank in the tables.

The CV analysis of Table 4 shows a decreasing pattern when we fix the value of the rate parameter λ for downward shifts and increasing pattern for upward shifts. This suggests that the lower-sided control chart is efficient for detecting large-size shifts in downward direction only. A similar behavior is observed when we fix the value of shape parameter α, that is, the chart is only efficient in detecting large-size shifts in the downward direction. For upper-sided chart, when we fix the value of λ, the CV values decrease for $\alpha > 0.75$ and increase for $\alpha < 0.75$ which implies that the chart can efficiently be used for detection of large size shifts in upward direction.

The quartile analysis from Table 5 shows that, for fixed λ, the ARL value is greater than the third quartile (Q3) or lies between second and third quartile (Q2 and Q3). This means that the ARL distribution is either highly or moderately skewed (positively). Similarly, fixing the value of α, the ARL distribution is observed highly skewed for large downward shift in λ and less skewed for comparatively small downward or upward shift in λ. The two-sided control chart shows similar characteristics. The upper-sided chart shows that for fixed λ, the distribution of ARL is moderately skewed as all the ARL values lie between Q2 and Q3. For a fixed α, the ARL distribution shows a similar pattern as it does for fixed value of λ. Similarly, one can compare the results listed in Tables S4–S10, which are given in the Supplementary data file.

7. Real data analysis

This section presents two real data applications to show the suitability of the proposed distribution and its application in quality control.

7.1. Rainfall data

The first data set has taken from [8], which is the daily rainfall (in mm) in the January for a location in Florida from 1907-2000. The mode of the original data set is zero. We transformed the data using $Y = \exp(-X)$ and the resulted data set is listed in Table 5, which represents the proportion of daily rainfall.

We compare the proposed UNH model with some other distributions, such as Kumaraswamy distribution [32]:

$$f(y; \alpha, \lambda) = \alpha\lambda y^{\alpha-1}(1 - y^{\alpha})^{\lambda-1}, \quad y \in (0, 1). \tag{85}$$

Topp-Leone distribution [17]:

$$f(y; \alpha, \lambda) = 2\alpha y^{\alpha-1}(1 - y)(2 - y)^{\alpha-1}, \quad y \in (0, 1), \tag{86}$$

reflected Generalized Topp-Leone (rGTL) distribution [7]:

$$f(y; \alpha, \lambda) = 2\alpha y^{\alpha-1}(1-y)(2-y)^{\alpha-1}, \quad y / \in (0, 1). \tag{87}$$

Beta distribution:

$$f(y; \alpha, \lambda) = \frac{1}{B(\alpha, \lambda)} y^{\alpha-1}(1 - y)^{\lambda-1}, \quad y \in (0, 1). \tag{88}$$

The values of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hanan Quinn Information Criterion (HQIC), MLEs with their standard errors, Kolmogorov-Smirnov (K-S) statistic p-values are listed in Tables 6 and 7, showed that the UNH distribution fits better than the other distributions. From Figure 2, it is clear that the proposed chart can effectively be used for monitoring the rainfall data.

7.2. Anxiety data analysis

The second data have been obtained from Bourguignon et al. [33], which is about the anxiety test performed in a group of 180 "normal" women, i.e., outside of a pathological clinic Townsville, Queensland, Australia. The data set is reproduced in Table 8.

The values of AIC, CAIC, BIC, HQIC, MLEs with their standard errors, Kolmogorov-Smirnov (K-S) statistic p-values are listed in Tables 9 and 10. From the tables, it is evident that the UNH distribution outperformed the other distributions. Furthermore,

Table 4. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the lower-sided chart with α_0=0.75, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5).

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.041	1.188	1.245	1.308	1.414	1.536	1.627	1.964	2.255
CV			0.199	0.397	0.444	0.485	0.541	0.591	0.621	0.701	0.746
Q1			0.089	0.156	0.177	0.199	0.234	0.273	0.302	0.404	0.491
Q2			0.215	0.376	0.427	0.479	0.564	0.658	0.727	0.974	1.183
Q3			0.429	0.751	0.853	0.959	1.129	1.317	1.454	1.948	2.366
ARL	**0.4**		1.121	1.718	2.048	2.491	3.489	5.184	7.012	21.689	58.019
CV			0.328	0.646	0.715	0.774	0.845	0.898	0.926	0.977	0.991
Q1			0.129	0.329	0.429	0.561	0.852	1.342	1.869	6.095	16.547
Q2			0.311	0.794	1.035	1.351	2.052	3.235	4.505	14.685	39.868
Q3			0.623	1.588	2.069	2.702	4.105	6.469	9.010	29.3692	79.737
ARL	**0.5**		1.140	1.893	2.347	2.994	4.586	7.659	11.411	53.154	213.829
CV			0.351	0.687	0.758	0.816	0.884	0.932	0.955	0.991	0.998
Q1			0.137	0.383	0.518	0.708	1.169	2.056	3.137	15.147	61.371
Q2			0.331	0.923	1.249	1.705	2.818	4.954	7.558	36.496	147.868
Q3			0.662	1.845	2.497	3.411	5.636	9.908	15.115	72.992	295.736
ARL	**0.6**		1.158	2.069	2.665	3.562	5.969	11.253	18.569	135.455	867.527
CV			0.369	0.719	0.790	0.848	0.912	0.955	0.973	0.996	0.999
Q1			0.144	0.436	0.612	0.873	1.569	3.091	5.197	38.828	249.428
Q2			0.348	1.050	1.474	2.103	3.781	7.448	12.521	93.543	600.977
Q3			0.695	2.100	2.947	4.206	7.561	14.896	25.043	187.086	1201.954
ARL	**0.9**		1.201	2.613	3.743	5.727	12.599	34.761	80.017	2745.419	97404.030
CV			0.409	0.786	0.856	0.909	0.959	0.986	0.994	0.999	0.999
Q1			0.161	0.596	0.926	1.499	3.479	9.855	22.875	789.664	28021.250
Q2			0.388	1.437	2.230	3.612	8.382	23.746	55.116	1902.633	67514.980
Q3			0.776	2.874	4.460	7.224	16.764	47.492	110.233	3805.266	135030
ARL	**1**		1.214	2.800	4.148	6.630	15.971	50.259	130.211	7945.941	550636.1
CV			0.419	0.802	0.871	0.922	0.968	0.990	0.996	0.999	0.999
Q1			0.166	0.651	1.043	1.759	4.449	14.315	37.315	2285.761	158408
Q2			0.399	1.569	2.513	4.239	10.720	34.489	89.909	5507.360	381671.5
Q3			0.798	3.138	5.025	8.479	21.440	68.979	179.817	11014.72	763343
ARL	**1.3**		1.247	3.382	5.514	10.016	31.651	149.313	561.104	224986.8	—
CV			0.445	0.839	0.905	0.949	0.984	0.997	0.999	0.999	1
Q1			0.178	0.821	1.438	2.735	8.961	42.811	161.276	64724.53	—
Q2			0.428	1.977	3.464	6.589	21.590	103.149	388.581	155948.6	—
Q3			0.856	3.955	6.928	13.179	43.180	206.298	777.163	311897.2	—
ARL	**1.5**		1.266	3.788	6.564	12.948	48.992	304.797	1485.845	2349055	—
CV			0.458	0.858	0.921	0.961	0.989	0.998	0.999	1	1
Q1			0.184	0.939	1.741	3.579	13.949	87.541	427.307	675780.9	—
Q2			0.444	2.262	4.194	8.624	33.611	210.922	1029.563	1628241	—
Q3			0.888	4.523	8.388	17.247	67.221	421.844	2059.125	3256481	—

Table 4. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the lower-sided chart with α_0=0.75, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5) (continued).

	λ	α	0.1	0.4	0.5	0.6	0.75	0.9	1	1.3	1.5
ARL	**2**		1.307	4.877	9.747	23.451	138.659	1753.213	16955.01	—	—
CV			0.485	0.892	0.947	0.978	0.996	0.999	0.999	1	1
Q1			0.199	1.254	2.658	6.602	39.746	504.224	4877.509	—	—
Q2			0.479	3.021	6.403	15.906	95.764	1214.888	11751.97	—	—
Q3			0.957	6.041	12.806	31.812	191.528	2429.776	23503.94	—	—
ARL	**2.5**		1.342	6.078	13.866	40.321	370.370	9693.294	193474	—	—
CV			0.505	0.914	0.963	0.988	0.999	0.999	1	1	1
Q1			0.210	1.600	3.843	11.455	106.405	2788.443	55658.86	—	—
Q2			0.507	3.856	9.260	27.600	256.375	6718.533	134105.6	—	—
Q3			1.014	7.711	18.521	55.201	512.749	13437.07	268211.2	—	—
ARL	**2.7**		1.354	6.591	15.819	49.513	541.553	19034.65	512333	—	—
CV			0.512	0.921	0.968	0.989	0.999	0.999	1	1	1
Q1			0.215	1.748	4.405	14.099	155.651	5475.784	147388.9	—	—
Q2			0.517	4.213	10.615	33.972	375.029	13193.47	355121.8	—	—
Q3			1.034	8.426	21.229	67.944	750.059	26386.93	710243.7	—	—
ARL	**3**		1.372	7.399	19.118	66.684	946.099	51931.07	2207735	—	—
CV			0.521	0.929	0.973	0.992	0.999	1	1	1	1
Q1			0.220	1.981	5.355	19.039	272.032	14939.49	635125.6	—	—
Q2			0.531	4.774	12.902	45.874	655.439	35995.53	1530285	—	—
Q3			1.062	9.548	25.804	91.748	1310.880	71991.060	3060570	—	—

Table 5. Daily rainfall (in mm) on the January for a location in Florida from (1907-2000).

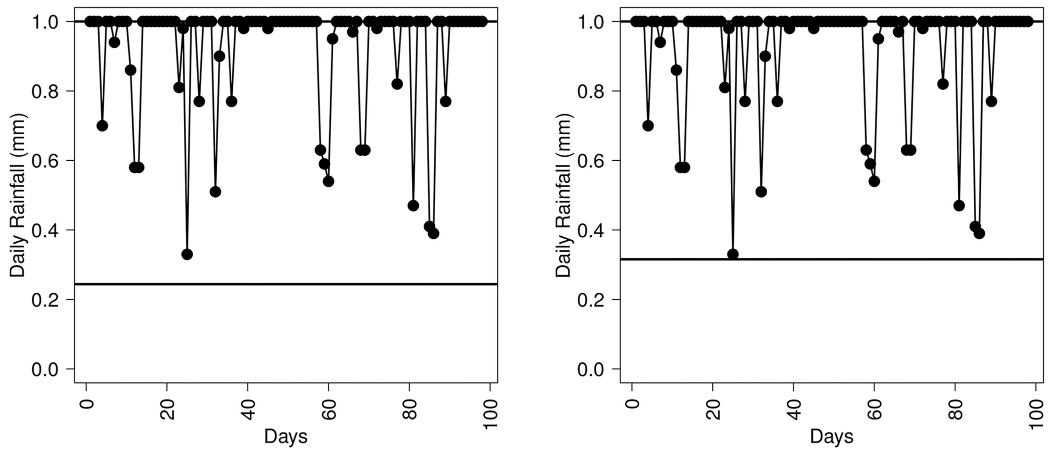
1.00	1.00	1.00	0.70	1.00	1.00	0.94	1.00	1.00	1.00	0.86	0.58	0.58	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.81	0.98	0.33	1.00	1.00	0.77
1.00	1.00	1.00	0.51	0.90	1.00	1.00	0.77	1.00	1.00	0.98	1.00	1.00	1.00
1.00	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	0.90	0.63	0.59	0.54	0.95	1.00	1.00	1.00	1.00	0.97	1.00	0.63	0.63
1.00	1.00	0.98	1.00	1.00	1.00	1.00	0.82	1.00	1.00	1.00	0.47	1.00	1.00
1.00	0.41	0.39	1.00	1.00	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00													

Table 6. AIC, BIC, CAIC, and HQIC computed after fitting different distributions on for rainfall data.

Statistic	UNH	Kumaraswamy	Topp-Leon	rGTL	Beta
AIC	–400.864	–329.714	–107.605	–99.080	–331.849
CAIC	–400.739	–329.589	–107.605	–98.955	–331.724
BIC	–395.674	–324.524	–105.01	–93.890	–326.659
HQIC	–398.764	–327.614	–106.555	–96.980	–329.749

Table 7. Maximum likelihood estimates with their standard errors (in parenthesis) and K-S test p-value for rainfall data.

Model	MLEs	K-S
UNH (α, λ)	$\hat{\alpha} = 0.513$, $\hat{\lambda}$=36.317 (0.039, 5.657)	0.717
Kumaraswamy(α, β)	$\hat{\alpha} = 5.045$, $\hat{\beta}$=0.428 (0.869, 0.050)	0.441
Topp-Leon (α)	$\hat{\alpha} = 8.568$ (0.861)	0.426
rGTL (α, υ)	$\hat{\alpha} = 0.443$, $\hat{\upsilon}$= 4.430 (0.147, 0.614)	0.920
Beta (α, λ)	$\hat{\alpha} = 4.512$, $\hat{\lambda}$= 0.439 (0.798, 0.051)	0.438



(a) UNH chart (b) Beta chart

Figure 2. Control charts for the rainfall data assuming UNH and beta distributions.

Table 8. Anxiety data set.

0.01	0.17	0.01	0.05	0.09	0.41	0.05	0.01	0.13	0.01	0.05	0.17	0.01	0.09
0.01	0.05	0.09	0.09	0.05	0.01	0.01	0.01	0.29	0.01	0.01	0.01	0.01	0.01
0.01	0.01	0.01	0.09	0.37	0.05	0.01	0.05	0.29	0.09	0.01	0.25	0.01	0.09
0.01	0.05	0.21	0.01	0.01	0.01	0.13	0.17	0.37	0.01	0.01	0.09	0.57	0.01
0.01	0.13	0.05	0.01	0.01	0.01	0.01	0.09	0.13	0.01	0.01	0.09	0.09	0.37
0.01	0.05	0.01	0.01	0.13	0.01	0.57	0.01	0.01	0.09	0.01	0.01	0.01	0.01
0.01	0.01	0.05	0.01	0.01	0.01	0.13	0.01	0.25	0.01	0.01	0.09	0.13	0.01
0.01	0.05	0.13	0.01	0.09	0.01	0.05	0.01	0.05	0.01	0.09	0.01	0.01	0.01
0.01	0.01	0.25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.37	0.25
0.05	0.05	0.25	0.05	0.05	0.01	0.05	0.01	0.01	0.01	0.17	0.29	0.57	0.01
0.01	0.05	0.01	0.01	0.01	0.17	0.29	0.57	0.01					
0.05	0.01	0.09	0.01	0.09	0.49	0.45	0.01	0.01	0.01	0.05	0.01	0.17	0.01
0.13	0.01	0.21	0.13	0.01	0.01	0.17	0.01	0.01	0.21	0.13	0.69	0.25	0.01
0.01	0.09	0.13	0.01	0.05	0.01	0.01	0.29	0.25	0.49	0.01	0.01		

Table 9. AIC, BIC, CAIC, and HQIC computed after fitting different distributions using anxiety data.

Statistic	UNH	rGTL-PS	Topp-Leon
AIC	-450.782	-443.914	-430.609
CAIC	-450.709	-443.842	-430.585
BIC	-444.522	-437.655	-427.479
HQIC	-448.241	-441.374	-429.339

Table 10. Maximum likelihood estimates with their standard errors (in parenthesis) and p-values of K-S test for anxiety data.

Model	**MLE**	**K-S**
UNH (α, λ)	$\hat{\alpha}$=8.794, $\hat{\lambda}$=0.025 (2.188,0.006)	0.356
rGTL-PS (α, υ)	$\hat{\alpha}$ = 0.537, $\hat{\upsilon}$=6.378 (0.223,1.090)	0.407
Topp-leon (α)	$\hat{\alpha}$=0.372 (0.028)	0.264

the UNH distribution has the lowest AIC and BIC values. Figure 3 indicates that anxiety level of many women fall on the lower limit of the proposed chart. This implies that these women need psychological therapy to improve their mind health.

8. Conclusion

In this article, a new distribution to accommodate the inflation of the ones is proposed. Furthermore, different properties and estimation methods are discussed in detail. From the simulation results using different methods of estimation, it is clear that the Maximum Product of Spacing (MPS), Maximum Likelihood Estimation (MLE), Anderson-Darling (AD), Cramér-Von-Mises (CVM), and PCE perform better in terms of Root Mean Squared Error (RMSE) than the rest of the methods. In addition to estimation methods, control charts are also proposed and their performance is studied using the ARL criterion. Two-real data

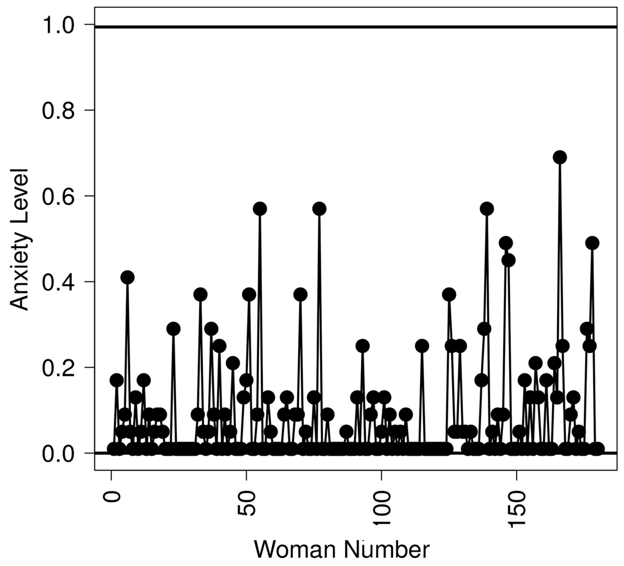


Figure 3. Control chart for anxiety data.

applications to show the practicality of the proposed distribution and utilization in process monitoring are also discussed. From the ARL study, it is noticed that for some combination of parameters, the $ARL_1 > ARL_0$ and hence, unbiased design of the control chart may be studied in the future.

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Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Sanku Dey: Conceptualization; Methodology; Resources; Software; Writing-Review and Editing.

Supplementary data

The supplementary data is available at: file:///C:/Users/pc/Downloads/Supplementary 20Data-Ali-65-SCI-2012-5167-1.pdf

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